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Poincaré duality in equivariant K-theory for $\mathbb{C}P(V)$

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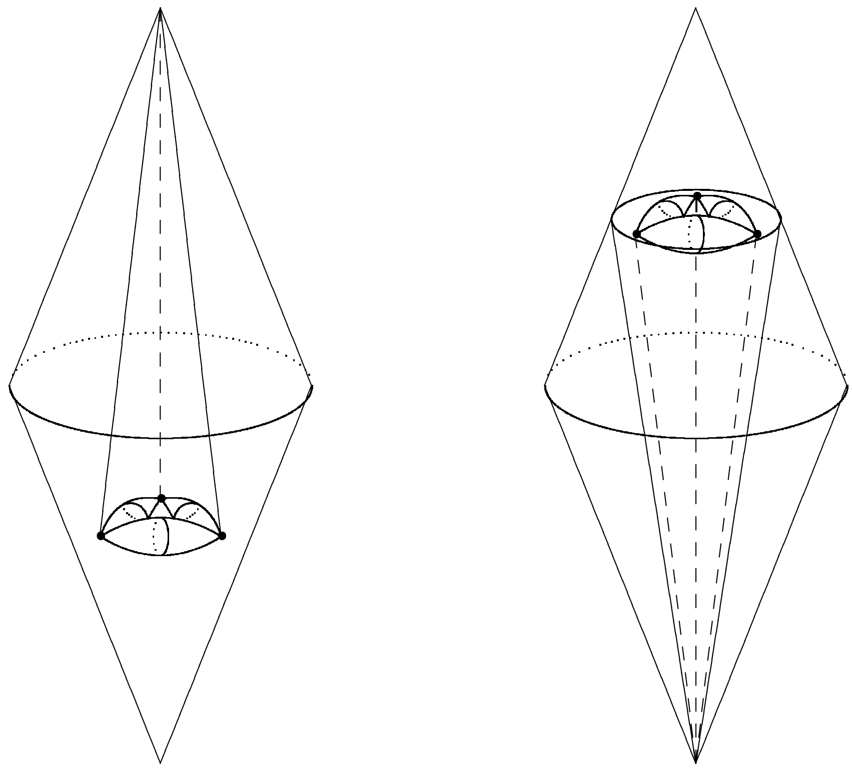

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Abstract

Henri Poincaré (1854 – 1912) has been described as "the father of Topology" [34] and "the last of the universalists in the field of mathematics" [26]. One of his famous results is the so-called *Poincaré duality*. In modern language, this *gives* isomorphisms

$$H^i(M) \xrightarrow{\cong} H_{n-i}(M)$$

between (co)homology groups of nice n-manifolds M. Equivariant K-theory is a celebrated cohomology theory due to Atiyah and Segal [55]. $\mathbb{C}P(V)$ is the G-space of lines in the complex representation V, and is the equivariant generalisation of complex projective space. In this thesis, we examine the concept of *Poincaré duality in equivariant K-theory for $\mathbb{C}P(V)$*. We work with finite groups of equivariance G and finite dimensional representations V.

We compute $K_G^*(\mathbb{C}P(V))$ and $K_*^G(\mathbb{C}P(V))$ (though we make no claim to originality here). We go on to identify a canonical element $\frac{1}{\chi(V\otimes z)} \in K_0^A(\mathbb{C}P(V))$ for abelian A. We continue with non-abelian groups. In our proofs we use the machinery of *equivariant stable homotopy theory*, developed by Adams [3] and May *et al* [41]. The outcome is an explicit, space level isomorphism

$$K_G^0(\mathbb{C}P(V)) \xrightarrow{\cong} K_0^G(\mathbb{C}P(V)). \qquad (*)$$

We consider Mackey functors and the Atiyah-Hirzebruch spectral sequence, and show how to use G-cellular structures to make deductions about $K_G^0(\mathbb{C}P(V))$ and $K_0^G(\mathbb{C}P(V))$. We continue by showing that our canonical element $\frac{1}{\chi(V\otimes z)} \in K_0^G(\mathbb{C}P(V))$ is a *fundamental class*. This permits us to conclude that our isomorphism $(*)$ really is *Poincaré duality in equivariant K-theory for $\mathbb{C}P(V)$*.

Preface . . .

. . . a.k.a. an introduction for non-mathematicians

"Doing a PhD is like banging your head against a brick wall. But there's no substitute for it if you want a thick skull."

J.P.C. Greenlees

The author distinctly recollects the conversation in which the remark above was made. It inspired many e-mails to end with the words "back to the head-banging" and consoled on more than one occasion when the skull was sore. It has become a standard answer to the question *what's it like to do a PhD?*

Greenlees' definition seldom fails to incite curiosity, which invariably runs its course to saying that *algebraic topology is all about holes.* This prompts a deal of response, best captured in the phrase

"*. . . but what the hell, holes are holes.*"

A. Maclean [43]

So it is ironic (natural?) that the current context is the only place in this thesis where the word *hole* appears. Can topologists *ever* explain what they do? This study began under the title *topics in equivariant algebraic topology.* Consulting the Oxford English Dictionary [56], *equivariant* is not listed. The present title still contains the undefined word so, in truth, the non-mathematician has as good a chance as anyone at making sense of what follows.

Acknowledgements

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Primary thanks go to Professor J.P.C. Greenlees, who was brave (foolish?) enough to take on the role of academic supervisor to the author. Without John's perennial guidance this thesis would not be what it is, and the time spent working towards it would be much bereft of interesting distraction – mathematical and otherwise. Indeed, John's contribution reaches far beyond academia: the author wishes to express his gratitude for all of the occasions on which John so enthusiastically made available his cooperation and friendship.

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The author wishes to acknowledge the utmost loving support of his parents which, unlike this thesis, has never wavered nor paused. Thanks must also go to the members of the *University of Sheffield Judo Club*, whose fighting spirit is never more evident than when "time at the bar" is called. Individual mention (of judoka and otherwise) is deserved by *C-T, F, H, M, P.*

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This manuscript was typeset using LaTeX 2_ε `<2003/12/01>` under *Red Hat Linux release 7.3 (Valhalla), Kernel 2.4.18-3.* The `.tex` file was prepared using *GNU Emacs 21.2.1.* The commutative diagrams and pictures were typeset using XY-pic version 3.7. TEX-nicalities were resolved by [37] and XY issues were resolved by painful consultation of [54, 53].

Chapter 1

Introduction

If we are to be interested in *Poincaré duality* then we should begin with a short review of the classical duality theorems.

1.1 Classical duality theorems

The adjective *classical* can be interpreted as *non-equivariant*. We are interested in duality displayed by manifolds which manifests itself as isomorphisms in the appropriate (co)homology groups. As our title suggests, our aim is to understand *Poincaré duality in the equivariant K-theory of $\mathbb{C}P(V)$* (Chapter 6). In the current section we outline the classical versions of the duality theorems which concern us. Spanier [61] and Whitehead [65] both provide excellent discussions of the development of duality.

1.1.1 Alexander duality

The *Alexander duality theorem* in its first form was published in Alexander's work [7] on generalising the famous *Jordan curve theorem*. The idea is that given a compact manifold X, nicely embedded in some sphere S^n, the complement $S^n \setminus X$ should be the "dual" of X. As such, one expects there should be an interplay between the cohomology groups of X and the homology groups of $S^n \setminus X$. Following generalisations, notably by Aleksandrov [5, 6], the classical Alexander duality theorem has an elegant statement.

Theorem 1.1.1 (Classical Alexander duality). *If X is a compact, locally contractible,* *nonempty, proper subspace of* S^n *then, for all* i*, there are isomorphisms*

$$\widetilde{H}_i(S^n \setminus X; \mathbb{Z}) \cong \widetilde{H}^{n-i-1}(X; \mathbb{Z}).$$

Proof. See [30 (Theorem 3.44)]. □

The theorem displays the duality between X and $S^n \setminus X$ in as pretty a way as one could hope, but the truly remarkable thing is that when we pass to homology, the Alexander dual $S^n \setminus X$ is *independent of the choice of embedding* $X \hookrightarrow S^n$. One can drop *locally contractible* from the hypotheses if one replaces the singular theory with the Čech theory. (Of course if X has the homotopy type of a CW-complex then these theories coincide. In this case one can remove the *locally contractible* hypothesis and still use the singular theory.) The shortcoming of the Alexander duality is that we have to have an embedding into some S^n – i.e. we have to choose an n.

1.1.2 Spanier-Whitehead duality

The standard references to classical *Spanier-Whitehead duality* are [59, 62, 60], in particular [62]. The Spanier-Whitehead duality takes place in the so-called *S-category* or the *suspension category.*

We omit full details about the S-category, contenting ourselves with only a brief glance. Given spaces X and Y we write $[X, Y]$ to mean *homotopy classes of maps* $X \longrightarrow Y$ and we write ΣX for the *suspension*[1] of X. Then we consider the sequence

$$[X, Y] \xrightarrow{\Sigma} [\Sigma X, \Sigma Y] \xrightarrow{\Sigma} [\Sigma^2 X, \Sigma^2 Y] \xrightarrow{\Sigma} \cdots$$

and define (the abelian group) $\{X, Y\}$ to be the direct limit

$$\{X, Y\} = \lim_{k} [\Sigma^k X, \Sigma^k Y].$$

The objects in the S-category are then topological spaces and the morphisms between X, Y are S-maps[2], that is precisely the elements of $\{X, Y\}$.

Spanier and Whitehead considered polyhedra X embedded in some S^n and defined an n-dual of X, $D_n(X)$, to be an S-deformation retract of the complement $S^n \setminus X$. We record

[1] See Definition 3.1.8.

[2] Given $f : X \longrightarrow Y$, we write $\{f\}$ for the S-map which is the class of f in $\{X, Y\}$.

here the duality theorem: it is stated concisely in [59 (§7)] and the details of the proofs can be found in [62].

Theorem 1.1.2 (Classical Spanier-Whitehead duality). *Given subpolyhedra $X, Y \subset S^n$ and n-duals $D_n(X), D_n(Y) \subset S^n$ there exists a unique map*

$$D_n : \{X, Y\} \longrightarrow \{D_n(Y), D_n(X)\}$$

with the properties

(i) *if $i : X \hookrightarrow Y$ and $i' : D_n(Y) \hookrightarrow D_n(X)$ are inclusions then $D_n\{i\} = \{i'\}$;*

(ii) *given $\{f\} \in \{X, Y\}$, $\{g\} \in \{Y, Z\}$ and n-duals $D_n(X)$, $D_n(Y)$, $D_n(Z)$ then*

$$D_n(\{g\}\{f\}) = D_n\{f\}D_n\{g\} \in \{Z, X\};$$

(iii) *using X, Y as n-duals for $D_n(X)$, $D_n(Y)$ respectively, the composite*

$$\{X, Y\} \xrightarrow{D_n} \{D_n(Y), D_n(X)\} \xrightarrow{D_n} \{X, Y\},$$

is the identity;

(iv) *D_n is a homomorphism;*

(v) *if we take $\Sigma D_n(X)$, $\Sigma D_n(Y)$ as $(n+1)$-dual to X, Y respectively, then we have $\Sigma D_n = D_{n+1}$;*

(vi) *if we take $D_n(X)$, $D_n(Y)$ as $(n+1)$-dual to ΣX, ΣY respectively, then we have $D_{n+1}\Sigma = D_n$;*

(vii) *given $f : X \longrightarrow Y$ we have a commutative diagram*

$$\begin{array}{ccc} H_p(X) & \xrightarrow{\quad f_* \quad} & H_p(Y) \\ \downarrow{\scriptstyle \mathfrak{D}_n} & & \downarrow{\scriptstyle \mathfrak{D}_n} \\ H^{n-p-1}(D_n(X)) & \xrightarrow{(D_n(f))^*} & H^{n-p-1}(D_n(Y)), \end{array}$$

where $\mathfrak{D}_n$ is the Alexander duality isomorphism of Theorem 1.1.1.

□

Remarks 1.1.3. **(i)** A *polyhedron* is a space covered by a simplicial complex. Whitehead [64 (Theorem 13)] has shown that a finite CW-complex has the same homotopy type as a finite polyhedron, so we have many candidates to exhibit Spanier-Whitehead duality.

(ii) As it stands, Theorem 1.1.2 still forces one to make a choice of n and then to embed in S^n. But there is a construction which addresses this: given *an* n-dual $D_n(X)$ one can repeatedly perform the so-called *desuspension* and define *the* Spanier-Whitehead dual $D(X) = \Sigma^{1-n} D_n(X)$. (The index is so chosen because an n-dual of S^m is $S^n \setminus S^m \simeq S^{n-m-1}$.) It is essentially properties (v) and (vi) of Theorem 1.1.2 which permit this. The interested reader can find more about desuspension in [1].

(iii) Spanier [60] went on to generalise the Spanier-Whitehead duality by considering *functional duals*. When one considers functional duals one has to think about *spectra* rather than spaces and we omit further detail here, other than to remark that if X is nice enough (e.g. X a finite CW-complex) then the functional dual is homotopy equivalent to the Spanier-Whitehead dual $D(X)$ of (ii).

(iv) Notice that the n-dual $D_n(X)$ of a space X is essentially the same thing as the Alexander dual of X. The improvement is that given a map $f : X \longrightarrow Y$, Spanier-Whitehead provide $D_n\{f\} : D_n(Y) \longrightarrow D_n(X)$ with the attractive properties listed in the theorem.

1.1.3 Poincaré duality

According to Whitehead [65 (p142)], the *Poincaré duality theorem* in its first form was published in Poincaré's work [51]. An introductory treatment is given by Milnor and Stasheff [49 (Appendix A)]. We present only the highlights here – details are postponed until §6.2.

Let $M = M^n$ be a (fixed) real manifold of dimension n. Some of the theory goes through *without* the requirement that M be compact but since the hypotheses of the Poincaré duality theorem include compactness, there is no harm in taking M compact from the outset.

Let $x \in M$ and consider the relative homology group $H_i(M, M \setminus \{x\}; \mathbb{Z})$. As we shall see in Lemma 6.2.1, this is infinite cyclic $\mathbb{Z}$ when $i = n$ and zero otherwise.

Definition 1.1.4. Here, $\mathbb{Z}$ coefficients and singular cohomology are understood.

(i) A *local orientation for M at $x \in M$* is a choice μ_x of one of the (two) possible generators for $H_n(M, M \setminus \{x\}) = \mathbb{Z}$.

(ii) An *orientation* for M is a function which assigns a local orientation μ_x for M at each $x \in M$ in such a way that there is a compact neighbourhood $N = N_x$ of x and a homology class $\mu_N \in H_n(M, M \setminus N)$ with $i^y_*(\mu_N) = \mu_y$ for all $y \in N$. Here, i^y_* is the natural homomorphism

$$H_n(M, M \setminus N) \longrightarrow H_n(M, M \setminus \{y\})$$

induced by the inclusion $i^y : (M, M \setminus N) \hookrightarrow (M, M \setminus \{y\})$.

(iii) An *oriented manifold* is a (compact) manifold together with a chosen orientation.

If M is oriented and compact, with local orientations μ_x, it follows [49 (Theorem A.8)] that there is *precisely* one $\mu \in H_n(M)$ with the property $i^x_*(\mu) = \mu_x$ for all $x \in M$.

Definition 1.1.5 (Fundamental classes). This μ is called the *(homology) fundamental class* of the oriented manifold M and is often written as $\mu = [M]$.

The final ingredient needed to state the theorem is the *cap product*. We adopt the definition given by May [45].

Definition 1.1.6 (Cap products). Here, $\mathbb{Z}$ coefficients and singular (co)homology are understood.

(i) Define the ((co)chain level) *cap product*

$$(-) \cap (-) : C^*(M) \otimes C_*(M) \longrightarrow C_*(M)$$

to be the composite

$$\begin{array}{ccc} C^*(M) \otimes C_*(M) & & \\ {\scriptstyle 1\otimes\Delta_*}\downarrow & \searrow^{\cap} & \\ C^*(M) \otimes C_*(M) \otimes C_*(M) & \xrightarrow{\varepsilon\otimes 1} & \mathbb{Z} \otimes C_*(M) \cong C_*(M). \end{array}$$

Here, ε is the evaluation of cochains on chains and Δ_* arises as follows. The diagonal map $\Delta : M \longrightarrow M \times M$ induces a map $\Delta_* : C_*(M) \longrightarrow C_*(M \times M)$. We use the Eilenberg-Zilber isomorphism [15 (VI, Corollary 1.4)] to view Δ_* as a map $C_*(M) \longrightarrow C_*(M) \otimes C_*(M)$. Notice that Δ_* restricts to maps

$$C_n(M) \longrightarrow \sum_{p \leq n} C_p(M) \otimes C_{n-p}(M)$$

so that, degree-wise, $\cap$ is given by maps $C^p(M) \otimes C_n(M) \longrightarrow C_{n-p}(M)$.

(ii) Define the *cap product*

$$(-) \cap (-) : H^*(M) \otimes H_*(M) \longrightarrow H_*(M)$$

to be the map obtained by passing to (co)homology in (i). Degree-wise, $\cap$ is given by maps $H^p(M) \otimes H_n(M) \longrightarrow H_{n-p}(M)$.

In Definition 1.1.6, in order to make sense of (ii) one needs to check that the (co)chain level cap product is a chain map, but that is done in [45 (p152)].

We are now in a position to record the duality theorem.

Theorem 1.1.7 (Classical Poincaré duality). *If $M = M^n$ is a compact, oriented n-manifold then*

$$H^i(M; \mathbb{Z}) \cong H_{n-i}(M; \mathbb{Z}), \qquad (1.1.8)$$

and this isomorphism is given by capping *with the fundamental class,*

$$a \longmapsto a \cap [M] \qquad (a \in H^i(M; \mathbb{Z})).$$

Proof. See [49 (p276)]. ☐

Remarks 1.1.9. **(i)** Unlike the other duality theorems in this section, the Poincaré duality theorem *depends upon a choice* of orientation. If we make a different choice of orientation for M then we may end up with a different fundamental class $[M]$ and accordingly the Poincaré duality gives a *different* isomorphism between $H^i(M)$ and $H_{n-i}(M)$. In this sense, the Poincaré duality isomorphism is *not* canonical.

(ii) In spite of (i) the Poincaré duality is desirable because it displays an interplay between the homology of M and the cohomology of M (rather than the homology of a *dual* of M and the cohomology of M).

(iii) Our Poincaré duality isomorphism (1.1.8) refers to integral singular (co)homology but in fact the theorem holds in greater generality – see §6.2.

1.2 The way ahead?

In the present section we set down some ideas relating to ways to proceed as well as dispelling what was, to the author, a myth for a substantial amount of time.

1.2.1 Poincaré duality vs. Fundamental classes

Often one uses Poincaré duality without giving a thought as to what the isomorphism in question is. For instance, if one has to compute the integral homology and cohomology groups of a compact, oriented manifold X, one can cut down on the work by only doing half and then using Poincaré duality to do the rest. But Theorem 1.1.7 is more subtle: not only does it assert the *existence* of an isomorphism, it tells us precisely what the isomorphism is.

If a Poincaré duality isomorphism $H^i(M;\mathbb{Z}) \xrightarrow{\cong} H_{n-i}(M;\mathbb{Z})$ is capping with a fundamental class $[M]$ then any given Poincaré duality isomorphism gives us a fundamental class – $[M]$ is the image of $1 \in H^0(M;\mathbb{Z})$. Conversely if we are given a fundamental class we know that capping with it gives a Poincaré duality isomorphism. In this sense one might say that a *Poincaré duality isomorphism* and a *fundamental class* are equivalent: to understand one it is necessary and sufficient to understand the other. For example, Adams [4 (III, Theorem 11.15)] examines the fundamental class $[\mathbb{C}P^n]$ in complex K-theory, and therefore he has simultaneously presented us with a Poincaré duality isomorphism.

1.2.2 Equivariant matters

As we hinted above, there are notions of Poincaré duality and fundamental classes in theories other than ordinary (co)homology. In §6.2.1 we shall consider Poincaré duality and fundamental classes in a generalised (co)homology theory. But what about the equivariant case? Could one write down an equivariant generalisation of Adams [4 (III, Theorem 11.15)]? Or how about an equivariant generalisation of the Poincaré duality isomorphism in Theorem 1.1.7?

It is now that we are beginning to tread on thin ice. In §1.1.3 we began by considering the homology groups of the pair $(M, M\setminus\{x\})$. If we agree to move to G-equivariant theory and proceed as above, we shall have to insist upon considering G-pairs, and this would severely limit our choice of $x \in M$. So already we see that care is required to set things up in the correct fashion.

1.2.3 A tentative plan

A natural question to ask might be *can we formulate an equivariant version of Poincaré duality?* By this we mean, given a G-manifold[3] M and a G-cohomology theory E_G could we expect an interplay between $E_G^*(M)$ and $E_*^G(M)$?

In view of §1.2.1, we are not interested in simply computing $E_*^G(M)$, $E_G^*(M)$ and observing isomorphisms. Instead we want to understand the interplay by writing down a Poincaré duality isomorphism, or put another way, by writing down a fundamental class.

An interplay between $E_G^*(M)$ and $E_*^G(M)$ for a random equivariant cohomology theory E_G and G-manifold M is too ambitious a task, as we shall explain in Chapter 6. As a compromise, we propose to investigate the issues arising when G is a finite group, E_G is equivariant K-theory and $M = \mathbb{C}P(V)$ is the equivariant generalisation of $\mathbb{C}P^{n-1}$ (see Definition 1.3.4).

1.2.4 The way to avoid?

> *"But topologists also like to do sums and see how things work out in concrete cases . . . "*

J.F. Adams [4 (p268)]

Although there exist treatments [41] of Poincaré duality in the equivariant case, [41] has been described [29 (p287)] as "encyclopedic" and the work is intended for use in the

[3] See Definition 1.3.10.

equivariant stable homotopy category (see §4.1). Likewise, there are treatments [21] of orientations in the equivariant setting, but by admission of the authors [21 (Abstract)] the "... focus here is on the geometric and homotopical aspects, rather than the cohomological aspects ... ".

In what follows we aim to pursue Adams' spirit above: the cons are that we restrict to considering equivariant K-theory and $\mathbb{C}P(V)$ but the pros are that we gain a truly explicit picture, at the level of spaces, of *Poincaré duality in equivariant K-theory for* $\mathbb{C}P(V)$.

1.3 $\mathbb{C}P(V)$

Before we can begin our study of duality we shall need to understand what we mean by $\mathbb{C}P(V)$.

Definition 1.3.1. Let V be a unitary representation of a group G. We then define the *sphere of* V, $S(V)$, the *disc of* V, $D(V)$, and the *one-point compactification of* V, S^V by

(i) $S(V) = \{v \in V \mid \|v\| = 1\}$;

(ii) $D(V) = \{v \in V \mid \|v\| \leq 1\}$;

(iii) $S^V = D(V)/S(V)$.

Since V is unitary, these each inherit the structure of a G-space from V.

Remark 1.3.2. In fact [8] *any* finite dimensional complex representation of a finite group is conjugate to a unitary representation, so we assume without loss that all of our finite dimensional representations of finite groups will be unitary.

A common point of view on S^V is to take $S^V = V \cup \{\infty\}$ with an open set of S^V being either an open set of V or a set of the form $S^V \setminus C$ for some compact set $C \subseteq V$. We justify:

Lemma 1.3.3. *There is a G-equivariant homeomorphism* $D(V)/S(V) \cong_G V \cup \{\infty\}$.

Proof. Let $\bar{\lambda}$ be a homeomorphism $[0,1] \xrightarrow{\cong} [0,\infty]$ in which $0 \longmapsto 0$ and $1 \longmapsto \infty$. Put $\lambda = \bar{\lambda}_{|[0,1)}$. Then given $v \in D(V)/S(V)$ define

$$f(v) = \begin{cases} \lambda(\|v\|)v & \text{if } v \notin S(V) \\ \infty & v \in S(V) \end{cases}$$

and f gives the required homeomorphism $D(V)/S(V) \longrightarrow V \cup \{\infty\}$. Equivariance follows from the fact that V is unitary. □

Definition 1.3.4 ($\mathbb{C}P(V)$). Let G be a group and let V be a finite dimensional unitary complex representation of G. Take T to be the circle group $T = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and write z for the natural representation of T. Writing $V \otimes z$ for $V \otimes_{\mathbb{C}} z$, we have a representation $V \otimes z$ of $G \times T$ and we define $\mathbb{C}P(V)$, the *complex projective space associated to* V, to be the quotient space

$$\mathbb{C}P(V) = S(V \otimes z)/T$$

with the quotient topology. Thus $\mathbb{C}P(V) = \{T(v \otimes 1) \mid v \in S(V)\}$: informally $\mathbb{C}P(V)$ is the space whose points are the *complex lines in* V. This is a G-space via the action

$$g \cdot T(v \otimes 1) = T(gv \otimes 1) \qquad (g \in G). \tag{1.3.5}$$

Remark 1.3.6. Often when people are in the habit of thinking about things equivariantly, they say "X" rather than "G-X". Nonetheless, if one likes to think of $\mathbb{C}P(V)$ as *lines in* V then one should make certain that one really is thinking of *lines* and **not** G-lines. (A *line in* V is $\ell = \mathrm{Span}_{\mathbb{C}}\{v\}$ with $0 \neq v \in V$ whereas a G-line would be a line ℓ for which the action of G on V restricts to an action on ℓ – in other words a G-line would be a one dimensional subrepresentation.)

The following two results are elementary but will be of use later. We adopt the convention here, as elsewhere, that by an n-dimensional $\mathbb{C}G$-module we mean a $\mathbb{C}G$-module whose dimension as a complex vector space is n.

Proposition 1.3.7. *Suppose that V is a $\mathbb{C}G$-module of dimension $n+1$ which decomposes as $V \cong_G W \oplus \alpha$ for some one-dimensional α. Then there is a G-equivariant homotopy equivalence $\mathbb{C}P(V) \setminus \mathbb{C}P(\alpha) \simeq_G \mathbb{C}P(W)$.*

Proof. The proposition is equivalent to proving that

$$A := \Big(S((W \oplus \alpha) \otimes z)/T\Big) \setminus \Big(S(\alpha \otimes z)/T\Big) \simeq_G S(W \otimes z)/T =: B.$$

Define maps $f : A \longrightarrow B$ and $g : B \longrightarrow A$ by

$$f(T((w_1, \dots, w_n, a) \otimes 1)) = T\left(\frac{(w_1, \dots, w_n)}{\|(w_1, \dots, w_n)\|} \otimes 1\right);$$
$$g(T((w_1, \dots, w_n) \otimes 1)) = T((w_1, \dots, w_n, 0) \otimes 1).$$

One checks that f, g are well defined and G-equivariant. Moreover, $fg = 1_B$ and one checks that $gf \simeq_G 1_A$ via the G-homotopy

$$h_\mu(T((w_1, \dots, w_n, a) \otimes 1)) = T\left(\frac{(w_1, \dots, w_n, \mu a)}{\|(w_1, \dots, w_n, \mu a)\|} \otimes 1\right) \qquad (\mu \in [0, 1]).$$

□

Proposition 1.3.8. *Suppose that α is a one dimensional $\mathbb{C}G$-module. With notation as in Definition 1.3.4, we have a $G \times T$-equivariant homeomorphism*

$$(G \times T)/\operatorname{Ker}(\alpha \otimes z) \cong_{G \times T} S(\alpha \otimes z).$$

In writing $\operatorname{Ker}(\alpha \otimes z)$ we are implicitly viewing $\alpha \otimes z$ as a homomorphism $G \times T \longrightarrow \mathbb{C}^\times$, and on the left hand side we understand / to mean taking quotient under group action. The reader will almost certainly prefer the pictures in Figure 1.1 to the proof below.

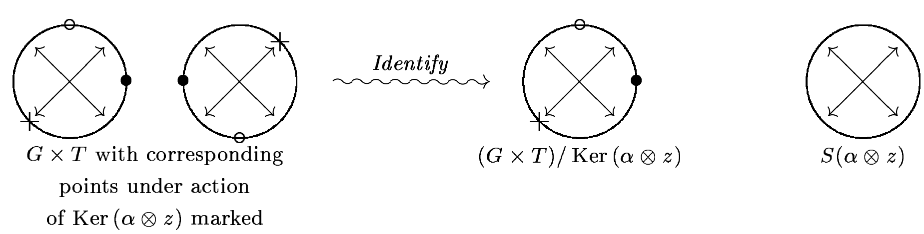


Figure 1.1. $(G \times T)/\operatorname{Ker}(\alpha \otimes z)$ and $S(\alpha \otimes z)$ where $G = C_2$ is cyclic of order two and α is the non-trivial one dimensional $\mathbb{C}G$-module. One easily checks that $\operatorname{Ker}(\alpha \otimes z) = \{(1, 1), (-1, -1)\}$.

Proof of Proposition 1.3.8. Writing H for $\mathrm{Ker}\,(\alpha \otimes z)$, define $F : (G \times T)/H \longrightarrow S(\alpha \otimes z)$ by

$$(g, \lambda)H \longmapsto \frac{\alpha(g) \otimes \lambda}{\|\alpha(g) \otimes \lambda\|}.$$

One checks that this is equivariant and well defined. Moreover, F is a continuous bijection from a compact space to a Hausdorff space, so it is a homeomorphism. □

Remark 1.3.9. In fact we can go a little further and identify $\mathrm{Ker}\,(\alpha \otimes z)$. One easily checks that we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Ker}\,(\alpha \otimes z) & \hookrightarrow & G \times T & \xrightarrow{\alpha \otimes z} & \mathbb{C}^{\times} \simeq T \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow = \\
0 & \longrightarrow & 0 & \longrightarrow & G & \hookrightarrow & G \times T & \xrightarrow{\varepsilon \otimes z} & \mathbb{C}^{\times} \simeq T
\end{array}$$

in which the rows are exact and the unspecified vertical maps are defined by the matrix $\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$. Noting that this matrix is non-singular it now follows from the Five Lemma [15 (IV, Lemma 5.10)] that $\mathrm{Ker}\,(\alpha \otimes z) \cong G$.

If we view $\mathbb{C}P(V)$ as *lines in* V, we observe that, ignoring equivariant structure, $\mathbb{C}P(V) \cong \mathbb{C}P^{n-1}$, where $n = \dim_{\mathbb{C}}(V)$. Often, people write n for n-dimensional complex space with trivial G-action: under this convention, $\mathbb{C}P(n) \cong \mathbb{C}P^{n-1}$. This explains why we view $\mathbb{C}P(V)$ as being an equivariant generalisation of $\mathbb{C}P^{n-1}$, and prompts us to think of $\mathbb{C}P(V)$ as a *G-manifold.*

Definition 1.3.10. **(i)** By a *G-manifold M* we mean a manifold M which is also a G-space.

(ii) By a *smooth G-manifold M* we mean a G-manifold whose underlying manifold is smooth and on which the action of G is smooth in the sense of Bredon [14].

The following proposition is clear.

Proposition 1.3.11. *$\mathbb{C}P(V)$ is a smooth G-manifold.* □

Chapter 2

G-cell structures for $\mathbb{C}P(V)$

In our quest to understand $\mathbb{C}P(V)$, we simplify the task by breaking $\mathbb{C}P(V)$ into simple pieces. The appropriate way of doing this is by looking at *cellular structures* for $\mathbb{C}P(V)$. This chapter, then, is essentially a list of cellular structures for a variety of $\mathbb{C}P(V)$. Our aim is to become accustomed to $\mathbb{C}P(V)$ by producing an arsenal of cellular structures. These will later enable us to perform equivariant (co)homology computations with ease. An encouraging side effect is the fact that many of the resulting pictures are pleasing to the eye.

Lewis [39] has discussed $(\mathbb{Z}/p)$-CW-structures for $\mathbb{C}P(V)$ when V is a representation of $\mathbb{Z}/p$ but we shall be interested in examples with more complicated groups of equivariance. Cole, Greenlees and Kriz [18 (§3)] discuss equivariant Schubert cells for Grassmanians but we shall be more down to earth and consider G-cells $G/H \times e^i$, where $H \leq G$ and e^i is an i-cell in the usual (non-equivariant) sense. For aesthetic and economic reasons we shall often write $\frac{G}{H}$ in place of G/H but we agree to mean left cosets by this. Precisely,

$$\frac{G}{H} = G/H = \{gH \mid g \in G\}.$$

Bredon [13] developed what is known as *equivariant ordinary cohomology* and we adopt his definition of a G-CW-complex.

Definition 2.0.1. Let G be a finite group. By a *G-CW-complex* we mean a CW-complex X and an action of G on X by cellular maps such that, for all $g \in G$, the fixed point set

$$X^g = \{x \in X \mid gx = x\}$$

is a subcomplex of X.

Inherent in the structure of a (non-equivariant) CW-complex are the *attaching maps* or *characteristic maps*: if $X^{(i)}$ is the i-skeleton, then for each $(i+1)$-cell e^{i+1} there is a map $f : S^i \longrightarrow X^{(i)}$, for which we view S^i as $\partial\overline{e^{i+1}}$.

In the equivariant scene we consider G-orbits of (non-equivariant) cells and call each of these a *G-cell* or an *equivariant cell* of the appropriate dimension. If each element in the G-orbit of a (non-equivariant) $(i+1)$-cell σ has isotropy subgroup $H \leq G$ we write $G/H \times e^{i+1}$ for the G-$(i+1)$-cell $\mathrm{Orbit}_G(\sigma)$. It is then necessary to specify an attaching map $S^i \longrightarrow X^{(i)}$ only for $eH \times \partial\overline{e^{i+1}}$ because the action of G specifies all we need to know.

2.1 A real interlude

Let us begin by considering a *real* example. The benefit is that one can then visualise all that is going on and we can obtain a complete picture. In the complex cases which follow, we may obtain only partial information. We shall need the appropriate analogue of Definition 1.3.4 for the real case.

Definition 1.3.4.$\mathbb{R}$. Let G be a finite group and let $V_{\mathbb{R}}$ be a finite dimensional real representation of G. Take $T_{\mathbb{R}}$ to be the real circle group $T_{\mathbb{R}} = \{\lambda \in \mathbb{R} \mid |\lambda| = 1\}$ and write $z_{\mathbb{R}}$ for the natural representation of $T_{\mathbb{R}}$. Then $V_{\mathbb{R}} \otimes z_{\mathbb{R}}$ is a representation of $G \times T_{\mathbb{R}}$ and we define $\mathbb{R}P(V_{\mathbb{R}})$, the *real projective space associated to* $V_{\mathbb{R}}$, to be the quotient space

$$\mathbb{R}P(V_{\mathbb{R}}) = S(V_{\mathbb{R}} \otimes z_{\mathbb{R}})/T_{\mathbb{R}},$$

with the quotient topology.

Now take $G = A_5$ to be the alternating group of even permutations of five objects. It is well known [52] that A_5 is the (direct) symmetry group of the icosahedron, a regular solid with twelve vertices, thirty edges and twenty faces, as shown in Figure 2.1 and described in [20].

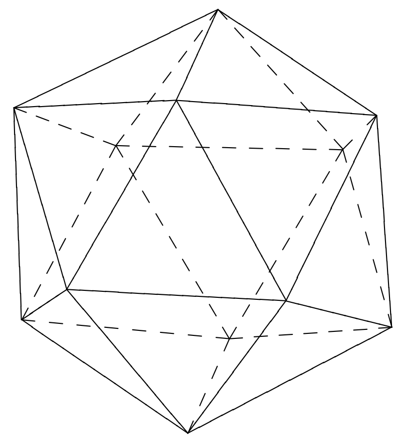


Figure 2.1. The icosahedron.

The fact that A_5 is the (direct) symmetry group of a platonic solid means that there is a three dimensional *real* representation of A_5 – call this $V_{\mathbb{R}}$. Looking at the character table of A_5 (Table A.11 on page 172) we see that there are two choices: either $V_{\mathbb{R}}$ has character γ or γ'. Let us fix $V_{\mathbb{R}}$ to have character γ.

Thus we have $\mathbb{R}P(V_{\mathbb{R}}) = \{T_{\mathbb{R}}x \mid x \in S(V_{\mathbb{R}})\} = \{\{x, -x\} \mid x \in S(V_{\mathbb{R}})\}$. Non-equivariantly this is homeomorphic to the real projective plane $\mathbb{R}P^2$, and we can draw a picture of a plane model $\mathbb{R}P^2$, equivalently of an icosahedron after identifying antipodal points. See Figure 2.2.

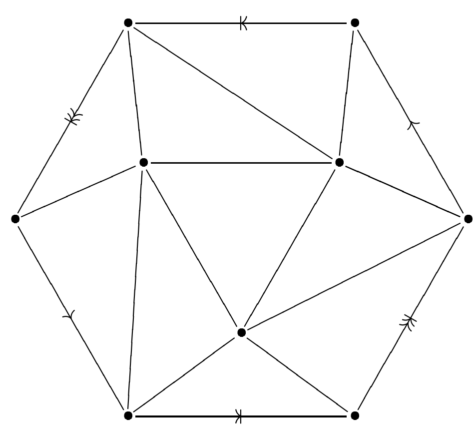


Figure 2.2. A non-equivariant cell structure for $\mathbb{R}P(V_{\mathbb{R}})$. Edges labelled with corresponding arrows are to be identified in the usual way.

Of course, what we are really interested in is an *equivariant* cellular structure for $\mathbb{R}P(V_{\mathbb{R}})$, and, in this geometric case, that is not so difficult to produce.

Proposition 2.1.1. *Let $X = \mathbb{R}P(V_{\mathbb{R}})$, $G = A_5$, I be the icosahedron and I^1 its first barycentric subdivision. Then X has the structure of a G-simplicial complex, as shown in Figure 2.3, with vertices, edges and faces as follows.*

Vertices: (a) *six vertices, shown as ♠, arising from the six antipodal pairs of vertices of I;*

(b) *fifteen vertices, shown as ♣, arising from the fifteen antipodal pairs of barycentres of edges of I;*

(c) *ten vertices, shown as ♡, arising from the ten antipodal pairs of barycentres of faces of I;*

Edges: (d) *thirty edges, shown as solid strokes, arising from the thirty antipodal pairs of half-edges of I;*

(e) i. *thirty edges, shown as dashed strokes, arising from the thirty antipodal pairs of edges of I^1 joining a vertex of I to the barycentre of a face of I;*

ii. *thirty edges, shown as dotted strokes, arising from the thirty antipodal pairs of edges of I^1 joining a vertex of I^1 but not of I to the barycentre of a face of I;*

Faces: (f) *sixty faces, arising from the sixty antipodal pairs of faces of I^1.*

Remark 2.1.2. Proposition 2.1.1 makes reference to a *G-simplicial complex* structure, which we have not defined. In view of Definition 2.0.1 it is clear what we should be talking about: *Let G be a finite group. By a G-*simplicial complex *we mean a simplicial complex X and an action of G on X by simplicial maps such that, for each $g \in G$, the fixed point set*

$$X^g = \{x \in X \mid gx = x\}$$

is a subcomplex of X.

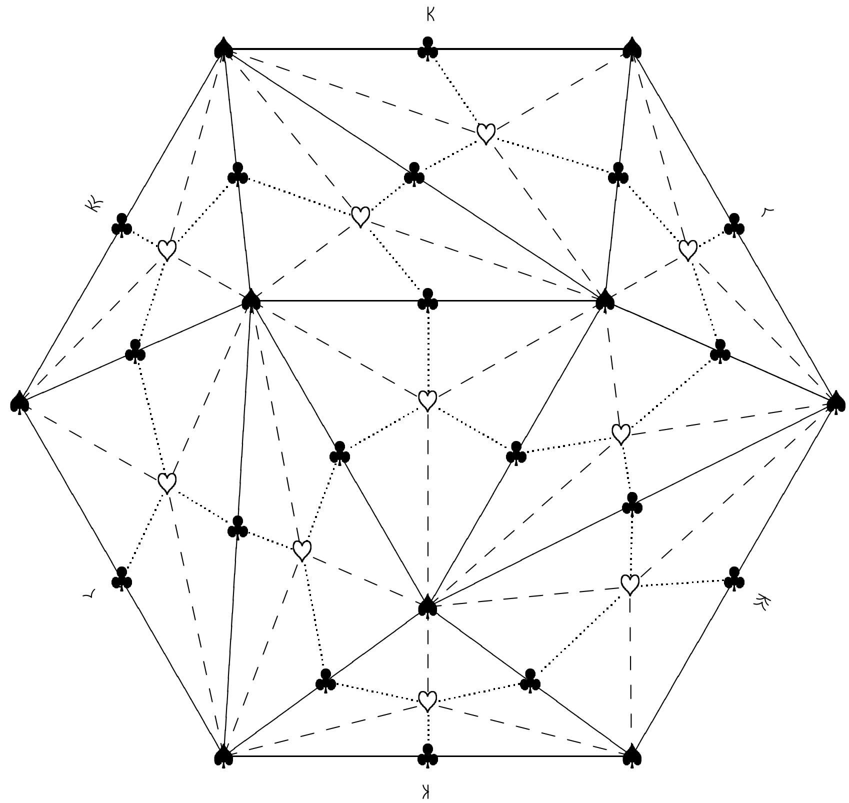


Figure 2.3. An equivariant cell structure for $\mathbb{R}P(V_{\mathbb{R}})$. Edges labelled with corresponding arrows are to be identified in the usual way, although the identification arrows have been displaced from their edges for clarity. Fixed points under rotations of order five, corresponding to case (a) of Proposition 2.1.1 are shown with ♠; ♣ represents points fixed by rotations of order two (case (b)) and ♡ represents points fixed by rotations of order three (case (c)). Note that each ♠ (resp. ♡) is fixed by a *unique* copy of C_5 (resp. C_3) but also by five (resp. three) copies of C_2. Thus the points ♠ have isotropy D_{10} and the ♡ have isotropy D_6. Each ♣ is fixed by precisely three rotations of order two which reside in a copy of V_4.

Proof of Proposition 2.1.1. The picture in Figure 2.3 shows a simplicial complex, and since the action of $g \in A_5$ is a symmetry of the icosahedron, the action is simplicial. It remains only to check the condition about fixed point sets. These are described in [20] but the geometry is obvious: recalling that A_5 acts via rotations, we need only observe that the axes of rotations pass through

(a) antipodal vertices of I, of which there are six. (These correspond to rotations of order five, and are shown as ♠ in Figure 2.3.)

(b) midpoints of antipodal edges of I, of which there are fifteen. (These correspond to rotations of order two, and are shown as ♣ in Figure 2.3.)

(c) barycentres of antipodal faces of I, of which there are ten. (These correspond to rotations of order three, and are shown as ♡ in Figure 2.3.)

This takes care of all fixed points sets $(V_{\mathbb{R}})^g$ but we need to consider the $(\mathbb{R}P(V_{\mathbb{R}}))^g$. In other words we have to ask *are there $v \in V_{\mathbb{R}}$ and $g \in G$ so that $gv = -v$?* Clearly any such g must be of order two, and any g of order two sends points in the plane perpendicular to the axis of rotation to their antipode, as in Figure 2.4. Note that this means that all of the edges in Figure 2.3 have isotropy C_2 and all of the faces are A_5-free. Observing that the double strokes in Figure 2.4 lie on edges of I^1 we see that the required fixed point sets are indeed subcomplexes. □

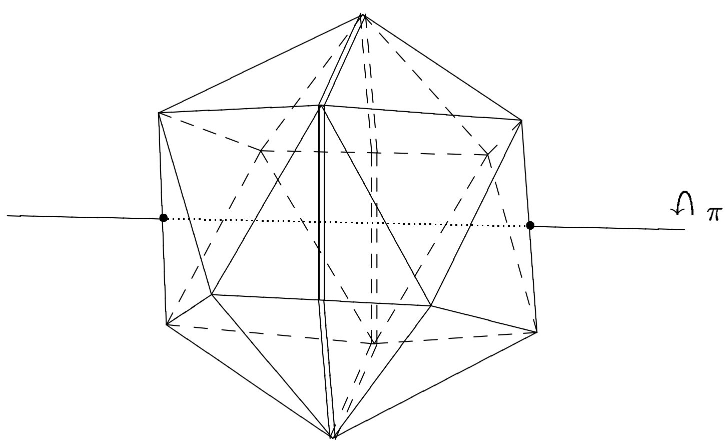


Figure 2.4. The icosahedron and a rotation of order two. Each point in the plane perpendicular to the axis is sent to its antipode. The intersection of that plane with the icosahedron is shown by lines with a double stroke.

Proposition 2.1.3. *Fix a simplex σ in Figure 2.3. Let τ be any other simplex of the same dimension as σ. Suppose further that σ, τ fall into the same class in the classification of Proposition 2.1.1. Then $\tau \in \operatorname{Orbit}(\sigma)$.*

Proof. Take a model of an icosahedron. Play with it. □

In view of Proposition 2.1.3, Figure 2.3 is redundant: it is the A_5-orbit of the simplex shown in Figure 2.5. Precisely,

$$\mathbb{R}P(V_{\mathbb{R}})/A_5 \text{ is as shown in Figure 2.5.} \tag{2.1.4}$$

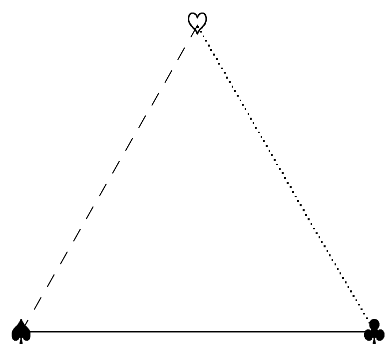


Figure 2.5. All the information contained in Figure 2.3 is given here, once we know the action of A_5.

A convenient way of expressing this equivariant information is to say that $\mathbb{R}P(V_{\mathbb{R}})$ has an A_5-equivariant simplicial complex structure

$$\left(\left(\frac{A_5}{D_{10}} \amalg \frac{A_5}{V_4} \amalg \frac{A_5}{D_6}\right) \times \Delta^0 \right) \bigcup_{f_1} \left(\left(\frac{A_5}{C_2} \amalg \frac{A_5}{C_2} \amalg \frac{A_5}{C_2}\right) \times \Delta^1 \right) \bigcup_{f_2} \left(\frac{A_5}{1} \times \Delta^2 \right)$$

where Δ^i is the standard i-simplex and the attaching maps

$$f_2 : e \times \partial\Delta^2 \longrightarrow \mathbb{R}P(V_{\mathbb{R}})^{(1)};$$
$$f_1 : \bigcup_{i=1}^{3} e_i C_2 \times \partial\Delta^1 \longrightarrow \mathbb{R}P(V_{\mathbb{R}})^{(0)}$$

are as specified in Figure 2.3. (For $i = 1, 2, 3$, $e_i C_2$ is the trivial element in the i^{th} $\frac{A_5}{C_2}$ of $(\frac{A_5}{C_2} \amalg \frac{A_5}{C_2} \amalg \frac{A_5}{C_2})$ and for $j = 0, 1$, $\mathbb{R}P(V_{\mathbb{R}})^{(j)}$ is the A_5-j-skeleton.)

2.2 Generalities for cell structures

We return now to the complex case. Illman [32] has shown that if G is finite then any smooth, compact G-manifold can be assigned the structure of a finite G-CW-complex. (In fact [41 (p10)] there is also a result for more general G.)

We shall restrict our attention to the case of *finite G-CW-complexes*, meaning that the underlying CW-complex is a finite CW-complex. In view of Definition 2.0.1 we are interested in the fixed point sets $\mathbb{C}P(V)^H$ for subgroups H of G. Lemma 2.2.3 below turns out to be crucial. First we need to introduce the notion of the *α-isotypic piece* of V.

Definition 2.2.1. Suppose that V is a finite dimensional representation of G and that α is a one dimensional representation of a subgroup $H \leq G$. Viewing α as a homomorphism $H \longrightarrow \mathbb{C}$ we define the *α-isotypic piece* of V, V_α, to be

$$V_\alpha = \{v \in V \mid h \cdot v = \alpha(h)v \text{ for all } h \in H\}.$$

The following proposition gives a convenient interpretation of V_α. The proof is elementary, and omitted.

Proposition 2.2.2. *Suppose V is a finite dimensional representation of G and that α is a one dimensional representation of a subgroup $H \leq G$. Then, writing m_α for the multiplicity of α in $V_{|_H}$, we have*

$$V_\alpha \cong \bigoplus_{i=1}^{m_\alpha} \alpha$$

as $\mathbb{C}H$-modules. If $m_\alpha = 0$ then the right hand side is to be interpreted as zero. □

We come now to the crucial result.

Lemma 2.2.3 (Key lemma). *If V is a representation of G then, for subgroups $H \leq G$, we have*

$$\mathbb{C}P(V)^H = \coprod_{\substack{\alpha \text{ a 1-dim.} \\ \mathbb{C}H\text{-module}}} \mathbb{C}P(V_\alpha).$$

Proof. Recall that we may view points of $\mathbb{C}P(V)$ as lines through the origin in V, so that points of $\mathbb{C}P(V)^H$ are H-fixed lines through the origin in V. Any such line is thus a

1-dimensional $\mathbb{C}H$-module and we deduce

$$\mathbb{C}P(V)^H = \bigcup_{\substack{\alpha \text{ a 1-dim.} \\ \mathbb{C}H\text{-module}}} \mathbb{C}P(V_\alpha).$$

We deduce that the union is disjoint because if $\alpha \subseteq V_\alpha$ and $\alpha \subseteq V_\beta$, Proposition 2.2.2 implies that $\alpha \cong \beta$. $\square$

Corollary 2.2.4. *Suppose that V, V', V'' are $\mathbb{C}G$-modules with $V \cong_{\mathbb{C}H} V' \oplus V''$ for a subgroup $H \leq G$. If the decompositions of V', V'' as $\mathbb{C}H$-modules contain no one-dimensional $\mathbb{C}H$-modules in common then*

$$\mathbb{C}P(V)^H = \mathbb{C}P(V')^H \amalg \mathbb{C}P(V'')^H.$$

Proof. From Lemma 2.2.3 we are required to show that

$$\coprod_{\substack{\alpha \text{ a 1-dim.} \\ \mathbb{C}H\text{-module}}} \mathbb{C}P(V_\alpha) = \left(\coprod_{\substack{\alpha' \text{ a 1-dim.} \\ \mathbb{C}H\text{-module}}} \mathbb{C}P(V'_{\alpha'}) \right) \amalg \left(\coprod_{\substack{\alpha'' \text{ a 1-dim.} \\ \mathbb{C}H\text{-module}}} \mathbb{C}P(V''_{\alpha''}) \right).$$

Fix such an α. Then *either* α is a $\mathbb{C}H$-submodule of V' *or* α is a $\mathbb{C}H$-submodule of V'' but *not* both. Thus *either* $V_\alpha = V'_\alpha$ *or* $V_\alpha = V''_\alpha$ but *not* both. It follows that *either* $\mathbb{C}P(V_\alpha) = \mathbb{C}P(V'_\alpha)$ *or* $\mathbb{C}P(V_\alpha) = \mathbb{C}P(V''_\alpha)$ but *not* both. Since this remains true for any such α we deduce

$$\coprod_{\substack{\alpha \text{ a 1-dim.} \\ \mathbb{C}H\text{-module}}} \mathbb{C}P(V_\alpha) \subseteq \left(\coprod_{\substack{\alpha' \text{ a 1-dim.} \\ \mathbb{C}H\text{-module}}} \mathbb{C}P(V'_{\alpha'}) \right) \amalg \left(\coprod_{\substack{\alpha'' \text{ a 1-dim.} \\ \mathbb{C}H\text{-module}}} \mathbb{C}P(V''_{\alpha''}) \right).$$

The reverse inclusion is clear. $\square$

Corollary 2.2.5. *Let H be a subgroup of G. Write $\alpha_{H_1}, \alpha_{H_2}, \ldots, \alpha_{H_r}$ for a complete list, without duplication, of 1-dimensional $\mathbb{C}H$-modules. Then the fixed point set $\mathbb{C}P(V)^H$ is, non-equivariantly, homeomorphic to a finite disjoint union of complex projective spaces*

$$\coprod_{i=1}^{r} \mathbb{C}P^{n_i - 1}$$

where n_i is the multiplicity of α_{H_i} in $V_{|H}$. (If $n_i = 0$ we interpret $\mathbb{C}P^{n_i-1}$ as $\emptyset$.) $\square$

Lemma 2.2.6. *Let W be a $\mathbb{C}G$-module for some finite group G. Let H be a subgroup of G. Let $\mathrm{Av}_H : W \longrightarrow W$ be the H-averaging map given by $\mathrm{Av}_H(w) = \frac{1}{|H|} \sum_{h \in H} hw$. Then*

(i) $W_{|_H} = W^H \oplus \mathrm{Ker}\,(\mathrm{Av}_H)$ *as* $\mathbb{C}H$*-modules;*

(ii) $\mathbb{C}P(W_{|_H})^H = \mathbb{C}P(W^H) \amalg \mathbb{C}P(\mathrm{Ker}\,(\mathrm{Av}_H))^H$.

Proof. (i) This is elementary and omitted.

(ii) It is easy to see that if α is a one dimensional subrepresentation of W^H then α is *not* a one dimensional subrepresentation of $\mathrm{Ker}\,(\mathrm{Av}_H)$. So we apply Corollary 2.2.4 to part (i), and use the obvious fact that $\mathbb{C}P(W^H)^H = \mathbb{C}P(W^H)$.

□

Example 2.2.7. Suppose that G is a finite abelian group so that all its simple representations are of unit dimension. Take $\alpha_1, \alpha_2, \ldots, \alpha_r$ to be a list of all simple representations of G, without duplicates. Write

$$V = n_1\alpha_1 \oplus n_2\alpha_2 \oplus \cdots \oplus n_r\alpha_r$$

for natural numbers $n_1, n_2, \ldots, n_r$. Then

$$\mathbb{C}P(V)^G = \mathbb{C}P\,(n_1\alpha_1) \amalg \mathbb{C}P\,(n_2\alpha_2) \amalg \cdots \amalg \mathbb{C}P\,(n_r\alpha_r)\,. \tag{2.2.8}$$

Consider the two extreme cases for an n-dimensional representation.

(i) $V = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_n$ with $\alpha_i \cong \alpha_j$ if and only if $i = j$. Then

$$\mathbb{C}P(V)^G = \mathbb{C}P(\alpha_1) \amalg \mathbb{C}P(\alpha_2) \amalg \cdots \amalg \mathbb{C}P(\alpha_n).$$

As a space, each $\mathbb{C}P(\alpha_i)$ is homeomorphic to $\mathbb{C}P^0$, that is a one point space. Thus

$$\mathbb{C}P(V)^G \cong \coprod_{i=1}^{n} \mathrm{pt}.$$

(ii) $V = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_n$ with $\alpha_i \cong \alpha_j$ for all i, j. Then

$$\mathbb{C}P(V)^G = \mathbb{C}P(V)$$

and, non-equivariantly, this is homeomorphic to $\mathbb{C}P^{n-1}$.

When we consider non-abelian groups G we cannot assume that all representations may be written as a sum of one-dimensionals and thus equation (2.2.8) will not hold in general. But recalling Definition 2.0.1 we are interested in $\mathbb{C}P(V)^H$ for *subgroups* H of G. One might hope to be able to say something about the *abelian* subgroups H and this is indeed the case.

Proposition 2.2.9. *Let V be a simple n-dimensional complex representation of G. It then follows that*

(i) *the centre $Z(G)$ acts as scalars on V;*

(ii) *we have $V_{|_{Z(G)}} \cong \bigoplus_{i=1}^{n} \alpha$ for some $\mathbb{C}Z(G)$-module α of dimension 1.*

Proof. Observe that multiplication by any central element of G is a $\mathbb{C}G$-map and appeal to Schur's lemma [33 (Lemma 9.1)] to deduce (i). Since $Z(G)$ is abelian we know that $V_{|_{Z(G)}}$ is a sum of one dimensionals, say $\alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_n$. For (ii), choose a basis $\{e_1, e_2, \ldots, e_n\}$ for $V_{|_{Z(G)}}$ such that $\alpha_i = \mathrm{Span}_{\mathbb{C}}\{e_i\}$. Define $f_{ij} : \alpha_i \longrightarrow \alpha_j$ by $e_i \longmapsto e_j$ and extending linearly. For central g and $\mu \in \mathbb{C}$ we have

$$\begin{aligned} f_{ij}(g \cdot \mu e_i) &= f_{ij}(\lambda_g \mu e_i) && \text{(by (i), } g \text{ acts as a scalar, } \lambda_g \text{ say)} \\ &= \lambda_g f_{ij}(\mu e_i) && \text{(linearity of } f\text{)} \\ &= g \cdot f_{ij}(\mu e_i) && (g \text{ acts as } \lambda_g), \end{aligned}$$

so that f_{ij} is a $\mathbb{C}Z(G)$-map. Since f_{ij} is non-zero it follows from Schur's lemma that f_{ij} is an isomorphism. □

By Proposition 2.2.9 (i), $Z(G)$ acts as scalars on V. So elements of $Z(G)$ take a line ℓ in V to itself. This proves the following corollary.

Corollary 2.2.10. *The centre $Z(G)$ acts trivially on $\mathbb{C}P(V)$.* □

2.2.1 The G-0-skeleton

Consider the fixed point sets $\mathbb{C}P(V)^H$ $(H \leq G)$. Corollary 2.2.5 tells us that these are, non-equivariantly, disjoint unions of $\mathbb{C}P^j$s for some j. When we consider the zero skeleton we are interested in the case when $j = 0$, which corresponds to having 1-dimensional subrepresentations of unit multiplicity in $V_{|_H}$.

Definition 2.2.11. **(i)** By the *subgroup lattice (modulo conjugacy)* of a finite group G we shall mean a graph with a vertex for each conjugacy class of subgroups H of G. Vertices are labelled (H) to indicate the conjugates of H in G, but if H is normal we write just H. Two vertices (H_1) and (H_2) are joined by an edge if and only if we have either $H_1 < H_2$ or $H_1 > H_2$ such that in the former case, $H_1 \leq H \leq H_2$ implies either $H = H_1$ or $H = H_2$, and similarly in the latter case. We shall often omit the "modulo conjugacy" and simply write *subgroup lattice*.

(ii) Let V be a finite dimensional representation of G. By the *degeneration lattice* of V we mean the graph obtained from the subgroup lattice of G by replacing each vertex (H) with $V_{|_H}$.

Example 2.2.12. Take $G = \langle r, s \mid r^3 = s^2 = 1, srs = r^{-1} \rangle$ to be the dihedral group D_6 of order six and let $V = \gamma$ be the unique two dimensional simple representation of G – see Table A.6 on page 171. In this example, the geometry is explicit: we have $V = V_{\mathbb{R}} \otimes \mathbb{C}$ where $V_{\mathbb{R}}$ is the two dimensional *real* representation arising from D_6 acting in the usual way (r is rotation by $\frac{2\pi}{3}$ about the origin, s is reflection in the x-axis) as the symmetries of a triangle in $\mathbb{R}^2$.

The subgroup lattice (modulo conjugacy) for D_6 and the degeneration lattice for V are shown in Figure 2.6.

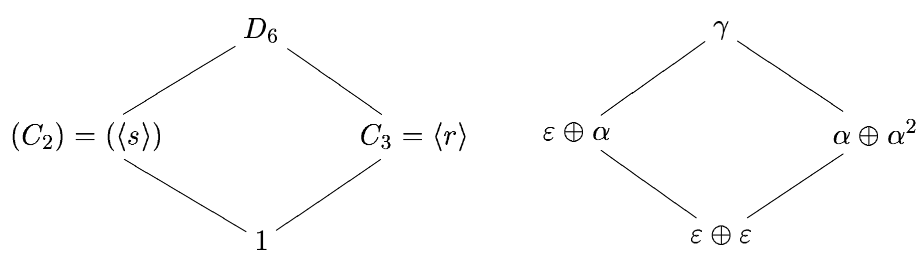


Figure 2.6. Subgroup lattice for D_6 (left) and degeneration lattice for γ (right). The notation in the degeneration lattice is taken from the appropriate character tables in Appendix A.

Much information about the G-0-skeleton of a G-CW-complex structure for $\mathbb{C}P(V)$ can be deduced from these degeneration lattices. In order to express the information concisely we shall need some notation. We write $\mathbb{C}P(V)^{(i)}$ for the G-i-skeleton and given $K, L \leq G$, we write $K \lesssim L$ to mean that K is conjugate to a subgroup of L.

Notation 2.2.13. Given $H \leq G$ write $\mathbb{C}P(V)^{(0)}_{\mathcal{I}(H)}$ for the isolated points of $\mathbb{C}P(V)$ with isotropy H. So $\mathbb{C}P(V)^{(0)}_{\mathcal{I}(H)} = n_H \frac{G}{H} \times e^0$ for some non-negative integer n_H and we write $\mathbb{C}P(V)^{(0)}_{\mathcal{I}}$ for

$$\mathbb{C}P(V)^{(0)}_{\mathcal{I}} = \coprod_{1 \neq H \leq G} \mathbb{C}P(V)^{(0)}_{\mathcal{I}(H)}.$$

Our aim in the current section is to write down an explicit description of $\mathbb{C}P(V)^{(0)}_{\mathcal{I}}$.

Definition 2.2.14. **(i)** Call a subgroup $H \leq G$ *locally* 0-*cell inducing* if there is a $\mathbb{C}H$-decomposition $V_{|_H} \cong \alpha \oplus V'$ where α is of dimension one and $\langle \alpha, V' \rangle = 0$.[1] Call such an α a *locally* 0-*cell inducing summand* of $V_{|_H}$.

(ii) Call a subgroup $H \leq G$ *globally* 0-*cell inducing* if H is locally 0-cell inducing and there is a locally 0-cell inducing summand α of $V_{|_H}$ which satisfies the following property: For all $K \leq G$ with $H \lesssim K$ there is no locally 0-cell inducing summand α' of $V_{|_K}$ with $\mathrm{Res}^K_H(\alpha') = \alpha$. Call such an α a *global* 0-*cell inducing summand* of $V_{|_H}$.

Notation 2.2.15. Write $c_G(H)$ for the number of conjugates of H in G and write $s(H)$ for the number of global 0-cell inducing summands of $V_{|_H}$.

We are now in position to state a preliminary result.

Lemma 2.2.16. *Let V be a representation of G of dimension n. Then $\mathbb{C}P(V)^{(0)}_{\mathcal{I}(H)}$ contains $s(H)$ (non-equivariant) zero cells and $\mathbb{C}P(V)^{(0)}_{\mathcal{I}}$ contains*

$$\sum_{\substack{\text{Globally 0-cell inducing} \\ \text{subgroups } H \leq G}} s(H) \tag{2.2.17}$$

(non-equivariant) 0-cells.

[1] Here, $\langle -, - \rangle$ denotes the inner product of characters.

Proof. Recall that $\mathbb{C}P(V)^{(0)}_{\mathcal{I}(H)}$ is the set of isolated fixed points of $\mathbb{C}P(V)$ which have isotropy H. Thus we count such points. By Corollary 2.2.5 there is a point fixed by H for each locally 0-cell inducing summand of $V_{|_H}$, but we must think a little harder. The point in question may also be fixed by some $K \geq H$ in which case we have a corresponding locally 0-cell inducing summand of $V_{|_K}$. We count points with *isotropy* H (not just *fixed* by H) so we must rule out this situation. A moment's thought shows that we must therefore count the *globally* 0-cell inducing summands of $V_{|_H}$, of which there are $s(H)$. Now, for (2.2.17), sum over all subgroups H using the above, and notice that $s(H) = 0$ if H is not globally 0-cell inducing. □

Remark 2.2.18. In practise it is more efficient to sum over conjugacy classes – in other words replace (2.2.17) with

$$\sum_{\substack{\text{Conjugacy classes of globally} \\ \text{0-cell inducing subgroups } H \leq G}} s(H) c_G(H). \tag{2.2.19}$$

As it stands, Lemma 2.2.16 serves little purpose, for it describes 0-skeleta in terms of *non-equivariant* cells. It is time now to redress this. We shall need the well known results in the following lemma. The proof is elementary and omitted.

Lemma 2.2.20. *Let H be a subgroup of the finite group G and let X be a G-space. Then we have*

(i) $(G/H)^H = N_G(H)/H$, *where $N_G(H)$ is the* normaliser *of H in G, $N_G(H) = \{g \in G \mid g^{-1}Hg = H\}$;*

(ii) $gX^H = X^{gHg^{-1}}$.

□

Proposition 2.2.21. *Let V be a representation of G of dimension n. Then*

$$\mathbb{C}P(V)^{(0)}_{\mathcal{I}} = \coprod_{\substack{\text{Conjugacy classes of globally} \\ \text{0-cell inducing subgroups } H \leq G}} \frac{s(H)}{|N_G(H)/H|} \frac{G}{H} \times e^0.$$

Proof. We know that isolated fixed points occur according to globally 0-cell inducing subgroups $H \leq G$ and fixed points corresponding to such an H have isotropy H. Suppose that $H_1, H_2, \ldots, H_r$ is a list, modulo conjugacy, of all the globally 0-cell inducing subgroups of G. Then for some non-negative integers $n_1, n_2, \ldots, n_r$ we have

$$\mathbb{C}P(V)^{(0)}_{\mathcal{I}} = \left(n_1 \frac{G}{H_1} \amalg n_2 \frac{G}{H_2} \amalg \cdots \amalg n_r \frac{G}{H_r} \right) \times e^0. \tag{2.2.22}$$

Now let us restrict attention to those points with isotropy H_i for some i with $1 \leq i \leq r$. We see that

$$\mathbb{C}P(V)^{(0)}_{\mathcal{I}(H_i)} = n_i \frac{G}{H_i} \times e^0,$$

and taking fixed points under the action of H_i gives

$$\left(\mathbb{C}P(V)^{(0)}_{\mathcal{I}(H_i)} \right)^{H_i} = n_i \left(\frac{G}{H_i} \right)^{H_i} \times e^0.$$

Clearly $\left(\mathbb{C}P(V)^{(0)}_{\mathcal{I}(H_i)} \right)^{H_i} = \mathbb{C}P(V)^{(0)}_{\mathcal{I}(H_i)}$ so using Lemma 2.2.20 we see that

$$\mathbb{C}P(V)^{(0)}_{\mathcal{I}(H_i)} = n_i \frac{N_G(H_i)}{H_i} \times e^0.$$

By Lemma 2.2.16, $\mathbb{C}P(V)^{(0)}_{\mathcal{I}(H_i)}$ contains $s(H_i)$ (non-equivariant) points, from which

$$n_i = \frac{s(H_i)}{|N_G(H_i)/H_i|}.$$

Substitute into (2.2.22) to deduce the proposition. □

2.3 Two dimensional examples

We present a selection of G-CW structures for a variety of G. We shall make use of these in Chapter 5. We are chiefly interested in simple representations of dimension greater than 1 so necessarily we take non-abelian groups of equivariance G and look at some two dimensional representations. We begin by considering the smallest such group, the dihedral (or equivalently, symmetric) group with six elements.

2.3.1 The dihedral group of order six

As in Example 2.2.12, take $G = \langle r, s \mid r^3 = s^2 = 1, srs = r^{-1} \rangle$ to be the dihedral group D_6 of order six and let $V = \gamma$ be the unique two dimensional simple representation of G – see Table A.6 on page 171. Recall from Figure 2.6 on page 42 the subgroup lattice for

G and the degeneration lattice for γ. We see that the globally 0-cell inducing subgroups of D_6 are C_2 and C_3.

It is left to the reader to verify that C_2 is self-normalising in D_6 whilst C_3 is normal. We apply Proposition 2.2.21 and see that

$$\begin{aligned}\mathbb{C}P(\gamma)^{(0)}_{\mathcal{I}} &= \left(\frac{2}{|N_{D_6}(C_2)/C_2|}\frac{D_6}{C_2} \amalg \frac{2}{|N_{D_6}(C_3)/C_3|}\frac{D_6}{C_3}\right) \times e^0 \\ &= \left(\frac{2}{|C_2/C_2|}\frac{D_6}{C_2} \amalg \frac{2}{|D_6/C_3|}\frac{D_6}{C_3}\right) \times e^0 \\ &= \left(2\frac{D_6}{C_2} \amalg \frac{D_6}{C_3}\right) \times e^0.\end{aligned}$$

Non-equivariantly this consists of $2 \cdot 3 + 2 = 8$ points. It is combinatorially convenient to draw the final picture by arranging these to lie at the vertices of a cube (see Figure 2.9). For the preceding computations the author finds the most intuitive picture to be a *plane model* of $\mathbb{C}P(\gamma) \cong S^2$.

Consider the diagram in Figure 2.7. Arrowheads indicate identification in the usual way. We represent the D_6-0-cells $D_6/C_3 \times e^0$, $D_6/C_2 \times e^0$ and $D_6/C_2' \times e^0$ by $\bullet$, $\times$ and $\circ$ respectively. We have taken the liberty of *choosing* to consider the $D_6/C_3 \times e^0$ cell at the poles and both of the $D_6/C_2 \times e^0$ cells on the equator. Given this, we are *forced* to introduce some D_6-1-cells and consideration of the degeneration lattice implies any such will necessarily be *free*. Once we make a choice of $e1 \times e^1$ the entire orbit $D_6/1 \times e^1$ is determined because we know how D_6 acts on the boundary (endpoints).

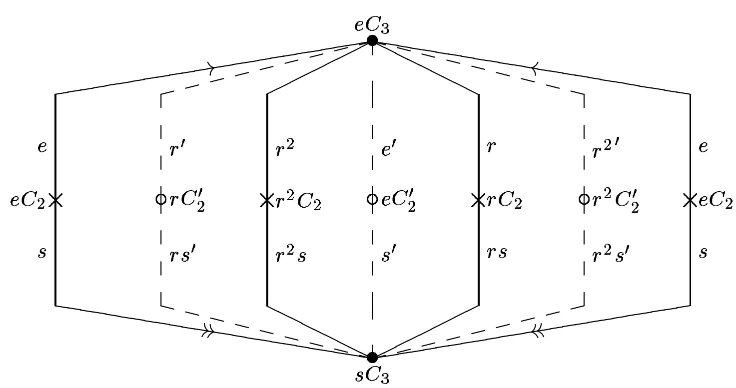


Figure 2.7. A possible 1-skeleton for a D_6-CW structure on $\mathbb{C}P(\gamma)$. We omit the "1" in $g1$ and write g, etc.

We proceed from Figure 2.7 by sewing a D_6-free D_6-2-cell into the panels. See Figure 2.8. It is then clear, by inspection, that we have a CW-structure on S^2. By construction this is a D_6-CW-structure on $\mathbb{C}P(\gamma)$.

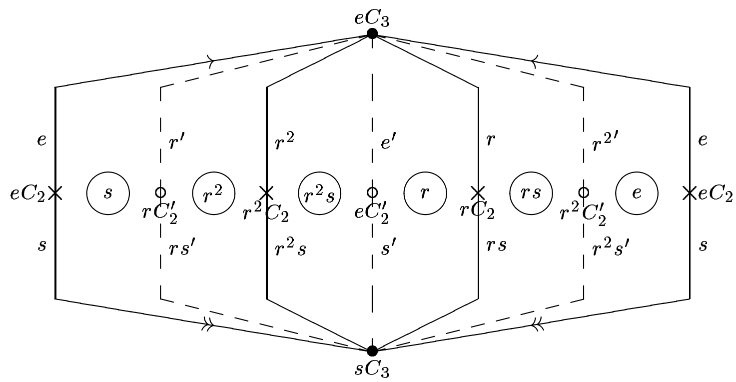


Figure 2.8. A D_6-CW-structure on $\mathbb{C}P(\gamma)$. Annotation of the D_6-2-cell is circled for emphasis.

The finished D_6-CW-complex is shown in Figure 2.9, where we have assembled the plane model and drawn the picture so that the D_6-0-cells lie at the vertices of a cube.

We summarise:

Proposition 2.3.1. *Let γ be the unique 2-dimensional simple representation of D_6. Then there is a D_6-CW-complex structure for $\mathbb{C}P(\gamma)$ as shown in Figures 2.8 and 2.9. Precisely, the D_6-CW-complex structure is*

$$\left((D_6/C_3 \amalg D_6/C_2 \amalg D_6/C_2') \times e^0\right) \bigcup_{f_1} \left((D_6/1 \amalg D_6/1') \times e^1\right) \bigcup_{f_2} \left(D_6/1 \times e^2\right)$$

where the attaching maps

$$f_2 : e \times \partial\overline{e^2} \longrightarrow \mathbb{C}P(\gamma)^{(1)};$$

$$f_1 : (e \times \partial\overline{e^1}) \cup (e' \times \partial\overline{e^1}) \longrightarrow \mathbb{C}P(\gamma)^{(0)}$$

are as specified by Figures 2.8 and 2.9. □

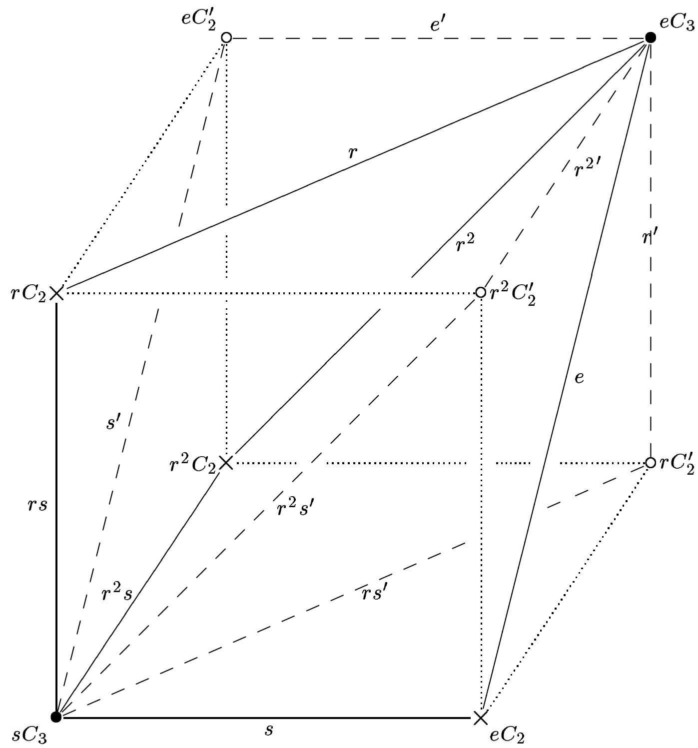


Figure 2.9. The D_6-CW-structure on $\mathbb{C}P(\gamma)$ inscribed in a cube. Note the dots are to emphasise the shape of the cube – *not* part of the CW-structure. For clarity we have not annotated the D_6-2-cell.

2.3.2 The dihedral group of order eight

Take $G = \langle r, s \mid r^4 = s^2 = 1, srs = r^{-1}\rangle$ to be the dihedral group D_8 of order eight and let $V = \gamma$ be the unique two dimensional simple representation of G – see Table A.7 on page 171. The subgroup lattice for D_8 and the degeneration lattice for γ are illustrated in Appendix B – see Figure B.2 on page 174. Writing $V_4 = \{e, r^2, s, r^2s\}$ and $V_4' = \{e, r^2, rs, r^3s\}$, we see that the globally 0-cell inducing subgroups of D_8 are V_4, V_4' and C_4.

Now V_4, V_4' and C_4 are normal in D_8 (they are subgroups of index two) so we apply Proposition 2.2.21 and see that

$$\begin{aligned}\mathbb{C}P(\gamma)^{(0)}_{\mathcal{I}} &= \left(\frac{2}{|N_{D_8}(V_4)/V_4|}\frac{D_8}{V_4} \amalg \frac{2}{|N_{D_8}(V_4')/V_4'|}\frac{D_8}{V_4'} \amalg \frac{2}{|N_{D_8}(C_4)/C_4|}\frac{D_8}{C_4}\right) \times e^0 \\ &= \left(\frac{D_8}{V_4} \amalg \frac{D_8}{V_4'} \amalg \frac{D_8}{C_4}\right) \times e^0.\end{aligned}$$

Non-equivariantly this consists of 6 points. As before we draw a plane model. Consider the diagram in Figure 2.10. Arrowheads indicate identification in the usual way. We represent the D_8-0-cells $D_8/V_4 \times e^0$, $D_8/V_4' \times e^0$ and $D_8/C_4 \times e^0$ by $\times$, $\circ$ and $\bullet$ respectively. We have taken the liberty of *choosing* to consider the $D_8/C_4 \times e^0$ cell at the poles and the $D_8/V_4 \times e^0$, $D_8/V_4' \times e^0$ cells on the equator. Given this, we are *forced* to introduce some D_8-1-cells and consideration of the degeneration lattice implies any such will necessarily *have isotropy group* $H := \langle r^2 \rangle$. Once we make a choice of $eH \times e^1$ the entire orbit $D_8/H \times e^1$ is determined because we know how D_8 acts on the boundary (endpoints).

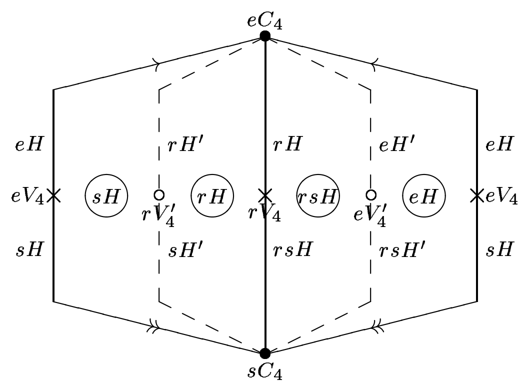


Figure 2.10. A D_8-CW-structure on $\mathbb{C}P(\gamma)$. Annotation of the D_8-2-cell is circled for emphasis. Recall that $H = \langle r^2 \rangle$.

We proceed by sewing a D_8-2-cell (necessarily with isotropy group $H = \langle r^2 \rangle$) into the panels of the figure. Once we have chosen $eH \times e^2$ its orbit is determined. It is then clear, by inspection, that we have a CW-structure on S^2. By construction this is a D_8-CW-structure on $\mathbb{C}P(\gamma)$. The assembled picture is shown in Figure 2.11.

We summarise:

Proposition 2.3.2. *Let γ be the unique 2-dimensional simple representation of D_8. Then there is a D_8-CW-complex structure for $\mathbb{C}P(\gamma)$ as shown in Figures 2.10 and 2.11. More*

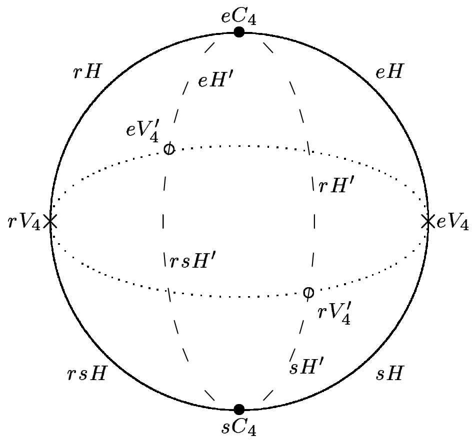


Figure 2.11. The D_8-CW-structure on $\mathbb{C}P(\gamma)$ inscribed in S^2. Note the dots are to emphasise the shape of the sphere – *not* part of the CW-structure. For clarity we have not annotated the D_8-2-cell.

precisely, if we write $V_4 = \{e, r^2, s, r^2s\}$, $V_4' = \{e, r^2, rs, r^3s\}$ *and* $H = \langle r^2 \rangle = H'$, *then the* D_8-CW*-complex structure is*

$$\Big((D_8/V_4 \amalg D_8/V_4' \amalg D_8/C_4) \times e^0\Big) \bigcup_{f_1} \Big((D_8/H \amalg D_8/H') \times e^1\Big) \bigcup_{f_2} \Big(D_8/H \times e^2\Big)$$

where the attaching maps

$$f_2 : eH \times \partial\overline{e^2} \longrightarrow \mathbb{C}P(\gamma)^{(1)};$$
$$f_1 : (eH \times \partial\overline{e^1}) \cup (eH' \times \partial\overline{e^1}) \longrightarrow \mathbb{C}P(\gamma)^{(0)}$$

are as specified by Figures 2.10 and 2.11. □

2.3.3 The quaternion group of order eight

Take

$$G = \langle i, j \mid i^4 = 1, i^2 = j^2, ji = i^3j \rangle$$

to be the quaternion group Q_8 of order eight. We follow standard notation and write k for ij and -1 for i^2. Let $V = \gamma$ be the unique two dimensional simple representation of G – see Table A.9 on page 172. The subgroup lattice for Q_8 and the degeneration lattice for γ are illustrated in Appendix B – see Figure B.3 on page 174. We see that the globally 0-cell inducing subgroups of Q_8 are $C_4^x := \langle x \rangle$ for $x = i, j, k$.

It is left to the reader to verify that *all* subgroups of Q_8 are normal in Q_8. We apply Proposition 2.2.21 and see that

$$\begin{aligned}
\mathbb{C}P(\gamma)^{(0)}_{\mathcal{I}} &= \left(\frac{2}{|N_{Q_8}(C_4^i)/C_4^i|}\frac{Q_8}{C_4^i} \amalg \frac{2}{|N_{Q_8}(C_4^j)/C_4^j|}\frac{Q_8}{C_4^j} \amalg \frac{2}{|N_{Q_8}(C_4^k)/C_4^k|}\frac{Q_8}{C_4^k}\right) \times e^0 \\
&= \left(\frac{Q_8}{C_4^i} \amalg \frac{Q_8}{C_4^j} \amalg \frac{Q_8}{C_4^k}\right) \times e^0.
\end{aligned}$$

Non-equivariantly this consists of 6 points. Once again we draw a plane model. Consider the diagram in Figure 2.12. Arrowheads indicate identification in the usual way. We represent the Q_8-0-cells $Q_8/C_4^i \times e^0$, $Q_8/C_4^j \times e^0$ and $Q_8/C_4^k \times e^0$ by $\bullet$, $\times$ and $\circ$ respectively. We have taken the liberty of *choosing* to consider the $Q_8/C_4^i \times e^0$ cell at the poles and the $Q_8/C_4^j \times e^0$, $Q_8/C_4^k \times e^0$ cells on the equator. Given this, we are *forced* to introduce some Q_8-1-cells and consideration of the degeneration lattice implies any such will necessarily *have isotropy group* $C_2 = \langle -1 \rangle$. Once we make a choice of $eC_2 \times e^1$ the entire orbit $Q_8/C_2 \times e^1$ is determined because we know how Q_8 acts on the boundary (endpoints).

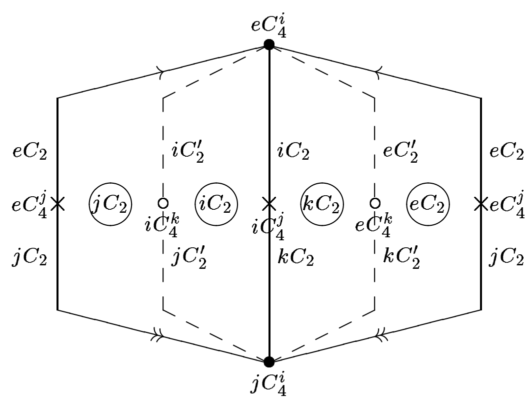


Figure 2.12. A Q_8-CW-structure on $\mathbb{C}P(\gamma)$. Annotation of the Q_8-2-cell is circled for emphasis.

We proceed just as before by sewing a Q_8-2-cell (necessarily with isotropy group C_2) into the panels of the figure. It is then clear, by inspection, that we have a CW-structure on S^2. By construction this is a Q_8-CW-structure on $\mathbb{C}P(\gamma)$. The assembled picture is shown in Figure 2.13.

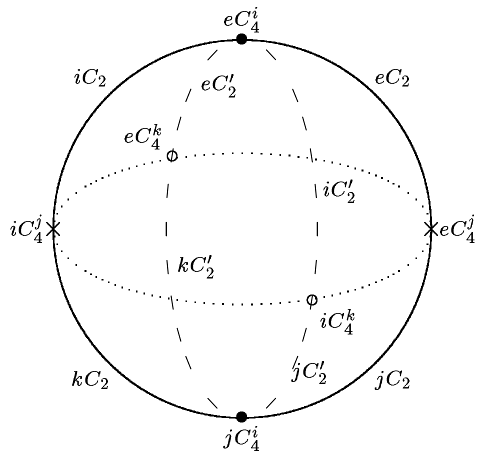


Figure 2.13. The Q_8-CW-structure on $\mathbb{C}P(\gamma)$ inscribed in S^2. Note the dots are to emphasise the shape of the sphere – *not* part of the CW-structure. For clarity we have not annotated the Q_8-2-cell.

We summarise:

Proposition 2.3.3. *Let γ be the unique 2-dimensional simple representation of Q_8. Then there is a Q_8-CW-complex structure for $\mathbb{C}P(\gamma)$ as shown in Figures 2.12 and 2.13. More precisely, the Q_8-CW-complex structure is*

$$\left((Q_8/C_4^i \amalg Q_8/C_4^j \amalg Q_8/C_4^k) \times e^0\right) \bigcup_{f_1} \left((Q_8/C_2 \amalg Q_8/C_2') \times e^1\right) \bigcup_{f_2} \left(Q_8/C_2 \times e^2\right)$$

where the attaching maps

$$f_2 : eC_2 \times \partial\overline{e^2} \longrightarrow \mathbb{C}P(\gamma)^{(1)};$$
$$f_1 : (eC_2 \times \partial\overline{e^1}) \cup (eC_2' \times \partial\overline{e^1}) \longrightarrow \mathbb{C}P(\gamma)^{(0)}$$

are as specified by Figures 2.12 and 2.13. □

Remarks 2.3.4. **(i)** The attentive reader will have noticed that in each of Propositions 2.3.1, 2.3.2 and 2.3.3 we have been able to construct a G-CW-complex whose G-0-skeleton consists of *only* the G-0-cells described by Proposition 2.2.21.

(ii) Looking at Figures 2.11 and 2.13 we see that the cell structures are highly similar: observe that each has the same number of G-i-cells ($i \in \mathbb{N}$) arranged in the same configuration. This is hardly surprising: Corollary 2.2.10 tells us that $Z(G)$ acts trivially on $\mathbb{C}P(V)$ so we might divert attention to the action of $G/Z(G)$. In the cases under consideration both $D_8/Z(D_8)$ and $Q_8/Z(Q_8)$ are isomorphic to V_4.

2.4 Three dimensional examples

We continue in the spirit of §2.3 by widening our study to *three* dimensional representations. The jump in dimension means we may obtain only partial information. Geometric intuition is often an invaluable aid. So consider the tetrahedron: its (direct) symmetry group is the alternating group A_4 of permutations of four objects.

2.4.1 The alternating group on four objects

Take G to be the alternating group A_4 (consisting of the even permutations of four objects). When necessary, we shall make use of standard notation by writing elements of A_4 as (products of) *cycles*. Let $V = \gamma$ be the unique three dimensional simple representation of A_4 – see Table A.10 on page 172. The subgroup lattice for A_4 and the degeneration lattice for γ are illustrated Appendix B – see Figure B.4 on page 175. We see that the globally 0-cell inducing subgroups of A_4 are V_4 and C_3.

It is left to the reader to verify that $N_{A_4}(C_3) = C_3$ whilst V_4 is normal. We apply Proposition 2.2.21 and see that

$$\begin{aligned}\mathbb{C}P(\gamma)^{(0)}_{\mathcal{I}} &= \left(\frac{3}{|N_{A_4}(V_4)/V_4|}\frac{A_4}{V_4} \amalg \frac{3}{|N_{A_4}(C_3)/C_3|}\frac{A_4}{C_3}\right) \times e^0 \\ &= \left(\frac{A_4}{V_4} \amalg \frac{A_4}{C_3} \amalg \frac{A_4}{C_3'} \amalg \frac{A_4}{C_3''}\right) \times e^0. \qquad (2.4.1)\end{aligned}$$

We turn our attention to the higher skeleta of $\mathbb{C}P(\gamma)$. We need to know the fixed points sets $\mathbb{C}P(\gamma)^H$ for subgroups H of A_4. We look at the degeneration lattice for γ, Figure B.4, and use Corollary 2.2.5 to write down the (non-equivariant) fixed point sets

$$\begin{aligned}\mathbb{C}P(\gamma)^{A_4} &= \emptyset; & \mathbb{C}P(\gamma)^{V_4} &= \coprod_{i=1}^{3} \mathbb{C}P^0; \\ \mathbb{C}P(\gamma)^{C_3} &= \coprod_{i=1}^{3} \mathbb{C}P^0; & \mathbb{C}P(\gamma)^{C_2} &= \mathbb{C}P^1 \amalg \mathbb{C}P^0.\end{aligned} \qquad (2.4.2)$$

We need to understand how these fixed point sets fit together. For instance, any subgroup C_2 of A_4 is also a subgroup of $V_4 \leq A_4$ so we must have $\mathbb{C}P(V)^{V_4} \subseteq \mathbb{C}P(V)^{C_2}$.

Proposition 2.4.3. *The space* $\bigcup_{C_2 \leq A_4} \mathbb{C}P(\gamma)^{C_2}$ *is connected.*

Proof. Take $g, h \in A_4$ to be distinct elements of order 2. From Lemma 2.2.6 (ii) we know that

$$\mathbb{C}P(\gamma)^{\langle g\rangle} = \mathbb{C}P(\gamma^{\langle g\rangle}) \amalg \mathbb{C}P(\mathrm{Ker}\,(\mathrm{Av}_{\langle g\rangle}));$$
$$\mathbb{C}P(\gamma)^{\langle h\rangle} = \mathbb{C}P(\gamma^{\langle h\rangle}) \amalg \mathbb{C}P(\mathrm{Ker}\,(\mathrm{Av}_{\langle h\rangle})).$$

Each of the four spaces on the right-hand-sides are connected (being $\mathbb{C}P^i$ for some i) so it suffices to show that their pairwise intersections are non-empty. It is now that we turn to geometric ideas.

Viewing A_4 as the direct symmetry group of the tetrahedron sitting inside $\mathbb{R}^3$, there is a three dimensional *real* representation $\gamma_{\mathbb{R}}$ with $\gamma_{\mathbb{R}} \otimes \mathbb{C} \cong_{\mathbb{C}G} \gamma$. Work for a moment in $\gamma_{\mathbb{R}}$. Any $x \in A_4$ of order two corresponds to a rotation of the tetrahedron of order two, as in Figure 2.14. Thus $(\gamma_{\mathbb{R}})^{\langle g\rangle}$ is an axis of rotation, so its vector space complement is of dimension two.

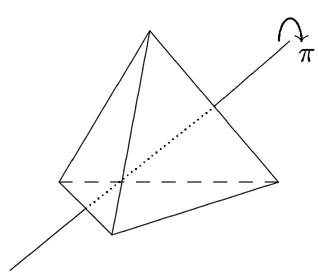


Figure 2.14. The action of $g \in A_4$ of order two on the tetrahedron.

Tensoring the previous paragraph with $\mathbb{C}$, we see $\gamma^{\langle g\rangle}$ is a one dimensional $\mathbb{C}\langle g\rangle$-module (so that $\mathbb{C}P(\gamma^{\langle g\rangle})$ is a single point) and its vector space complement is of complex dimension two. But we know from Lemma 2.2.6 (i) that the complement of $\gamma^{\langle g\rangle}$ is actually the $\mathbb{C}\langle g\rangle$-module $\mathrm{Ker}\,(\mathrm{Av}_{\langle g\rangle})$, which must therefore be of complex dimension two. Since

$g \neq h$, it follows that $\mathrm{Ker}\,(\mathrm{Av}_{\langle g\rangle}) \neq \mathrm{Ker}\,(\mathrm{Av}_{\langle h\rangle})$ so that $\mathrm{Ker}\,(\mathrm{Av}_{\langle g\rangle}) \cap \mathrm{Ker}\,(\mathrm{Av}_{\langle h\rangle})$ is a complex vector space of dimension one. Thus

$$\mathbb{C}P(\mathrm{Ker}\,(\mathrm{Av}_{\langle g\rangle})) \cap \mathbb{C}P(\mathrm{Ker}\,(\mathrm{Av}_{\langle h\rangle}))$$

is precisely one point – in particular is non-empty.

To complete the proof, it suffices (by symmetry) to show that

$$\mathbb{C}P(V^{\langle g\rangle}) \subseteq \mathbb{C}P(\mathrm{Ker}\,(\mathrm{Av}_{\langle h\rangle})).$$

But this is obvious: the axis of a rotation of the tetrahedron of order two is perpendicular to the other two such axes, and now tensor the statement *a rotation of order two (in $\gamma_{\mathbb{R}}$) necessarily fixes any line in the plane perpendicular to the axis of rotation* with $\mathbb{C}$. □

Proposition 2.4.4. *The fixed point set* $\bigcup_{C_2 \leq A_4} \mathbb{C}P(V)^{C_2}$ *is homeomorphic to the quotient space* $(S^2 \amalg S^2 \amalg S^2)/\sim$. *The* $\sim$ *is defined as follows: Let* N_i *and* S_i *be the north and south poles of the* i^{th} *copy of* S^2 $(i = 1, 2, 3)$. *Then*

$$N_1 \sim S_3; \qquad S_1 \sim N_2; \qquad S_2 \sim N_3.$$

Proof. We summarise what we know about the fixed point set $\bigcup_{C_2 \leq A_4} \mathbb{C}P(V)^{C_2}$. Let g, h be distinct elements of A_4 of order two. Then

(i) by (2.4.2), $\mathbb{C}P(\gamma)^{\langle g\rangle} \cong S^2 \amalg \mathrm{pt}$;

(ii) Lemma 2.2.6 (ii) tells us that $\mathbb{C}P(\gamma)^{\langle g\rangle} = \mathbb{C}P(\gamma^{\langle g\rangle}) \amalg \mathbb{C}P(\mathrm{Ker}\,(\mathrm{Av}_{\langle g\rangle}))$;

(iii) From the previous proof, we note $\mathbb{C}P(\mathrm{Ker}\,(\mathrm{Av}_{\langle g\rangle}))$ is the S^2 and $\mathbb{C}P(\gamma^{\langle g\rangle})$ is the pt.

(iv) Also from the previous proof,

$$\mathbb{C}P(\mathrm{Ker}\,(\mathrm{Av}_{\langle g\rangle})) \cap \mathbb{C}P(\mathrm{Ker}\,(\mathrm{Av}_{\langle h\rangle})) \cong \mathrm{pt} \text{ and } \mathbb{C}P(\gamma^{\langle g\rangle}) \subseteq \mathbb{C}P(\mathrm{Ker}\,(\mathrm{Av}_{\langle h\rangle})).$$

There are three elements of order two in A_4 and thus three fixed point sets of the form (i). Call the three elements f, g, h. Let us consider first the three copies of S^2. Write S^2_g for the copy of S^2 in the fixed point set $\mathbb{C}P(\gamma)^{\langle g\rangle}$, etc. By (iii) and (iv), each of S^2_f, S^2_g, S^2_h meets each of the other two in precisely one point. The only options are

- a wedge $(S^2 \amalg S^2 \amalg S^2)/\approx$ where $N_1 \approx N_2 \approx N_3$;
- the quotient $(S^2 \amalg S^2 \amalg S^2)/\sim$, as pictured in Figure 2.15.

Suppose we have the former configuration. Now consider the copies of pt arising from (i) – call them $\mathrm{pt}_f, \mathrm{pt}_g, \mathrm{pt}_h$. By (i), pt_f is disjoint from S^2_f but by (iii) and (iv), $S^2_g \supseteq \mathrm{pt}_f \subseteq S^2_h$ so that pt_f must lie at one, and so all, of the north poles. This contradicts the fact that pt_f is disjoint from S^2_f. Thus the former configuration does not occur.

Hence we must have the three copies of S^2 configured as in Figure 2.15. The configuration of $\mathrm{pt}_f, \mathrm{pt}_g, \mathrm{pt}_h$ is now determined, for instance pt_f lies in both S^2_g and S^2_h, so must be the unique point of intersection $S^2_g \cap S^2_h$ (which is indeed disjoint from S^2_f). Similarly for $\mathrm{pt}_g, \mathrm{pt}_h$. □

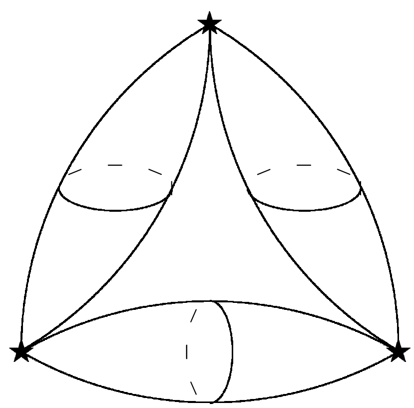


Figure 2.15. The fixed point set $\bigcup_{C_2 \leq A_4} \mathbb{C}P(\gamma)^{C_2}$. The $\mathrm{pt}_f, \mathrm{pt}_g, \mathrm{pt}_h$ are each represented by ★ and the S^2_f, S^2_g, S^2_h are each shown as a sausage shape.

Proposition 2.4.5. *The singular set* $\bigcup_{1 \neq H \leq A_4} \mathbb{C}P(\gamma)^H$ *is as shown in Figure 2.16, precisely*

$$\bigcup_{1 \neq H \leq A_4} \mathbb{C}P(\gamma)^H = \Big((S^2 \amalg S^2 \amalg S^2)/\sim \Big) \amalg \coprod_{i=0}^{11} \mathrm{pt},$$

where $\sim$ *is as defined in Proposition 2.4.4.*

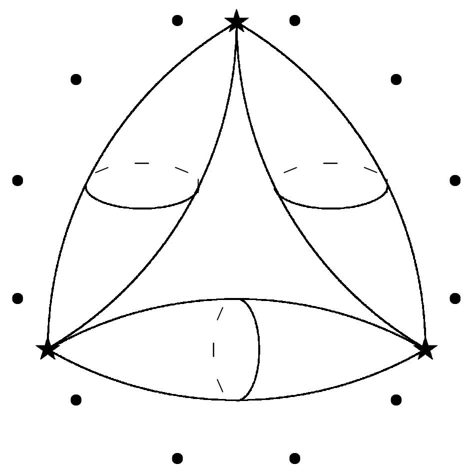


Figure 2.16. The singular set $\bigcup_{1\neq H\leq A_4} \mathbb{C}P(\gamma)^H$.

Proof. We must consider each fixed point set $\mathbb{C}P(\gamma)^H$ for $1 \neq H \leq A_4$. The case $H = C_2$ was dealt with in Proposition 2.4.4. This automatically covers the case $H = V_4$ because $C_2 \leq V_4$. All that remains is $H = C_3$ and $H = A_4$. By (2.4.2) there is nothing to consider for the latter and $\mathbb{C}P(\gamma)^{C_3} \cong \mathrm{pt} \amalg \mathrm{pt} \amalg \mathrm{pt}$. Since there are four copies of C_3 in A_4 we see that we must add twelve points to $\bigcup_{C_2\leq A_4} \mathbb{C}P(\gamma)^{C_2}$.

These twelve points must be disjoint from $\bigcup_{C_2\leq A_4} \mathbb{C}P(\gamma)^{C_2}$ for any point $\mathbb{C}P(\gamma)^{C_2} \cap \mathbb{C}P(\gamma)^{C_3}$ would have isotropy group containing an element of order two and an element of order three. The only such subgroup of A_4 is A_4 itself, and $\mathbb{C}P(\gamma)^{A_4} = \emptyset$ by (2.4.2).

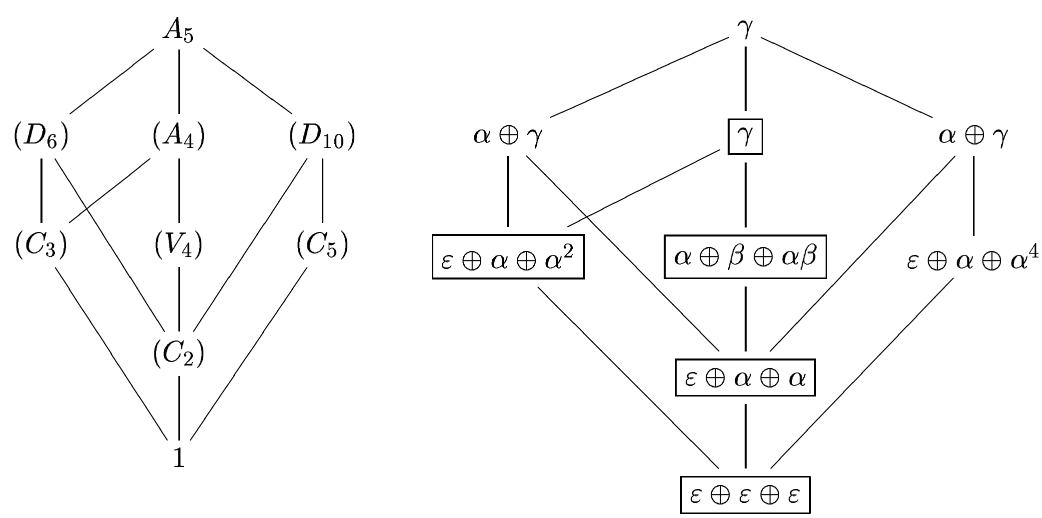
It remains only to answer *are the twelve points mutually disjoint?* Equation (2.4.1) tells us that the answer is *yes.* □

At this point we abandon further pursuit of an A_4-cellular structure on $\mathbb{C}P(\gamma)$. We feel satisfied with the picture in Figure 2.16 – not only is it interesting and aesthetically pleasing in its own right, but the process of completing an A_4-cellular structure would be rather *ad-hoc.* Nonetheless, we can use our limited picture to proceed with one more example of interest.

2.4.2 The alternating group on five objects

Take G to be the alternating group A_5 (consisting of the even permutations of five objects). When necessary, we shall make use of standard notation by writing elements of A_5 as (products of) *cycles*. Let V be the three dimensional simple representation of A_5 with character γ – see Table A.11 on page 172. The subgroup lattice for A_5 and the degeneration lattice for γ are illustrated in Appendix B – see Figure B.5 on page 175.

At this point we could apply Proposition 2.2.21. But much of the work has been done in §2.4.1 so we avoid using the proposition. Let us examine in more detail the subgroup lattice for A_5 and the degeneration lattice for γ (Figure B.5). For convenience we reproduce them side by side below.



Looking at the boxed entries, we see that we have a copy of the degeneration lattice from the A_4 case of §2.4.1. Thus for each copy of A_4 inside A_5 we obtain a contribution to the singular set as in Figure 2.16.

Proposition 2.4.6. *The singular set* $\bigcup_{1 \neq H \leq A_5} \mathbb{C}P(\gamma)^H$ *is as shown in Figure 2.17.*

Proof. Denote by X the A_4-singular set of Figure 2.16. There are five copies of A_4 inside A_5 (each is the stabiliser of some $i \in \{1, 2, 3, 4, 5\}$) and, as discussed above, each gives

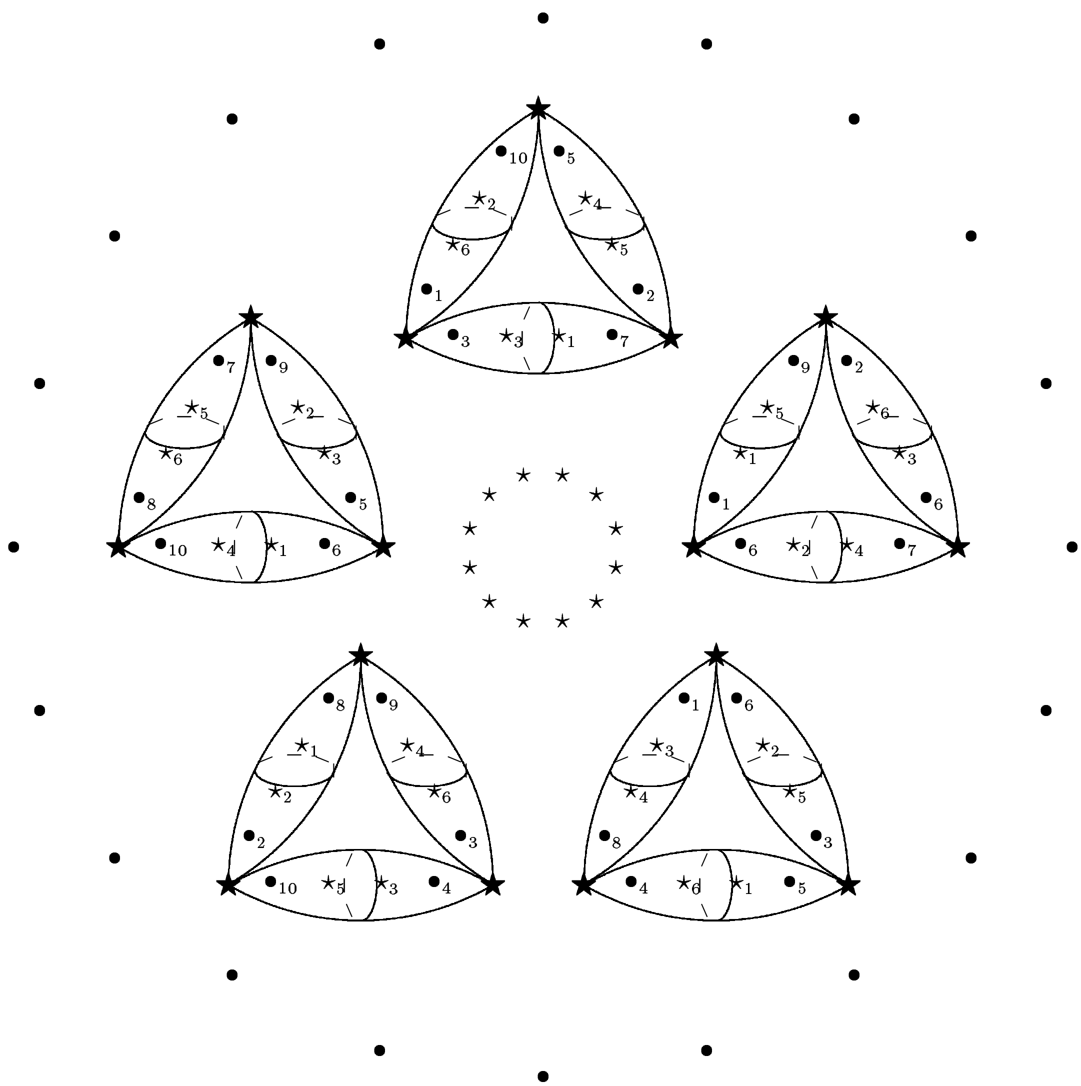


Figure 2.17. The singular set $\bigcup_{1 \neq H \leq A_5} \mathbb{C}P(\gamma)^H$. Each of the fifteen copies of S^2 has isotropy C_2; the points $\bigstar$ have isotropy V_4. The points marked $\bullet$ are fixed by C_3 – those inside a copy of S^2 have isotropy D_6 and those not inside a copy of S^2 have isotropy C_3. The points marked $\star$ are fixed by C_5 – those inside a copy of S^2 have isotropy D_{10} and those not inside a copy of S^2 have isotropy C_5. Finally, $\bullet_i$ is to be identified with $\bullet_j$ if and only if $i = j$, likewise for $\star_i$ and $\star_j$.

rise to a copy of X. This accounts for all of the boxed entries in the degeneration lattice above.

Fix a copy of D_6. Looking at the degeneration lattice above, and using Corollary 2.2.5 we see that $\mathbb{C}P(V)^{D_6} = \mathbb{C}P(\alpha) \cong \mathbb{C}P^0$. Since $A_4 \geq C_3 \leq D_6$ we note that this D_6-fixed point has already been counted as one of the C_3-fixed points in a copy of X. Similarly we have already considered D_{10} fixed points. It remains only to consider points fixed by C_5 but *not* by D_{10}. Pick a copy of C_5. Looking at the diagram above, and using Corollary 2.2.5 we see that $\mathbb{C}P(V)^{C_5} \cong \mathbb{C}P^0 \amalg \mathbb{C}P^0 \amalg \mathbb{C}P^0$. One of these three points was fixed by D_{10}, and that has been taken care of already, so our copy of C_5 contributes two fresh fixed points. The icosahedron has twelve vertices and each antipodal pair gives a copy of C_5, so that there are six copies of C_5 in A_5. It follows that $\mathbb{C}P(\gamma)^{C_5} \setminus \mathbb{C}P(\gamma)^{D_{10}}$ is twelve disjoint points. Thus our singular set consists of five copies of X and $\coprod_1^{12} \mathrm{pt}$ arranged in some configuration. The configuration is worked out from the subgroup structure of A_5. For example, there are five copies of C_2 in each D_{10}. Thus each D_{10} fixed point lies in five copies of S^2 arising from C_2 fixed points.

The wise philosophy "persuasion backed up by sound logic is ultimately more effective than coercion" of Kano [36] is to be followed whenever possible, but in this instance the author found the least confusing way to proceed was to coerce. Tables 2.1 and 2.2 on pages 61 and 62 explicitly list all elements and all proper subgroups of A_5, and from these it is elementary though tedious to verify that the diagram in Figure 2.17 is correct. □

At this point we abandon further pursuit of an A_5-cellular structure on $\mathbb{C}P(V)$ for the same reasons as in §2.4.1. We are content that we have examined sufficiently many examples that we may now continue by considering equivariant K-theory and its application to $\mathbb{C}P(V)$.

	Element of type		
id	**(a b)(c d)**	**(a b c)**	**(a b c d e)**
id	(1 2)(3 4)	(3 4 5)	(1 2 3 4 5)
	(1 2)(3 5)	(3 5 4)	(1 3 5 2 4)
	(1 2)(4 5)	(2 4 5)	(1 4 2 5 3)
	(1 3)(2 4)	(2 5 4)	(1 5 4 3 2)
	(1 3)(2 5)	(2 3 5)	(1 2 3 5 4)
	(1 3)(4 5)	(2 5 3)	(1 3 4 2 5)
	(1 4)(2 3)	(2 3 4)	(1 5 2 4 3)
	(1 4)(2 5)	(2 4 3)	(1 4 5 3 2)
	(1 4)(3 5)	(1 4 5)	(1 3 2 4 5)
	(1 5)(2 3)	(1 5 4)	(1 2 5 3 4)
	(1 5)(2 4)	(1 3 5)	(1 4 3 5 2)
	(1 5)(3 4)	(1 5 3)	(1 5 4 2 3)
	(2 3)(4 5)	(1 3 4)	(1 2 4 3 5)
	(2 4)(3 5)	(1 4 3)	(1 4 5 2 3)
	(2 5)(3 4)	(1 2 5)	(1 3 2 5 4)
		(1 5 2)	(1 5 3 4 2)
		(1 2 4)	(1 3 4 5 2)
		(1 4 2)	(1 4 2 3 5)
		(1 2 3)	(1 5 3 2 4)
		(1 3 2)	(1 2 5 4 3)
			(1 5 2 3 4)
			(1 2 4 5 3)
			(1 3 5 4 2)
			(1 4 3 2 5)
1 element	*15 elements*	*20 elements*	*24 elements*

Table 2.1. The sixty elements of A_5. The elements are grouped so that the union of any box with id gives a cyclic subgroup of A_5.

Subgroup of type					
V_4	**D_6**		**D_{10}**		**A_4**
id	id	id	id	id	$\mathrm{Stab}_{A_5}(1)$
(1 2)(3 4)	(3 4 5)	(1 3 4)	(1 2 3 4 5)	(1 2 4 3 5)	$\mathrm{Stab}_{A_5}(2)$
(1 3)(2 4)	(3 5 4)	(1 4 3)	(1 3 5 2 4)	(1 4 5 2 3)	$\mathrm{Stab}_{A_5}(3)$
(1 4)(2 3)	(1 2)(4 5)	(2 5)(3 4)	(1 4 2 5 3)	(1 3 2 5 4)	$\mathrm{Stab}_{A_5}(4)$
id	(1 2)(3 4)	(1 4)(2 5)	(1 5 4 3 2)	(1 5 3 4 2)	$\mathrm{Stab}_{A_5}(5)$
(1 2)(3 5)	(1 2)(3 5)	(1 3)(2 5)	(1 2)(3 5)	(1 2)(4 5)	
(1 3)(2 5)	id	id	(1 3)(4 5)	(1 4)(3 5)	
(1 5)(2 3)	(2 4 5)	(1 4 5)	(1 4)(2 3)	(1 3)(2 4)	
id	(2 5 4)	(1 5 4)	(1 5)(2 4)	(1 5)(2 3)	
(1 2)(4 5)	(1 3)(4 5)	(1 4)(2 3)	(2 5)(3 4)	(2 5)(3 4)	
(1 4)(2 5)	(1 3)(2 4)	(2 3)(4 5)	id	id	
(1 5)(2 4)	(1 3)(2 5)	(1 5)(2 3)	(1 2 3 5 4)	(1 3 4 5 2)	
id	id	id	(1 3 4 2 5)	(1 4 2 3 5)	
(1 3)(4 5)	(2 3 5)	(1 2 3)	(1 5 2 4 3)	(1 5 3 2 4)	
(1 4)(3 5)	(2 5 3)	(1 3 2)	(1 4 5 3 2)	(1 2 5 4 3)	
(1 5)(3 4)	(1 4)(2 5)	(1 2)(4 5)	(1 2)(3 4)	(1 3)(2 4)	
id	(1 4)(3 5)	(1 3)(4 5)	(1 3)(4 5)	(1 4)(2 5)	
(2 3)(4 5)	(1 4)(2 3)	(2 3)(4 5)	(1 5)(2 3)	(1 5)(3 4)	
(2 4)(3 5)	id	id	(1 4)(2 5)	(1 2)(3 5)	
(2 5)(3 4)	(2 3 4)	(1 2 4)	(2 4)(3 5)	(2 3)(4 5)	
	(2 4 3)	(1 4 2)	id	id	
	(1 5)(2 4)	(1 2)(3 5)	(1 3 2 4 5)	(1 5 2 3 4)	
	(1 5)(3 4)	(1 4)(3 5)	(1 2 5 3 4)	(1 2 4 5 3)	
	(1 5)(2 3)	(2 4)(3 5)	(1 4 3 5 2)	(1 3 5 4 2)	
	id	id	(1 5 4 2 3)	(1 4 3 2 5)	
	(1 3 5)	(1 2 5)	(1 3)(2 5)	(1 5)(2 4)	
	(1 5 3)	(1 5 2)	(1 2)(4 5)	(1 2)(3 4)	
	(2 4)(3 5)	(1 2)(3 4)	(1 4)(2 3)	(1 3)(2 5)	
	(1 3)(2 4)	(1 5)(3 4)	(1 5)(3 4)	(1 4)(3 5)	
	(1 5)(2 4)	(2 5)(3 4)	(2 4)(3 5)	(2 3)(4 5)	
5 subgroups	*10 subgroups*		*6 subgroups*		*5 subgroups*

Table 2.2. The non-cyclic subgroups of A_5.

Chapter 3

Equivariant K-theory of $\mathbb{C}P(V)$

Throughout Chapter 3, by G we understand a compact Lie group and by A an abelian compact Lie group, unless stated otherwise. Let V be a complex representation of G of complex dimension $\dim_{\mathbb{C}} V = n$. A G-equivariant cohomology theory graded over the integers $\mathbb{Z}$ is called a *naive* cohomology theory. If such a theory E extends to one graded over $RO(G)$, the real representation ring of G, and admits *suspension isomorphisms*

$$\widetilde{E}_G^{i+R}(S^R \wedge X) \cong \widetilde{E}_G^i(X)$$

for all *real* representations R, it is then called a *genuine* or $RO(G)$*-graded* equivariant cohomology theory. For more details about $RO(G)$-graded theory, the reader is referred to [40], which asserts that many naturally occurring naive theories *do* extend – for example complex K-theory $K_G^*(-)$ and complex bordism $MU_G^*(-)$ are both genuine theories.

Recall from Remarks 1.1.3 (ii) that the Spanier-Whitehead dual is $D(X) = \Sigma^{1-n}(S^n \setminus X)$. Equivariantly, this becomes $D(X) = \Sigma^{1-V}(S^V \setminus X)$, so if we are to hope for duality we would ask that

$$E_G^i(X) \cong E_{-i}^G(D(X)) = E_{-i}^G(\Sigma^{1-V}(S^V \setminus X)) \cong E_{V-i-1}^G(S^V \setminus X).$$

For this to make sense we require a *genuine* theory E, which explains our present interest in genuine theories.

Some genuine equivariant cohomology theories E are *complex stable*, meaning that

$$\widetilde{E}_G^{i+|C|}(S^C \wedge X) \cong \widetilde{E}_G^i(X)$$

for all *complex* representations C. (Here, $|C|$ is the underlying *real* vector space of C with trivial G-action, i.e. $|C| = \dim_{\mathbb{R}}(C)$.) In fact complex K-theory and complex bordism are [18] complex stable but equivariant K-theory is even more special: we have the celebrated *equivariant Bott periodicity* theorem (§3.2) to tell us that equivariant K-theory is furthermore 2-periodic, $\widetilde{K}_G^{i+|V|}(X) \cong \widetilde{K}_G^i(X)$ for complex representations V. It is with equivariant K-theory that our interest resides.

Recall [16] the *Brown representability* theorem tells us that any generalised cohomology theory[1] satisfying the wedge axiom can be represented by a *spectrum*. Likewise [46 (XIII)] in the equivariant setting. Equivariant K-theory, then, can be described in terms of *equivariant spectra*. We shall follow that approach as and when required, but for now we describe equivariant K-theory in perhaps its most geometric and intuitive form – in terms of G-vector bundles. In fact the two descriptions coincide provided we restrict attention to finite G-CW-complexes. (We shall be precise about this in §4.2.)

3.1 Introduction to equivariant K-theory

A survey to introduce the reader to equivariant K-theory may be necessary, and in any event we wish to record results for use later. Segal [55] provides all that we could wish for, and more, in his beautiful article. We summarise his first two sections here.

Definition 3.1.1. If X is a G-space, a *G-vector bundle* E (or just *G-bundle*) on X is a G-space E (called the *total space*) together with a G-map $p : E \longrightarrow X$ such that

(i) $p : E \longrightarrow X$ is a complex vector bundle on X, that is the *fibres* $E_x := p^{-1}(x)$ for $x \in X$ are finite dimensional complex vector spaces, and the situation is locally trivial in the usual sense [10];

(ii) for any $g \in G$ and $x \in X$ the group action $g : E_x \longrightarrow E_{gx}$ is a homomorphism of vector spaces.

The reader will observe that we have written E for both the bundle and its total space. That is entirely standard.

[1] A *generalised cohomology theory* satisfies all of the Eilenberg-Steenrod axioms [24] expect for the dimension axiom. For the G-Eilenberg-Steenrod axioms see Illman [31 (p5)].

As two easy examples of G-bundles, notice that a G-bundle over a point is just a finite dimensional vector space with a G-action, that is a finite dimensional $\mathbb{C}G$-module. And if G acts trivially on X then a G-vector bundle on X is just a family of representations parametrised by $x \in X$ and varying continuously with x.

Given two G-bundles E and E' over a common G-space X one may form their *sum* $E \oplus E'$ which has fibres $(E \oplus E')_x = (E_x) \oplus (E'_x)$ and their *tensor product* $E \otimes E'$ with fibres $(E \otimes E')_x = (E_x) \otimes (E'_x)$. A *homomorphism* f between E and E' is a continuous G-map $f : E \longrightarrow E'$ inducing homomorphisms of vector spaces $f_x : E_x \longrightarrow E'_x$ for all $x \in X$. If all of the f_x are isomorphisms of vector spaces then we say that f is a *G-bundle isomorphism* $E \longrightarrow E'$.

Before we proceed to defining equivariant K-theory, we record one more important result in bundle theory – the *clutching lemma.*

Lemma 3.1.2 (Clutching lemma). *Let X be the union of compact G-spaces X_1 and X_2, and write $A = X_1 \cap X_2$. Suppose that E_i is a G-vector bundle on X_i $(i = 1, 2)$ and that $f : (E_1)_{|A} \longrightarrow (E_2)_{|A}$ is an isomorphism. Then there is a unique G-vector bundle E on X with isomorphisms $E_{|X_i} \cong E_i$ $(i = 1, 2)$ which are compatible with f.*

Proof. As Segal [55] points out, the details in this equivariant setting go through just as in the non-equivariant case [10 (§1)]. □

We aim to define groups $K^*_G(X)$ for compact G-spaces X, and Definition 3.1.3 below is the key ingredient.

Definition 3.1.3. Let X be a G-space. Then by $K_G(X)$ we mean the Grothendieck group associated to the semi-group of isomorphism classes of G-bundles on X under $\oplus$. Thus the elements of $K_G(X)$ are formal differences $E - E'$ of G-bundles E, E' on X modulo the equivalence relation

$$E - E' \sim F - F' \iff (E \oplus F' \oplus \xi \cong F \oplus E' \oplus \xi \text{ for some } G\text{-bundle } \xi \text{ on } X).$$

Example 3.1.4. Write pt for the one-point space, which is automatically a G-space for any group G. Then

$$\begin{aligned} K_G(\mathrm{pt}) &= \text{Grothendieck group}(G\text{-bundles over pt}) \\ &= \text{Grothendieck group}(\text{finite dimensional } \mathbb{C}G\text{-modules}) \\ &= R(G), \end{aligned}$$

the complex *representation ring* of G. Additively, this is

$$\bigoplus_{\substack{\text{Simple} \\ \mathbb{C}G\text{-modules}}} \mathbb{Z},$$

and a multiplicative structure is induced by taking tensor products of G-bundles. (We shall say more in Proposition 3.1.13 below.)

Recall that T is the circle group with natural representation z, so that $K_{G\times T}(\mathrm{pt}) = R(G \times T)$. We shall later be very interested in the representation ring $R(G \times T)$ which we now discuss. Adams [2 (Theorem 3.65)] tells us that

$$R(G \times T) \cong R(G) \otimes R(T)$$

so that we are now interested in $R(T)$. But [2 (Corollary 3.77)] asserts that $R(T) \cong \mathbb{Z}[z, z^{-1}]$ so that we may conclude

$$R(G \times T) \cong R(G)[z, z^{-1}]. \tag{3.1.5}$$

The following proposition provides a basic link between equivariant K-theory and the non-equivariant theory.

Proposition 3.1.6. *Suppose G is a compact Lie group and N is a normal subgroup of G which acts freely on the compact G-space X. Then the quotient $q : X \longrightarrow X/N$ induces an isomorphism*

$$q^* : K_{G/N}(X/N) \xrightarrow{\cong} K_G(X).$$

Proof. This is [55 (Proposition 2.1)]. □

Definition 3.1.7. Two G-bundles E, E' on X are *stably equivalent* if there are $\mathbb{C}G$-modules W, W' such that $E \oplus (W \times X) \cong E' \oplus (W' \times X)$. Here, $W \times X$ is the trivial G-bundle over X whose every fibre is W. The equivalence classes of G-bundles on X modulo stable equivalence form an abelian group under $\oplus$, which we denote by $\widetilde{K}_G(X)$.

Definition 3.1.8. If X is a compact G-space with G-fixed basepoint x_0 and A a closed G-subspace, we define, for natural numbers q,

$$\begin{aligned}\widetilde{K}_G^{-q}(X) &= \widetilde{K}_G(\Sigma^q X);\\ \widetilde{K}_G^{-q}(X,A) &= \widetilde{K}_G(\Sigma^q(X \cup_A CA));\\ \widetilde{K}_G^{-q}(X,x_0) &= \widetilde{K}_G^{-q}(X).\end{aligned}$$

Here, $\Sigma^q X$ means the q^{th} *reduced suspension* of X,

$$\Sigma^q X = \begin{cases} X & q=0 \\ CX \cup_X CX & q=1 \\ \Sigma(\Sigma^{q-1}X) & q>1 \end{cases},$$

where CX is the *reduced cone* on X, $CX = X \times [0,1]/(X \times \{0\}) \cup (\{x_0\} \times [0,1])$, and if A is a subspace of both the spaces X, Y then $X \cup_A Y$ is obtained from $X \amalg Y$ by gluing along the inclusions $Y \longleftarrow A \longrightarrow X$.

As discussed in [24 (Chapter 10)], any cohomology theory defined on compact spaces with basepoint gives rise to a cohomology theory on locally compact spaces *without* basepoint. Write X_+ for the one-point compactification of X, so if X was already compact then $X_+ = X \amalg \mathrm{pt}$.

At this point we deviate slightly from Segal. Segal continues by defining equivariant K-theory on (locally) compact G-spaces. We shall later be concerned with the so called *represented* theory. This agrees with Segal provided one restricts to finite G-CW-complexes. Thus we continue to pursue Segal's approach but restrict to the case of interest.

Definition 3.1.9. For a finite G-CW-complex X and closed G-subcomplex A, we define

$$\begin{aligned}K_G^{-q}(X) &= \widetilde{K}_G^{-q}(X_+);\\ K_G^{-q}(X,A) &= \widetilde{K}_G^{-q}(X_+, A_+).\end{aligned}$$

Remarks 3.1.10. **(i)** From these definitions it follows that $K_G^{-q}(X,\emptyset) = K_G^{-q}(X)$.

(ii) In the case when X is compact, $K_G^0(X)$ coincides with $K_G(X)$. Indeed, there is a homomorphism $K_G(X) \longrightarrow \widetilde{K}_G(X_+)$ defined by extending G-bundles over X by giving them the zero fibre at the disjoint point, x_0 say. It turns out that there is an inverse given by assigning to a G-bundle E on X_+ the element $E_{|X} - (E_{x_0} \times X) \in K_G(X)$. (Here, $E_{|X}$ is E viewed as a bundle over X and E_{x_0} is the fibre of E at x_0.)

Notation 3.1.11. For a compact G-space X we name

(i) $K^i_G(X)$ the i^{th} *equivariant K-cohomology* of X;

(ii) $K^i_G(X, A)$ the i^{th} *equivariant K-cohomology* of the pair (X, A);

and for a based, compact G-space X we name

(iii) $\widetilde{K}^i_G(X)$ the i^{th} *equivariant reduced K-cohomology* of X;

(iv) $\widetilde{K}^i_G(X, A)$ the i^{th} *equivariant reduced K-cohomology* of the pair (X, A).

As is standard, we write K^*_G for $K^*_G(\mathrm{pt})$, and more generally E^* in place of $E^*(\mathrm{pt})$ for any cohomology theory E, equivariant or otherwise. Similarly in the reduced case.

We have not yet defined $K^i_G(X)$ for positive i, but we shall come to that in §3.2. First, with a view to following Atiyah's suggestion [9 (Remark (3) following Proposition 4.9)] we record the following corollary.

Corollary 3.1.12. *Suppose G is a compact Lie group and N is a normal subgroup of G which acts freely on the compact G-CW-complex X. Then the quotient $q : X \longrightarrow X/N$ induces an isomorphism*

$$q^* : K^*_{G/N}(X/N) \xrightarrow{\cong} K^*_G(X).$$

Proof. This is an obvious consequence of Proposition 3.1.6. □

We record one final result in this introductory section for use later.

Proposition 3.1.13. *Let G be a compact Lie group and X a compact G-CW-complex. Then $K^*_G(X)$ has the structure of a module over $K^*_G(\mathrm{pt}) = R(G)$, and this structure is given by*

$$\rho \cdot E = \mathrm{pr}^*(\rho) \otimes E \qquad\qquad (\rho \in R(G),\ E \in K^*_G(X)).$$

Here pr *is the G-map $X \longrightarrow \mathrm{pt}$, and this induces $\rho \overset{\mathrm{pr}^*}{\longmapsto} X \times \rho$.*

Proof. See [55 (§2, Example (i))]. □

3.2 Equivariant Bott periodicity

The equivariant Bott periodicity theorem is a striking and fundamental result in equivariant K-theory. There are several associated statements: we present the highlights here. The interested reader will want to look at [46 (XIV, §3)] for more details. We begin our tour with the *Thom isomorphism.*

Theorem 3.2.1 (Thom isomorphism). *Let E be a vector bundle over a compact G-CW-complex X. Write E^T for the Thom space $E^T = D(E)/S(E)$, where $D(E)$ and $S(E)$ denote the disc and sphere bundles of E respectively*[2]*. Then there is an isomorphism*

$$\tau : K_G^*(X) \xrightarrow{\cong} K_G^*(E^T).$$

□

We shall use this Thom isomorphism to give a definition of the *Euler class*, but in order to do so we need to know explicitly what τ is. For that, the reader is referred to Segal [55 (§3)].

If we take X to be a point then a bundle over X is just a representation V. Recall from Lemma 1.3.3 that the one-point compactification of V is $S^V = D(V)/S(V) \cong V \cup \{\infty\}$. We take ∞ as basepoint for S^V and, working in the reduced theory, we have an isomorphism

$$\tau : \widetilde{K}_G^0(S^0) \xrightarrow{\cong} \widetilde{K}_G^0(S^V). \tag{3.2.2}$$

Definition 3.2.3. Given a representation V of G we define the *Euler class* $\chi(V)$ of V as follows. Let i_V be the inclusion $S^0 \hookrightarrow S^V$ at the north and south poles. (Precisely, i_V preserves basepoints and carries the other point of S^0 to $0 \in S^V$.) Using the Thom isomorphism (3.2.2) gives the diagram

$$\begin{array}{ccc} \widetilde{K}_G^0(S^0) & \xleftarrow{(i_V)^*} & \widetilde{K}_G^0(S^V) \\ & \nwarrow^{m} & \uparrow{\scriptstyle \cong}\,{\scriptstyle \tau} \\ & & \widetilde{K}_G^0(S^0). \end{array}$$

Recalling that $\widetilde{K}_G^0(S^0) = R(G)$, we write $1 \in \widetilde{K}_G^0(S^0)$ corresponding to the trivial one-dimensional complex representation $\varepsilon \in R(G)$. We then define $\chi(V) = m(1)$.

[2] One obtains $D(E)$ and $S(E)$ from E by replacing each fibre with its disc and sphere, respectively.

Theorem 3.2.4 (Equivariant Bott periodicity). *Let X be a compact, finite G-CW-complex and V a $\mathbb{C}G$-module. Then multiplying by the* Bott class $\tau(1) \in \widetilde{K}_G(S^V)$ *gives a natural isomorphism*

$$\widetilde{K}_G(X_+) = K_G(X) \xrightarrow{\cong} K_G(V \times X) = \widetilde{K}_G(S^V \wedge X_+).$$

Moreover, if $\dim_{\mathbb{C}}(V) = n$ *then*

$$\chi(V) = (i_V)^* \tau(1) = 1 - \lambda V + \lambda^2 V - \cdots + (-1)^n \lambda^n V \in R(G),$$

where $\lambda^i V$ denotes the i^{th} exterior power *of V.* □

By Definition 3.1.8 and equivariant Bott periodicity, we see that for a finite G-CW-complex X we have $\widetilde{K}_G^{-q-2}(X_+) = \widetilde{K}_G^{-q}(\Sigma^2 X_+) = \widetilde{K}_G^{-q}(S^2 \wedge X_+) \cong \widetilde{K}_G^{-q}(X_+)$ for $q \in \mathbb{N}$. This motivates the following definition.

Definition 3.2.5. For $i \in \mathbb{N}$ we define $K_G^i(X) = K_G^\delta(X)$ where $\delta = \begin{cases} 0 & i \text{ even} \\ -1 & i \text{ odd} \end{cases}$. Similarly for pairs and in the reduced case.

This justifies our comments at the start of the Chapter that equivariant K-theory is 2-periodic and complex stable (since $\dim_{\mathbb{R}}(V)$ is even for complex representations V).

A crucial property of the Euler class, which we shall certainly use later, is the following proposition.

Proposition 3.2.6. *If $V = V_1 \oplus V_2$ is a $\mathbb{C}G$-module then $\chi(V) = \chi(V_1)\chi(V_2)$.*

Proof. The proof is easy once we know [38 (XIX, Proposition 1.4)] that

$$\lambda^n V \cong \bigoplus_{p+q=n} \lambda^p V_1 \otimes \lambda^q V_2.$$

□

We conclude this section with an example of equivariant Bott periodicity in action.

Proposition 3.2.7. *For odd i we have $\widetilde{K}_G^i(S^0) = 0$.*

Proof. By equivariant Bott periodicity it suffices to prove the proposition when $i = -1$. By definition,

$$\widetilde{K}_G^{-1}(S^0) = \widetilde{K}_G(S^1 \wedge S^0) = \widetilde{K}_G(S^1).$$

Thus it suffices to prove that $\widetilde{K}_G(S^1) = 0$.

Take a G-bundle E over S^1. Write S^1_+ for the upper hemisphere and S^1_- for the lower hemisphere, so that $S^1_+ \cap S^1_- = S^0$. Put $E_+ = E_{|S^1_+}$ and similarly for E_-. Since S^1_+ is contractible we have $E_+ \cong V \times S^1_+$ for some representation V. Likewise $E_- \cong V \times S^1_-$ for the *same* representation V since E_1 and E_2 agree on S^0.

Consider now the possible isomorphisms $f : E_{|S^0} \longrightarrow E_{|S^0}$. This map tells us how to glue the ends of E_+ and E_- together and is equivalent to two G-maps $f_1, f_2 : V \longrightarrow V$. We may as well assume that f_1 is the identity. Decomposing V as $V = n_1V_1 \oplus \cdots \oplus n_sV_s$ with V_i simple and $n = \sum_{i=1}^{s} n_i \dim(V_i)$, Schur's lemma implies that f_2 corresponds to a diagonal $n \times n$ matrix with the diagonal entries non-zero complex scalars, so $f_2 \in GL_n(\mathbb{C})$. Since $GL_n(\mathbb{C})$ is path connected we see that f_2 is homotopic to the identity and we may take f to be the identity.

By the clutching lemma (Lemma 3.1.2) we see that there is only one G-bundle on S^1 whose restriction to $E_\pm$ is $V \times S^1_\pm$ and which is compatible with the identity on $V \times S^0$. Since the bundle $V \times S^1$ has this property we conclude that $E \cong V \times S^1$.

Now given two bundles $E \cong V \times S^1$ and $F \cong W \times S^1$ we see that they are obviously stably equivalent, since

$$(V \times S^1) \oplus (W \times S^1) \cong (W \times S^1) \oplus (V \times S^1).$$

Thus $\widetilde{K}_G(S^1) = 0$ as required. □

3.3 $K^0_G(\mathbb{C}P(V))$

Recall that $\mathbb{C}P(V) = S(V \otimes z)/T$ and $\lambda \in T$ acts on $v \otimes 1 \in V \otimes z$ by $\lambda \cdot (v \otimes 1) = v \otimes \lambda = (\lambda v \otimes 1)$. This action is clearly free, and since moreover T is a normal subgroup of $G \times T$ we appeal to Corollary 3.1.12 and deduce that

$$K^*_G(\mathbb{C}P(V)) \cong K^*_{G\times T}(S(V \otimes z)). \tag{3.3.1}$$

We aim to proceed as in [17 (§6)]. Recall that given a pair of based spaces (X, A) there is a cofibre sequence

$$A \hookrightarrow X \hookrightarrow X \cup_A CA \hookrightarrow \Sigma A \hookrightarrow \cdots \tag{3.3.2}$$

which induces a long exact sequence in cohomology. Consider the cofibre sequence of the based pair $(D(V\otimes z)_+, S(V\otimes z)_+)$, viz

$$S(V\otimes z)_+ \longrightarrow D(V\otimes z)_+ \longrightarrow D(V\otimes z)_+ \cup_{S(V\otimes z)_+} C(S(V\otimes z)_+) \longrightarrow \cdots.$$

As an aid to intuition, the reader may like to consider a low dimensional example, where we can draw pictures as in Figure 3.1.

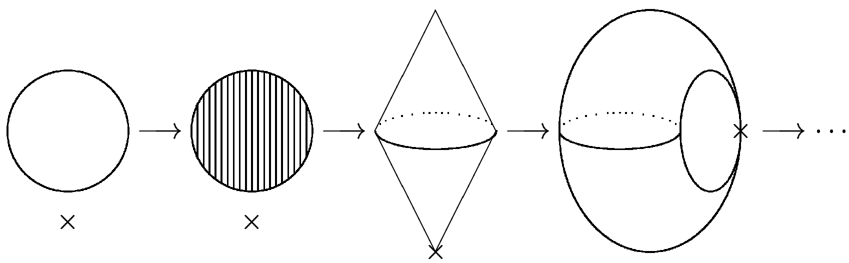


Figure 3.1. Cofibre sequence of the pair $(D(V\otimes z)_+, S(V\otimes z)_+)$.

Now, discs are contractible, $D(V\otimes z)_+ \simeq \mathrm{pt}_+ \cong S^0$ and so are cones, $X \cup_A CA \simeq X/A$. Viewing $D(V)/S(V)$ as S^V we can then write the cofibre sequence of the pair $(D(V\otimes z)_+, S(V\otimes z)_+)$ as

$$S(V\otimes z)_+ \longrightarrow S^0 \longrightarrow S^{V\otimes z} \longrightarrow \Sigma(S(V\otimes z)_+) \longrightarrow \cdots. \tag{3.3.3}$$

Under $\widetilde{K}^*_{G\times T}(-)$ the cofibre sequence (3.3.3) induces the long exact sequence

$$\begin{aligned} \cdots \longrightarrow\ & \widetilde{K}^{-1}_{G\times T}(S(V\otimes z)_+) \\ \longrightarrow\ & \widetilde{K}^0_{G\times T}(S^{V\otimes z}) \longrightarrow \widetilde{K}^0_{G\times T} \longrightarrow \widetilde{K}^0_{G\times T}(S(V\otimes z)_+) \\ \longrightarrow\ & \widetilde{K}^1_{G\times T}(S^{V\otimes z}) \longrightarrow \cdots. \end{aligned} \tag{3.3.4}$$

Lemma 3.3.5. **(i)** *Multiplication by the Euler class $\chi(V \otimes z)$ is an injection*

$$R(G \times T) \longrightarrow R(G \times T);$$

(ii) $\widetilde{K}^1_{G\times T}(S^{V\otimes z}) = 0$.

Proof. (i) Suppose V has complex dimension n. Take $W_1, W_2 \in R(G \times T)$ and suppose that

$$\chi(V \otimes z)W_1 = \chi(V \otimes z)W_2 \in R(G \times T). \tag{3.3.6}$$

We shall show that $W_1 = W_2$. By equivariant Bott periodicity,

$$\begin{aligned} \chi(V \otimes z) &= 1 - V \otimes z + \lambda^2(V \otimes z) - \cdots + (-1)^n \lambda^n(V \otimes z) \\ &= 1 - Vz + \lambda^2(V)z^2 - \cdots + (-1)^n \lambda^n(V)z^n \end{aligned} \tag{3.3.7}$$

is a polynomial in z with constant term 1. From equation (3.1.5) we have

$$W_1 = a_{-p}z^{-p} + \cdots + a_q z^q \text{ and } W_2 = b_{-r}z^{-r} + \cdots + b_s z^s \tag{3.3.8}$$

where $a_i, b_j \in R(G)$ and p, q, r, s are non-negative integers. Use equations (3.3.7) and (3.3.8) to compare the lowest powers of z in equation (3.3.6). We observe that $a_{-p}z^{-p} = b_{-r}z^{-r}$ from which $p = r$ and $a_{-p} = b_{-r}$.

It now follows that

$$\chi(V \otimes z)(W_1 - a_{-p}z^{-p}) = \chi(V \otimes z)(W_2 - b_{-r}z^{-r}) \tag{3.3.9}$$

and we use equations (3.3.7) and (3.3.8) to compare the lowest powers of z in equation (3.3.9). We deduce that $a_{-p+1} = b_{-r+1}$. Continuing in this way we conclude that $W_1 = W_2$ as required.

(ii) By equivariant Bott periodicity, $\widetilde{K}^1_{G\times T}(S^{V\otimes z}) \cong \widetilde{K}^1_{G\times T}(S^0)$ and the latter is zero by Proposition 3.2.7.

□

Proposition 3.3.10. *There is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{K}^0_{G\times T}(S^{V\otimes z}) & \longrightarrow & R(G \times T) & \longrightarrow & K^0_G(\mathbb{C}P(V)) & \longrightarrow & 0, \\ & & \tau \uparrow \cong & \nearrow_{\chi(V\otimes z)} & & & & & \\ & & R(G \times T) & & & & & & \end{array} \tag{3.3.11}$$

in which τ is the Thom isomorphism (3.2.2) and in which the row is an exact sequence of rings.

Proof. Appeal to Lemma 3.3.5 and equation (3.3.1) to deduce the row from the long exact sequence (3.3.4). And the triangle is just the definition of $\chi(V \otimes z)$. □

Proposition 3.3.12. *Let G be a compact Lie group, V a complex representation of G of dimension n and z the natural representation of the circle group T. Then we have an isomorphism of rings*

$$K_G^0(\mathbb{C}P(V)) \cong \frac{R(G)[z]}{(\chi(V \otimes z))}. \tag{3.3.13}$$

Proof. The First Isomorphism Theorem for Rings and the exact row of (3.3.11) tell us that there is a ring isomorphism $K_G^0(\mathbb{C}P(V)) \cong R(G \times T)/\operatorname{Im}(\phi)$, where ϕ is the first non-zero horizontal map of (3.3.11). And from (3.3.11) it is evident that $\operatorname{Im}(\phi)$ is the ideal of $R(G\times T)$ generated by $\chi(V\otimes z)$. Next, recall from (3.1.5) that $R(G\times T) = R(G)[z, z^{-1}]$ and so it remains only to observe that, in the quotient, z^{-1} comes for free. Since $\chi(V\otimes z)$ is a polynomial in z with constant term 1, say $\chi(V \otimes z) = 1 + z\chi'(V \otimes z)$, in the quotient we have

$$0 = 1 + z\chi'(V \otimes z)$$

and $-\chi'(V \otimes z)$ is an inverse for z. □

Remark 3.3.14. After devoting this effort to $K_G^0(\mathbb{C}P(V))$ the reader may be curious about $K_G^1(\mathbb{C}P(V))$, but it turns out that $K_G^1(\mathbb{C}P(V)) = 0$. Indeed, the long exact sequence (3.3.4) reads

$$\cdots \widetilde{K}^1_{G\times T} \longrightarrow \widetilde{K}^1_{G\times T}(S(V\otimes z)_+) \longrightarrow \widetilde{K}^2_{G\times T}(S^{V\otimes z}) \xrightarrow{\chi} \widetilde{K}^2_{G\times T} \longrightarrow \cdots .$$

The map labelled χ is seen to be injective, since it forms a side of a triangle as in (3.3.11) and we may then appeal to Lemma 3.3.5, which also tells us that $\widetilde{K}^1_{G\times T} = 0$. It then follows that $K_G^1(\mathbb{C}P(V)) = 0$.

Example 3.3.15. Suppose $G = C_2$, the cyclic group with two elements. We present, in Table A.1 (page 169), the character table of C_2, from which one observes that $R(C_2) = \mathbb{Z}[\alpha]/(\alpha^2 - 1)$. We consider the 2-dimensional complex representations of G. Theorem 3.2.4 and Proposition 3.2.6 tell us that

- $K^0_{C_2}(\mathbb{C}P(\varepsilon \oplus \varepsilon)) \cong \mathbb{Z}[\alpha, z]/(\alpha^2 - 1, (1-z)^2)$;
- $K^0_{C_2}(\mathbb{C}P(\varepsilon \oplus \alpha)) \cong \mathbb{Z}[\alpha, z]/(\alpha^2 - 1, (1-z)(1-\alpha z))$;
- $K^0_{C_2}(\mathbb{C}P(\alpha \oplus \alpha)) \cong \mathbb{Z}[\alpha, z]/(\alpha^2 - 1, (1-\alpha z)^2)$.

The reader will notice that here we have begun to foster the common omission of the symbol $\otimes$. This trend is set to continue. We present one more result before we specialise to the abelian case in the next section.

Lemma 3.3.16. *$K^0_G(\mathbb{C}P(V))$ is free and finitely generated as a module over the representation ring $R(G)$.*

Proof. Suppose that V has complex dimension n. We argue that $\{1, z, \ldots, z^{n-1}\}$ is a finite $R(G)$-basis for $K^0_G(\mathbb{C}P(V))$. Consider equation (3.3.13). Clearly $1, z, \ldots, z^{n-1}$ are independent in the quotient, so it suffices to show that we can express powers of z as an $R(G)$-linear combination of $1, z, \ldots, z^{n-1}$. Inductively all we need show is that this is possible for z^n. By equivariant Bott periodicity,

$$\begin{aligned} \chi(V \otimes z) &= 1 - V \otimes z + \lambda^2(V \otimes z) - \cdots + (-1)^n \lambda^n(V \otimes z) \\ &= 1 - Vz + \lambda^2(V)z^2 - \cdots + (-1)^n \lambda^n(V) z^n \end{aligned}$$

and so in $K^0_G(\mathbb{C}P(V))$ we have

$$\lambda^n(V) z^n = (-1)^n \Big(1 - Vz + \lambda^2(V)z^2 - \cdots + (-1)^{n-1}\lambda^{n-1}(V) z^{n-1}\Big).$$

Now $\lambda^n(V)$ is, by [42 (XVI, Theorem 11)], the determinant,

$$\lambda^n(V) = \det(V). \tag{3.3.17}$$

This is invertible in $R(G)$ and we are done. ☐

3.3.1 An $R(A)$-basis for $K^0_A(\mathbb{C}P(V))$

We now draw about us the safety net of taking $G = A$ to be abelian. As we shall see, this permits a choice of a convenient basis for $K^0_A(\mathbb{C}P(V))$.

Proposition 3.3.18. *Suppose $V = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_n$ is a sum of one-dimensional complex representations α_i $(1 \leq i \leq n)$ of the abelian compact Lie group A. Then we have an $R(A)$-basis for $K_A^0(\mathbb{C}P(V))$ given by*

$$\left\{1, y^{\alpha_1}, y^{\alpha_1}y^{\alpha_2}, \ldots, y^{\alpha_1}\cdots y^{\alpha_{n-1}}\right\}, \tag{3.3.19}$$

where $y^{\alpha_i} = 1 - \alpha_i z$ $(1 \leq i \leq n)$.

Proof. This is clear. □

Remark 3.3.20. This basis, (3.3.19), came about from writing $V = \alpha_1 \oplus \cdots \oplus \alpha_n$ with the summands *in that order*, that is, from the complete flag

$$\mathcal{F} = \Big(0 \subset \alpha_1 \subset \alpha_1 \oplus \alpha_2 \subset \cdots \subset \alpha_1 \oplus \cdots \oplus \alpha_n = V\Big) \tag{3.3.21}$$

for V. Our hypothesis of $G = A$ being abelian ensures that such an $\mathcal{F}$ exists but we are in the unfortunate position of having to *choose* from at worst $n!$ such $\mathcal{F}$. In §3.5 we shall see that this turns out to be much less of a bother than one may initially suppose. Unless stated otherwise, given $V = \alpha_1 \oplus \cdots \oplus \alpha_n$ we shall *always* choose the flag $\mathcal{F}$ above. $\mathcal{F}$ is then reflected in the notation for our basis (3.3.19).

3.4 Equivariant K-homology

We devote now some attention to the equivariant K-*hom*ology of $\mathbb{C}P(V)$. Just as with cohomology, there will come a point when we restrict to taking $G = A$ an abelian group, but, until then we take G to be any compact Lie group. One can [50] perform a bare-hands construction of equivariant K-homology, but that is not our aim here. As before, we shall be interested in the *represented* theory. We shall come to that in due course (Chapter 4) but for now we content ourselves with a preliminary or "working" definition of equivariant K-homology.

The usual point of view is that cohomology is considered the dual construction to homology. Our preliminary definition, justified in §4.2, permits us to do just the opposite: given its geometric interpretation, we prefer to describe equivariant K-cohomology first, and then think of equivariant K-homology as the dual construction.

Preliminary Definition 3.4.1. Suppose that $K_G^0(X)$ is free and finitely generated as a module over $R(G)$. Then we define the *zeroth equivariant K-homology* of X to be

$$K_0^G(X) = \mathrm{Hom}_{R(G)}(K_G^0(X), R(G))$$

as a module over $R(G)$.

For justification of this Preliminary Definition, see Remark 4.2.12. Note that if $X = \mathbb{C}P(V)$, then the hypotheses in Preliminary Definition 3.4.1 are satisfied. (This follows from Lemma 3.3.16.)

Corollary 3.4.2. *We have $K_0^{G\times T}(\mathrm{pt}) \cong R(G\times T)$ and $K_0^G(\mathrm{pt}) \cong R(G)$.*

Proof. In view of Example 3.1.4 this is obvious. □

If based X and ΣX satisfy the conditions in Preliminary Definition 3.4.1, we make sense of $\widetilde{K}_i^G(X)$ for $i \in \mathbb{Z}$ by setting

$$\widetilde{K}_i^G(X) = \begin{cases} \widetilde{K}_0^G(X) & i \text{ even} \\ \widetilde{K}_0^G(\Sigma X) & i \text{ odd} \end{cases}.$$

Lemma 3.4.3. *For i odd we have $\widetilde{K}_i^G(\mathrm{pt}_+) \cong \widetilde{K}_i^G(S^{V\otimes z}) = 0$.*

Proof. The equivariant Bott periodicity statements of §3.2 all have their homological analogues, so it clearly suffices to prove the lemma for $\widetilde{K}_{-1}^G(\mathrm{pt}_+)$. Recalling Proposition 3.2.7 we see that

$$\begin{aligned} \widetilde{K}_{-1}^G(\mathrm{pt}_+) &= \widetilde{K}_0^G(\Sigma \mathrm{pt}_+) = \widetilde{K}_0^G(S^1) \\ &\cong \mathrm{Hom}_{R(G)}(\widetilde{K}_G^0(S^1), R(G)) \\ &= \mathrm{Hom}_{R(G)}(\widetilde{K}_G^{-1}(\mathrm{pt}_+), R(G)) \\ &= \mathrm{Hom}_{R(G)}(0, R(G)) = 0. \end{aligned}$$

□

Lemma 3.4.4. *We have $\widetilde{K}_0^G(\mathbb{C}P(V)_+) \cong \widetilde{K}_{-1}^{G\times T}(S(V\otimes z)_+)$.*

Proof. Delayed until §4.2. □

Applying the covariant functor $\widetilde{K}_*^{G\times T}(-)$ to the cofibre sequence (3.3.3) we obtain a long exact sequence in homology, viz

$$\begin{array}{l} \cdots \longrightarrow \widetilde{K}_1^{G\times T}(S^{V\otimes z}) \\ \hookrightarrow \widetilde{K}_0^{G\times T}(S(V\otimes z)_+) \longrightarrow \widetilde{K}_0^{G\times T} \longrightarrow \widetilde{K}_0^{G\times T}(S^{V\otimes z}) \\ \hookrightarrow \widetilde{K}_{-1}^{G\times T}(S(V\otimes z)_+) \longrightarrow \widetilde{K}_{-1}^{G\times T} \longrightarrow \cdots . \end{array} \tag{3.4.5}$$

Proposition 3.4.6 (inferior version). *There is a short exact sequence of rings*

$$0 \longrightarrow R(G\times T) \longrightarrow \widetilde{K}_0^{G\times T}(S^{V\otimes z}) \longrightarrow K_0^G(\mathbb{C}P(V)) \longrightarrow 0. \tag{3.4.7}$$

Proof. Pick out the relevant portion of the long exact sequence (3.4.5) and appeal to Lemmas 3.4.3 and 3.4.4. To deduce injectivity of the first non-zero map, observe that it corresponds to multiplication by the Euler class $\chi(V\otimes z)$ and apply Lemma 3.3.5 (i). $\square$

Consider the multiplicative set $S = \{(\chi(V\otimes z))^n\}_{n\geq 0} \subseteq R(G\times T)$. As in [12 (Chapter 3)] we can form the *module of fractions* $S^{-1}R(G\times T)$ of $R(G\times T)$ with respect to S. Now $\frac{1}{\chi(V\otimes z)} \in S^{-1}R(G\times T)$ and we write

$$\frac{1}{\chi(V\otimes z)}R(G\times T) = \left\{\frac{1}{\chi(V\otimes z)}r \mid r\in R(G\times T)\right\} \subseteq S^{-1}R(G\times T).$$

Note that $\rho \longmapsto \frac{\rho}{\chi(V\otimes z)}$ is an injection $R(G\times T) \longrightarrow \frac{1}{\chi(V\otimes z)}R(G\times T)$ because $\chi(V\otimes z)$ is not a zero divisor in $R(G\times T)$. (This follows from Lemma 3.3.5 (i).) Hence $\frac{1}{\chi(V\otimes z)}R(G\times T)$ is an $R(G\times T)$-submodule of $S^{-1}R(G\times T)$ which is isomorphic to $R(G\times T)$ and is generated by $\frac{1}{\chi(V\otimes z)} \in \frac{1}{\chi(V\otimes z)}R(G\times T)$.

In view of [12 (Proposition 3.3)] the reader will observe that we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R(G\times T) & \xrightarrow{\chi(V\otimes z)} & R(G\times T) & \longrightarrow & K_0^G(\mathbb{C}P(V)) & \longrightarrow & 0 \\ & & \big\downarrow = & & \vdots\!\uparrow \chi(V\otimes z)\ \cong & \nearrow & & & \\ & & R(G\times T) & \hookrightarrow & \frac{1}{\chi(V\otimes z)}R(G\times T) & & & & \end{array}$$

in which the lower row is the inclusion $R(G\times T) \hookrightarrow \frac{1}{\chi(V\otimes z)}R(G\times T)$ given by $\rho \longmapsto \frac{\chi(V\otimes z)\rho}{\chi(V\otimes z)}$, and *either* route consisting of solid arrows is exact.

Proposition 3.4.6 (superior version). *We may rewrite (3.4.7) as*

$$0 \longrightarrow R(G\times T) \hookrightarrow \frac{1}{\chi(V\otimes z)}R(G\times T) \longrightarrow K_0^G(\mathbb{C}P(V)) \longrightarrow 0. \tag{3.4.8}$$

□

The diagram above justifies the truth of the proposition but not the adjective *superior*, which prompts the question *why the change of notation?* Suppose that V, W are representations of G. Presently, §3.5, we shall consider the inclusions $V \hookrightarrow V\oplus W \hookleftarrow W$. They induce a commutative diagram

$$\begin{array}{ccccccccc}
 & & R(G\times T) & \hookrightarrow & \frac{1}{\chi(V\otimes z)}R(G\times T) & \longrightarrow & K_0^G(\mathbb{C}P(V)) & & \\
 & \nearrow & \| & & \downarrow & & \downarrow & \searrow & \\
0 & \longrightarrow & R(G\times T) & \hookrightarrow & \frac{1}{\chi((V\oplus W)\otimes z)}R(G\times T) & \longrightarrow & K_0^G(\mathbb{C}P(V\oplus W)) & \longrightarrow & 0 \\
 & \searrow & \| & & \uparrow & & \uparrow & \nearrow & \\
 & & R(G\times T) & \hookrightarrow & \frac{1}{\chi(W\otimes z)}R(G\times T) & \longrightarrow & K_0^G(\mathbb{C}P(W)) & &
\end{array}$$

in which the three rows are exact. The superior notation ensures that the vertical maps are *inclusions*, and this will aid our reasoning. Moreover, chasing $\frac{1}{\chi(V\otimes z)}$ around the upper-right-hand square demonstrates

$$\frac{1}{\chi(V\otimes z)} = \frac{\chi(W\otimes z)}{\chi((V\oplus W)\otimes z)} \in K_0^A(\mathbb{C}P(V\oplus W)). \tag{3.4.9}$$

This is an intuitive notation – and [57 (p46)] "you can ask no more of notation than it should be an effective substitute for thought".

Corollary 3.4.10. *We have a ring isomorphism*

$$K_0^G(\mathbb{C}P(V)) \cong \frac{\frac{1}{\chi(V\otimes z)}R(G)[z]}{R(G)[z]}.$$

□

Remark 3.4.11. The curious reader may once again be concerned with odd degrees. But $K_1^G(\mathbb{C}P(V)) = 0$ – this is obvious from Remark 3.3.14 and our Preliminary Definition 3.4.1.

3.4.1 An $R(A)$-basis for $K_0^A(\mathbb{C}P(V))$

Recall from §3.3.1 that we have an $R(A)$-basis for $K_A^0(\mathbb{C}P(V))$ provided we take the group of equivariance to be abelian – let us suppose henceforth that this is so.

Notation 3.4.12. With notation as in §3.3.1 we write $\{\beta_i^{\mathcal{F}}\}_{0\le i\le n-1}$ for the *dual basis* to (3.3.19). That is, for $0 \le i \le n-1$, each $\beta_i^{\mathcal{F}}$ is a homomorphism $K_G^0(\mathbb{C}P(V)) \longrightarrow R(G)$ defined by

$$\beta_i^{\mathcal{F}}(1) = \begin{cases} 1 & i = 0 \\ 0 & i > 0 \end{cases} ;$$

$$\beta_i^{\mathcal{F}}(y^{\alpha_1}\cdots y^{\alpha_j}) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \ne j \end{cases} \qquad (1 \le j \le n-1),$$

and extending linearly. Given $f \in K_0^A(\mathbb{C}P(V))$, $x \in K_A^0(\mathbb{C}P(V))$ we write $\langle x, f\rangle$ for the *Kronecker pairing*, $\langle x, f\rangle = f(x)$. Thus $\langle y^{\alpha_1}\cdots y^{\alpha_j}, \beta_i^{\mathcal{F}}\rangle = \delta_{ij}$ and we have a basis

$$\left\{\beta_0^{\mathcal{F}}, \beta_1^{\mathcal{F}}, \ldots, \beta_{n-1}^{\mathcal{F}}\right\} \tag{3.4.13}$$

for $K_0^A(\mathbb{C}P(V))$.

3.5 Independence of $\mathcal{F}$

Throughout this section we require that $G = A$ be abelian. The reader will recall, with disappointment, that our bases (3.3.19), (3.4.13) depended crucially upon the flag

$$\mathcal{F} = \Big(0 \subset \alpha_1 \subset \alpha_1 \oplus \alpha_2 \subset \cdots \subset \alpha_1 \oplus \cdots \oplus \alpha_n\Big).$$

Our immediate goal is to redress this dilemma by proving that *the sum* $\sum_{i=0}^{n-1} \beta_i^{\mathcal{F}}$ *is independent of the flag* $\mathcal{F}$. Precisely, we have Theorem 3.5.10. The argument is by induction on the complex dimension n of V. We present, §3.5.2, the 2-dimensional case for its illuminating clarity. The general case, §3.5.3, then involves nothing more than heavier notation.

3.5.1 The initial step

We take $V = \alpha$ to be a one-dimensional representation. Then there is no choice: the only complete flag for V is $(0 \subset \alpha_1 = V)$ giving $\{1\}$ as an $R(A)$-basis for $K_A^0(\mathbb{C}P(\alpha))$ – this corresponds to the fact that setting the Euler class equal to zero means $z = \alpha^{-1} \in R(A)$, and $K_A^0(\mathbb{C}P(\alpha)) \cong R(A)$. The dual basis for $K_0^A(\mathbb{C}P(V))$ is $\{\beta_0\}$. The initial step is obvious but since it is the foundation of the inductive argument we record it.

Lemma 3.5.1. *We have* $\beta_0 = \frac{1}{\chi(\alpha\otimes z)} \in K_0^A(\mathbb{C}P(\alpha))$. □

3.5.2 The 2-dimensional case

We take $V = \alpha \oplus \beta$ and we *choose* the flag

$$\mathcal{F} = \Big(0 \subset \alpha \subset \alpha \oplus \beta = V\Big),$$

so that $K_A^0(\mathbb{C}P(V))$ has basis $\{1, 1 - \alpha z\}$ and $K_0^A(\mathbb{C}P(V))$ has basis $\{\beta_0^{\mathcal{F}}, \beta_1^{\mathcal{F}}\}$. We have inclusions

$$\mathbb{C}P(\alpha) \xhookrightarrow{j_\beta} \mathbb{C}P(\alpha \oplus \beta) \xhookleftarrow{j_\alpha} \mathbb{C}P(\beta)$$

and we write ι_α for $\frac{1}{\chi(\beta\otimes z)} \in K_0^A(\mathbb{C}P(\beta))$ and ι_β for $\frac{1}{\chi(\alpha\otimes z)} \in K_0^A(\mathbb{C}P(\alpha))$. Strictly, by ι_β we mean the equivalence class $\left[\frac{1}{\chi(\alpha\otimes z)}\right]$ in $K_0^A(\mathbb{C}P(\alpha))$, that is those elements which are homologous to $\frac{1}{\chi(\alpha\otimes z)}$. We shall only insist upon writing the square brackets if necessary. Likewise for ι_α.

Lemma 3.5.2. *Suppose that* $1 - \alpha\beta^{-1}$ *is not a zero divisor in* $R(A \times T)$. *Then we have* $\beta_0^{\mathcal{F}} + \beta_1^{\mathcal{F}} = \frac{1}{\chi((\alpha\oplus\beta)\otimes z)} \in K_0^A(\mathbb{C}P(\alpha \oplus \beta))$.

Proof. We aim to express $(j_\alpha)_*(\iota_\alpha), (j_\beta)_*(\iota_\beta) \in K_0^A(\mathbb{C}P(\alpha \oplus \beta))$ in terms of the basis $\{\beta_0^{\mathcal{F}}, \beta_1^{\mathcal{F}}\}$. We shall see in Remark 4.2.15 that for a G-map f between G-spaces X, Y, given $c \in K_G^0(Y), h \in K_0^G(X)$, we have

$$\langle c, f_*(h)\rangle = \langle f^*(c), h\rangle \in R(G). \tag{3.5.3}$$

Hence,

$$\langle 1, (j_\alpha)_*(\iota_\alpha)\rangle = \langle (j_\alpha)^*(1), \iota_\alpha\rangle = \langle 1, \iota_\alpha\rangle.$$

The attentive reader will have noticed that in $\langle 1, \iota_\alpha \rangle$ the "1" is $1 \in K_A^0(\mathbb{C}P(\beta))$ whereas the other two occurrences are $1 \in K_A^0(\mathbb{C}P(\alpha \oplus \beta))$. By the initial step we see that

$$\langle 1, (j_\alpha)_*(\iota_\alpha) \rangle = 1.$$

Next,

$$\begin{aligned} \langle 1 - \alpha z, (j_\alpha)_*(\iota_\alpha) \rangle &= \langle (j_\alpha)^*(1 - \alpha z), \iota_\alpha \rangle \\ &= \langle (j_\alpha)^*(\alpha\beta^{-1}(1 - \beta z) + 1 - \alpha\beta^{-1}), \iota_\alpha \rangle \\ &= \langle 1 - \alpha\beta^{-1}, \iota_\alpha \rangle \\ &= 1 - \alpha\beta^{-1}. \end{aligned}$$

The third equality follows because $1 - \beta z = 0 \in K_0^A(\mathbb{C}P(\beta))$ and we now deduce that

$$(j_\alpha)_*(\iota_\alpha) = \beta_0^{\mathcal{F}} + (1 - \alpha\beta^{-1})\beta_1^{\mathcal{F}}. \tag{3.5.4}$$

In similar spirit, we find that

$$\langle 1, (j_\beta)_*(\iota_\beta) \rangle = 1, \qquad \langle 1 - \alpha z, (j_\beta)_*(\iota_\beta) \rangle = 0$$

which tells us that

$$(j_\beta)_*(\iota_\beta) = \beta_0^{\mathcal{F}}. \tag{3.5.5}$$

From equation (3.4.9) we see that

$$(j_\alpha)_*(\iota_\alpha) = \frac{\chi(\alpha \otimes z)}{\chi((\alpha \oplus \beta) \otimes z)}, \qquad (j_\beta)_*(\iota_\beta) = \frac{\chi(\beta \otimes z)}{\chi((\alpha \oplus \beta) \otimes z)}. \tag{3.5.6}$$

Since $1 - \alpha z = \alpha\beta^{-1}(1 - \beta z) + (1 - \alpha\beta^{-1})$ in $R(A \times T)$ we have

$$\frac{\chi(\alpha \otimes z)}{\chi((\alpha \oplus \beta) \otimes z)} = \alpha\beta^{-1} \frac{\chi(\beta \otimes z)}{\chi((\alpha \oplus \beta) \otimes z)} + (1 - \alpha\beta^{-1}) \frac{1}{\chi((\alpha \oplus \beta) \otimes z)}$$

in $K_0^A(\mathbb{C}P(\alpha \oplus \beta))$. Using equations (3.5.4), (3.5.5), (3.5.6) and rearranging yields

$$(1 - \alpha\beta^{-1})(\beta_0^{\mathcal{F}} + \beta_1^{\mathcal{F}}) = (1 - \alpha\beta^{-1}) \frac{1}{\chi((\alpha \oplus \beta) \otimes z)}.$$

The lemma follows. □

3.5.3 The n-dimensional case

This time we take $V = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_n$ and we *choose* the flag

$$\mathcal{F} = \Big(0 \subset \alpha_1 \subset \alpha_1 \oplus \alpha_2 \subset \cdots \subset \alpha_1 \oplus \cdots \oplus \alpha_n = V\Big).$$

We have bases (3.3.19) for $K^0_A(\mathbb{C}P(V))$ and (3.4.13) for $K_0^A(\mathbb{C}P(V))$, the inclusions

$$j_i : \mathbb{C}P(\alpha_1 \oplus \cdots \oplus \alpha_{i-1} \oplus \alpha_{i+1} \oplus \cdots \oplus \alpha_n) \hookrightarrow \mathbb{C}P(V)$$

and we write

$$\begin{aligned} \iota_i &= \frac{1}{\chi((\alpha_1 \oplus \cdots \oplus \alpha_{i-1} \oplus \alpha_{i+1} \oplus \cdots \oplus \alpha_n) \otimes z)} \\ &\in K_0^A(\mathbb{C}P(\alpha_1 \oplus \cdots \oplus \alpha_{i-1} \oplus \alpha_{i+1} \oplus \cdots \oplus \alpha_n)) \end{aligned}$$

for $1 \leq i \leq n$.

Lemma 3.5.7. *Let $n > 1$. Suppose that $1 - \alpha_{i-1}\alpha_i^{-1}$ is not a zero divisor in $R(A \times T)$ for $i = 2, \ldots, n$. Then we have*

$$\beta_0^{\mathcal{F}} + \cdots + \beta_{n-1}^{\mathcal{F}} = \frac{1}{\chi(V \otimes z)} \in K_0^A(\mathbb{C}P(V)).$$

Proof. We induct on the complex dimension of V. So suppose the lemma holds for representations of dimensions smaller than n. We use (3.5.3) and the inductive hypothesis to compute

$$\langle y^{\alpha_1} \cdots y^{\alpha_i}, (j_n)_*(\iota_n) \rangle = \begin{cases} 1 & 0 \leq i \leq n-2 \\ 0 & i = n-1 \end{cases}$$

from which

$$(j_n)_*(\iota_n) = \beta_0^{\mathcal{F}} + \cdots + \beta_{n-2}^{\mathcal{F}}. \tag{3.5.8}$$

Similarly,

$$\langle y^{\alpha_1} \cdots y^{\alpha_i}, (j_{n-1})_*(\iota_{n-1}) \rangle = \begin{cases} 1 & 0 \leq i \leq n-2 \\ 1 - \alpha_{n-1}\alpha_n^{-1} & i = n-1 \end{cases}$$

from which

$$(j_{n-1})_*(\iota_{n-1}) = \beta_0^{\mathcal{F}} + \cdots + \beta_{n-2}^{\mathcal{F}} + (1 - \alpha_{n-1}\alpha_n^{-1})\beta_{n-1}^{\mathcal{F}}. \tag{3.5.9}$$

Appealing to equations (3.4.9), (3.5.8) and (3.5.9),

$$\frac{\chi(\alpha_{n-1}\otimes z)}{\chi(V\otimes z)} = \alpha_{n-1}\alpha_n^{-1}\frac{\chi(\alpha_n\otimes z)}{\chi(V\otimes z)} + (1-\alpha_{n-1}\alpha_n^{-1})\frac{1}{\chi(V\otimes z)},$$

which rearranges to

$$(1-\alpha_{n-1}\alpha_n^{-1})(\beta_0^{\mathcal{F}}+\cdots+\beta_{n-1}^{\mathcal{F}}) = (1-\alpha_{n-1}\alpha_n^{-1})\frac{1}{\chi(V\otimes z)}$$

and the lemma follows. □

Theorem 3.5.10. *Let V be a complex representation, $\dim_{\mathbb{C}} V = n$, of the abelian compact Lie group A. Take the complete flag*

$$\mathcal{F} = \Big(0 \subset \alpha_1 \subset \alpha_1\oplus\alpha_2 \subset \cdots \subset \alpha_1\oplus\cdots\oplus\alpha_n = V\Big)$$

for V and let the corresponding bases be

$$\{1, y^{\alpha_1}, y^{\alpha_1}y^{\alpha_2}, \ldots, y^{\alpha_1}\cdots y^{\alpha_{n-1}}\} \text{ for } K_A^0(\mathbb{C}P(V));$$
$$\{\beta_0^{\mathcal{F}}, \ldots, \beta_{n-1}^{\mathcal{F}}\} \text{ for } K_0^A(\mathbb{C}P(V))$$

as in (3.3.19) and (3.4.13). Then

$$\sum_{i=0}^{n-1}\beta_i^{\mathcal{F}} = \frac{1}{\chi(V\otimes z)},$$

and since the right-hand side is a topological invariant of V, the sum on the left-hand side is independent of the flag $\mathcal{F}$.

Proof. Let T^n be the n-torus, and write z_i for the representation of T^n in which

$$(\lambda_1,\ldots,\lambda_n) \longmapsto \lambda_i.$$

Proceeding as in §3.1, we find that $R(T^n) = \mathbb{Z}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$. Note that $1 - z_i z_j^{-1}$ is not a zero divisor provided $i \neq j$ so Lemma 3.5.7 completes the proof if $A = T^n$ and $V = z_1\oplus\cdots\oplus z_n$. The result now follows in general, for the pullback of $z_1\oplus\cdots\oplus z_n$ along the homomorphism $\boldsymbol{\alpha}: A \longrightarrow T^n$, in which $\boldsymbol{\alpha}(a) = (\alpha_1(a),\ldots,\alpha_n(a))$, is $\alpha_1\oplus\cdots\oplus\alpha_n$. □

Example 3.5.11. Theorem 3.5.10 is about the *sum* of the $\beta_i^{\mathcal{F}}$. We cannot recast the theorem to say that the individual $\beta_i^{\mathcal{F}}$ are independent of $\mathcal{F}$. For take $A = C_2$ and consider $V = \alpha \oplus \varepsilon \oplus \varepsilon$ (notation as in Table A.1, page 169). We have flags

$$\mathcal{F} = \Big(0 \subset \alpha \subset \alpha \oplus \varepsilon \subset \alpha \oplus \varepsilon \oplus \varepsilon\Big);$$
$$\mathcal{F}' = \Big(0 \subset \varepsilon \subset \varepsilon \oplus \varepsilon \subset \varepsilon \oplus \varepsilon \oplus \alpha\Big)$$

for V which give bases

- $\{1, 1-\alpha z, (1-\alpha z)(1-z)\}$ for $K^0_{C_2}(\mathbb{C}P(V))$ (from $\mathcal{F}$);
- $\{1, 1-z, (1-z)(1-z)\}$ for $K^0_{C_2}(\mathbb{C}P(V))$ (from $\mathcal{F}'$);
- $\{\beta_0^{\mathcal{F}}, \beta_1^{\mathcal{F}}, \beta_2^{\mathcal{F}}\}$ and $\{\beta_0^{\mathcal{F}'}, \beta_1^{\mathcal{F}'}, \beta_2^{\mathcal{F}'}\}$ for $K_0^{C_2}(\mathbb{C}P(V))$.

Immediately, $\beta_1^{\mathcal{F}}(1-\alpha z) = 1$. But writing

$$1 - \alpha z = \alpha \cdot (1-z) + (1-\alpha) \cdot 1$$

we see that $\beta_1^{\mathcal{F}'}(1-\alpha z) = \alpha \cdot 1 + (1-\alpha)\cdot 0 = \alpha$.

Notation 3.5.12. We now write $\sum_{i=0}^{n-1} \beta_i$ where before we were constrained to write $\sum_{i=0}^{n-1} \beta_i^{\mathcal{F}}$. Theorem 3.5.10 guarantees that this is not ambiguous.

Though we have relieved ourselves of depending upon $\mathcal{F}$, we proceed one step further and find a result which is useful not only for concrete computations but is inspiring when we come to consider the non-abelian case in §4.4.

Proposition 3.5.13. *Let $\{\rho_a\}_{a\in A}$ be a set containing all simple complex representations of the abelian group A, without duplication. Take $n \geq 2$ and $V = \alpha_1 \oplus \cdots \oplus \alpha_n$, so each α_i is some ρ_{a_i}. Suppose that $a_1, \ldots, a_m \in A$ where the positive integer m satisfies $m < n$. Then*

$$\left(\sum_{i=0}^{n-1} \beta_i\right)(y^{\rho_{a_1}} y^{\rho_{a_2}} \cdots y^{\rho_{a_m}}) = 1.$$

Proof. For each fixed n we induct on m. Take the basis (3.3.19) for $K^0_A(\mathbb{C}P(V))$ arising from the flag (3.3.21) for V. When $m = 1$ we have

$$\left(\sum_{i=0}^{n-1} \beta_i\right)(1 - \rho_{a_1} z) = \left(\sum_{i=0}^{n-1} \beta_i\right)(\rho_{a_1}\alpha_1^{-1}(1-\alpha_1 z) + 1 - \rho_{a_1}\alpha_1^{-1}) = 1$$

as claimed.

Now suppose the proposition holds for $m = 1, \ldots, k-1 < n-1$ and consider $y := y^{\rho_{a_1}} \cdots y^{\rho_{a_k}}$. If y is a basis element we are done. If y is *not* such an element, we let $\eta(y)$ be the unique integer such that we may write y in the form

$$y = y^{\alpha_1} y^{\alpha_2} y^{\alpha_3} \cdots y^{\alpha_{\eta(y)}} y^{\rho_{b_1}} y^{\rho_{b_2}} \cdots y^{\rho_{b_t}}$$

where $t = k - \eta(y)$, $b_1, \ldots, b_t \in A$ and $\rho_{b_i} \ngeq \alpha_{\eta(y)+1}$ for all $1 \le i \le t$. A priori it could happen that $\eta(y) = 0$, but in all cases $\eta(y) < k$ else y would be a basis element. Recalling that n is fixed we assume, without loss, that $\eta(y)$ is maximal over such y. Then we have

$$\begin{aligned}
\left(\sum_{i=0}^{n-1} \beta_i\right)(y) &= \left(\sum_{i=0}^{n-1} \beta_i\right)\left(y^{\alpha_1} \cdots y^{\alpha_{\eta(y)}} y^{\rho_{b_1}} \cdots y^{\rho_{b_t}}\right) \\
&= \left(\sum_{i=0}^{n-1} \beta_i\right)\Big(\rho_{b_1}\alpha_{\eta(y)+1}^{-1} \cdot y^{\alpha_1} \cdots y^{\alpha_{\eta(y)}} y^{\alpha_{\eta(y)+1}} y^{\rho_{b_2}} \cdots y^{\rho_{b_t}} \\
&\qquad + (1 - \rho_{b_1}\alpha_{\eta(y)+1}^{-1}) \cdot y^{\alpha_1} \cdots y^{\alpha_{\eta(y)}} y^{\rho_{b_2}} \cdots y^{\rho_{b_t}}\Big) \\
&= \rho_{b_1}\alpha_{\eta(y)+1}^{-1}\left(\sum_{i=0}^{n-1} \beta_i\right)\left(y^{\alpha_1} \cdots y^{\alpha_{\eta(y)+1}} y^{\rho_{b_2}} \cdots y^{\rho_{b_t}}\right) + 1 - \rho_{b_1}\alpha_{\eta(y)+1}^{-1} \\
&= \rho_{b_1}\alpha_{\eta(y)+1}^{-1} + 1 - \rho_{b_1}\alpha_{\eta(y)+1}^{-1} = 1.
\end{aligned}$$

The third equality follows from the inductive hypothesis. The fourth follows because the maximality of $\eta(y)$ means that $y^{\alpha_1} \cdots y^{\alpha_{\eta(y)+1}} y^{\rho_{b_2}} \cdots y^{\rho_{b_t}}$ must be a basis element. This completes the inductive step and the proof. ☐

Remarks 3.5.14. **(i)** The hypothesis "$m < n$" is necessary. Ignoring it, ones finds that

$$\left(\sum_{i=0}^{n-1} \beta_i\right)(y^{\alpha_1} \cdots y^{\alpha_n}) = \left(\sum_{i=0}^{n-1} \beta_i\right)(\chi(V \otimes z)) = 0.$$

(ii) The proposition gives an entirely *algebraic* proof that $\sum_{i=0}^{n-1} \beta_i$ is independent of flag. The lack of dependence on topology comes at a price: Proposition 3.5.13 is weaker than Theorem 3.5.10 which told us that $\sum_{i=0}^{n-1} \beta_i$ was equal to $\frac{1}{\chi(V \otimes z)}$. But in the algebraist's favour, Proposition 3.5.13 has an important corollary.

Corollary 3.5.15. *Suppose $V = \alpha_1 \oplus \cdots \oplus \alpha_n$. Write $\mathcal{B}$ for the basis*

$$\{1, 1-z, (1-z)^2, \ldots, (1-z)^{n-1}\}$$

for $K_A^0(\mathbb{C}P(V))$ and $\{\beta_0^{\mathcal{B}}, \ldots, \beta_{n-1}^{\mathcal{B}}\}$ for the dual basis for $K_0^A(\mathbb{C}P(V))$. Then with the usual notation,

$$\frac{1}{\chi(V \otimes z)} = \sum_{i=0}^{n-1} \beta_i = \sum_{i=0}^{n-1} \beta_i^{\mathcal{B}}.$$

Proof. By Proposition 3.5.13 we see that

$$\left\langle (1-z)^j, \sum_{i=0}^{n-1} \beta_i \right\rangle = 1$$

for $j = 0, \ldots, n-1$. The corollary follows. □

Remark 3.5.16. The basis $\mathcal{B}$ in Corollary 3.5.15 does *not* depend on V having a decomposition as a sum of one-dimensional $\mathbb{C}G$-modules. Thus, if $\dim_{\mathbb{C}}(V) = n$,

$$\mathcal{B} = \{1, 1-z, (1-z)^2, \ldots, (1-z)^{n-1}\}$$

is an $R(G)$-basis for $K_G^0(\mathbb{C}P(V))$ for G compact Lie, with dual basis $\{\beta_0^{\mathcal{B}}, \beta_1^{\mathcal{B}}, \ldots, \beta_{n-1}^{\mathcal{B}}\}$ for $K_0^G(\mathbb{C}P(V))$.

3.6 Perfect pairings

In this section we discuss the (algebraic) notion of a *perfect pairing*. We find, still under the restriction of $G = A$ being abelian, that $\frac{1}{\chi(V \otimes z)}$ gives us a perfect pairing in a sense made precise below. The description as $\sum_{i=0}^{n-1} \beta_i$ is convenient for computation, allowing us to be extraordinarily explicit. The aim is to understand the topological concept of *duality* via the perfect pairing we construct here.

Definition 3.6.1. Let M and N be modules over the commutative ring R. A bilinear map $b : M \otimes N \longrightarrow R$ is called a *perfect pairing* if

$$m \longmapsto \Big(f : N \longrightarrow R : n \longmapsto b(m \otimes n)\Big)$$

defines an isomorphism of R-modules $M \xrightarrow{\cong} \mathrm{Hom}_R(N, R)$.

We shall construct a perfect pairing $K^0_A(\mathbb{C}P(V)) \otimes K^0_A(\mathbb{C}P(V)) \longrightarrow R(A)$. In view of Preliminary Definition 3.4.1 we will then have an isomorphism of $R(A)$-modules

$$K^0_A(\mathbb{C}P(V)) \cong K^A_0(\mathbb{C}P(V)). \tag{3.6.2}$$

Definition 3.6.3. Given $\Lambda \in K^A_0(\mathbb{C}P(V))$ we write $\lceil -,- \rceil_\Lambda$ for the composite

$$\begin{array}{ccccc} K^0_A(\mathbb{C}P(V)) \otimes K^0_A(\mathbb{C}P(V)) & \longrightarrow & K^0_A(\mathbb{C}P(V)) & \longrightarrow & R(A) \\ x \otimes y & \longmapsto & xy & \longmapsto & \lceil x,y \rceil_\Lambda = \langle xy, \Lambda \rangle, \end{array}$$

and in the case $\Lambda = \frac{1}{\chi(V \otimes z)}$ we write just $\lceil -,- \rceil$ in place of $\lceil -,- \rceil_\Lambda$.

Proposition 3.6.4. *The pairing*

$$\lceil -,- \rceil : K^0_A(\mathbb{C}P(V)) \otimes K^0_A(\mathbb{C}P(V)) \longrightarrow K^0_A(\mathbb{C}P(V)) \longrightarrow R(A) \tag{3.6.5}$$

is perfect.

Proof. We consider the map

$$\Pi : K^0_A(\mathbb{C}P(V)) \longrightarrow \mathrm{Hom}_{R(A)}\left(K^0_A(\mathbb{C}P(V)), R(A)\right) = K^A_0(\mathbb{C}P(V))$$

given by $x \overset{\Pi}{\longmapsto} \left(y \longmapsto \langle xy, \frac{1}{\chi(V\otimes z)}\rangle\right)$.

Recall that our standard basis for $K^0_A(\mathbb{C}P(\alpha_1 \oplus \cdots \oplus \alpha_n))$ is

$$\left\{1, y^{\alpha_1}, y^{\alpha_1}y^{\alpha_2}, \ldots, y^{\alpha_1}\cdots y^{\alpha_{n-1}}\right\},$$

arising from our standard flag $\mathcal{F}$, i.e. (3.3.21).

We see from the flag

$$\mathcal{F}' = \left(0 \subset \alpha_n \subset \alpha_n \oplus \alpha_{n-1} \subset \cdots \subset \alpha_n \oplus \cdots \oplus \alpha_1 = V\right)$$

that another basis for $K^0_A(\mathbb{C}P(V))$ is

$$\left\{1, y^{\alpha_n}, y^{\alpha_n}y^{\alpha_{n-1}}, \ldots, y^{\alpha_n}\cdots y^{\alpha_2}\right\}.$$

Suppose $1 < k \le n$. We claim that, for $j \ge 0$,

$$y^{\alpha_n}\cdots y^{\alpha_k} \overset{\Pi}{\longmapsto} \left(y^{\alpha_1}\cdots y^{\alpha_j} \longmapsto \begin{cases} 0 & j \ge k \\ 1 & j < k \end{cases}\right).$$

Indeed, if $j \geq k$ the product $y^{\alpha_n} \cdots y^{\alpha_k} \cdot y^{\alpha_1} \cdots y^{\alpha_j}$ contains a factor $\chi(V \otimes z)$ and so is zero. And if $j < k$ we appeal to Proposition 3.5.13. And of course,

$$1 \overset{\Pi}{\longmapsto} \left(y^{\alpha_1} \cdots y^{\alpha_j} \longmapsto \begin{cases} 0 & j \geq n \\ 1 & j < n \end{cases} \right),$$

so that we find

$$\begin{aligned} \Pi(y^{\alpha_n} \cdots y^{\alpha_2}) &= \beta_0^{\mathcal{F}}; \\ \Pi(y^{\alpha_n} \cdots y^{\alpha_3}) &= \beta_0^{\mathcal{F}} + \beta_1^{\mathcal{F}}; \\ &\vdots \\ \Pi(y^{\alpha_n}) &= \beta_0^{\mathcal{F}} + \cdots + \beta_{n-2}^{\mathcal{F}}; \\ \Pi(1) &= \beta_0^{\mathcal{F}} + \cdots + \beta_{n-1}^{\mathcal{F}}. \end{aligned}$$

Hence Π, taking a basis of $K_A^0(\mathbb{C}P(V))$ to a basis of $K_0^A(\mathbb{C}P(V))$, is an isomorphism and $\lceil -, - \rceil$ is perfect. □

Recall that we hope to understand Poincaré duality. Proposition 3.6.4 gives us an isomorphism $K_A^0(\mathbb{C}P(V)) \overset{\cong}{\longrightarrow} K_0^A(\mathbb{C}P(V))$, and we shall see in Chapter 6 that this *is* what we require. For now we shall simply call $\lceil -, - \rceil$ a *duality pairing*.

3.6.1 C_2-equivariant examples

We have a recipe which tells us how to calculate $\lceil x, y \rceil$ for cohomology elements $x, y \in K_A^0(\mathbb{C}P(V))$: we simply tensor and add up the coefficients of the product in the basis (3.3.19). In principle we can do this for *any* finite dimensional representation V of *any* abelian group A – though the working may be tedious. Even in the $A = C_2$ case the working is tedious, but is sufficiently manageable that we can be explicit.

So we aim to compute the duality pairing $\lceil -, - \rceil$ for any representation V of $A = C_2$. Evidently it suffices to compute on basis elements. Consider the character table of C_2 – Table A.1 on page 169. Suppose that V is the representation with character $\varepsilon^{\oplus n} \oplus \alpha^{\oplus m}$. Our basis for $K_{C_2}^0(\mathbb{C}P(V)) = \frac{R(C_2)[z]}{(1-z)^n(1-\alpha z)^m}$ is $\{1, u, u^2, \ldots, u^n, u^n v, u^n v^2, \ldots, u^n v^{m-1}\}$ where $u = 1 - z$ and $v = 1 - \alpha z$. As usual, denote the dual basis for $K_0^{C_2}(\mathbb{C}P(V))$ by $\{\beta_0, \ldots, \beta_{n+m-1}\}$.

Proposition 3.6.6. *We have*

$$\begin{aligned} u^{n+r}v^s &= \binom{r}{0}\alpha^{-r}u^n v^{r+s} + \binom{r}{1}\alpha^{-r+1}(1-\alpha^{-1})u^n v^{r+s-1} + \cdots \\ &\cdots + \binom{r}{p}\alpha^{-r+p}(1-\alpha^{-1})^p u^n v^{r+s-p} + \cdots \\ &\cdots + \binom{r}{r-1}\alpha^{-1}(1-\alpha^{-1})^{r-1}u^n v^{1+s} + \binom{r}{r}(1-\alpha^{-1})^r u^n v^s. \end{aligned}$$

Proof. The proof is by induction on r. Initially, take $r = 1$. Then we have

$$\begin{aligned} u^{n+1}v^s &= u^n v^s(\alpha^{-1}(1-\alpha z) + 1 - \alpha^{-1}) \\ &= \alpha^{-1}u^n v^{s+1} + (1-\alpha^{-1})u^n v^s \end{aligned}$$

as required.

Now suppose inductively that the claim holds for $r = 1, \ldots, k$. Then we have

$$\begin{aligned} u^{n+k+1}v^s &= (1-z)\Bigg(\alpha^{-k}u^n v^{k+s} + \cdots + \binom{k}{p}\alpha^{-k+p}(1-\alpha^{-1})^p u^n v^{k-p+s} + \cdots \\ &\quad \cdots + (1-\alpha^{-1})^k u^n v^s\Bigg) \\ &= \left(\alpha^{-1}v + 1 - \alpha^{-1}\right)\Bigg(\alpha^{-k}u^n v^{k+s} + \cdots \\ &\quad \cdots + \binom{k}{p}\alpha^{-k+p}(1-\alpha^{-1})^p u^n v^{k-p+s} + \cdots + (1-\alpha^{-1})^k u^n v^s\Bigg). \end{aligned}$$

Picking out the coefficient of $u^n v^{k-p+s+1}$, we find it is

$$\alpha^{-k+p-1}(1-\alpha^{-1})^p\left(\binom{k}{p} + \binom{k}{p-1}\right) = \alpha^{-k+p-1}(1-\alpha^{-1})^p\binom{k+1}{p}$$

as required to complete the inductive step and proof. □

Proposition 3.6.7. *We have*

$$\alpha^{-r} + \binom{r}{1}\alpha^{-r+1}(1-\alpha^{-1}) + \cdots + \binom{r}{p}\alpha^{-r+p}(1-\alpha^{-1})^p + \cdots + (1-\alpha^{-1})^r = 1.$$

Proof. The left hand side is just the binomial expansion of $(\alpha^{-1} + (1-\alpha^{-1}))^r$. □

Proposition 3.6.8. *The duality pairing*

$$\lceil -,-\rceil : K^0_{C_2}(\mathbb{C}P(V)) \otimes K^0_{C_2}(\mathbb{C}P(V)) \longrightarrow K^0_{C_2}(\mathbb{C}P(V)) \longrightarrow R(C_2)$$

on elements of the basis $\{1, u, \ldots, u^n, u^n v, \ldots, u^n v^{m-1}\}$ *for* $K^0_{C_2}(\mathbb{C}P(\varepsilon^{\oplus n} \oplus \alpha^{\oplus m}))$ *(where* $u = 1 - z, v = 1 - \alpha z$*) is given as follows:*

$$\lceil u^i, u^j \rceil = \begin{cases} 1 & i + j < n + m \\ 1 - \alpha^{-(m+q)} - \binom{m+q}{1}\alpha^{-(m+q)+1}(1-\alpha^{-1}) - \cdots \\ \cdots - \binom{m+q}{p}\alpha^{-(m+q)+p}(1-\alpha^{-1})^p - \cdots & i + j = n + m + q \\ \cdots - \binom{m+q}{q}\alpha^{-m}(1-\alpha^{-1})^q \end{cases};$$

$$\lceil u^i, u^n v^j \rceil = \begin{cases} 1 & i + j < m \\ 0 & j \geq m \\ 1 - \alpha^{-i} - \binom{i}{1}\alpha^{-i+1}(1-\alpha^{-1}) - \cdots \\ \cdots - \binom{i}{p}\alpha^{-i+p}(1-\alpha^{-1})^p - \cdots & i + j = m + q \text{ and } j < m \\ \cdots - \binom{i}{q}\alpha^{-i+q}(1-\alpha^{-1})^q \end{cases};$$

and for $i, j > 0$,

$$\lceil u^n v^i, u^n v^j \rceil = \begin{cases} 1 & i + j < m - n \\ 0 & i + j \geq m \\ 1 - \alpha^{-n} - \binom{n}{1}\alpha^{-n+1}(1-\alpha^{-1}) - \cdots \\ \cdots - \binom{n}{p}\alpha^{-n+p}(1-\alpha^{-1})^p - \cdots & \begin{array}{l} m - n \leq i + j \\ = m - n + q < m \end{array} \\ \cdots - \binom{n}{q}\alpha^{-n+q}(1-\alpha^{-1})^q \end{cases}.$$

Proof. In view of Propositions 3.6.6 and 3.6.7, this is elementary. □

We apply these somewhat unwieldy formulae and obtain the following tables, which explicitly give the duality pairing for some n-dimensional representations of C_2.

	1	u	u^2	u^3	$\cdots$	u^{n-3}	u^{n-2}	u^{n-1}
1	1	1	1	1		1	1	1
u	1	1	1	1		1	1	0
u^2	1	1	1	1		1	0	0
u^3	1	1	1	1		0	0	0
$\vdots$					$\iddots$			
u^{n-3}	1	1	1	0		0	0	0
u^{n-2}	1	1	0	0		0	0	0
u^{n-1}	1	0	0	0		0	0	0

Table 3.1. Duality pairing for $V = \varepsilon^{\oplus n}$, with basis $\{1, u, \ldots, u^{n-1}\}$ for $K^0_{C_2}(\mathbb{C}P(V))$, where $u = 1 - z$.

	1	u	u^2	u^3	$\cdots$	u^{n-3}	u^{n-2}	u^{n-1}
1	1	1	1	1		1	1	1
u	1	1	1	1		1	1	e
u^2	1	1	1	1		1	e	$2e$
u^3	1	1	1	1		e	$2e$	$4e$
$\vdots$					$\iddots$			
u^{n-3}	1	1	1	e		$2^{n-6}e$	$2^{n-5}e$	$2^{n-4}e$
u^{n-2}	1	1	e	$2e$		$2^{n-5}e$	$2^{n-4}e$	$2^{n-3}e$
u^{n-1}	1	e	$2e$	$4e$		$2^{n-4}e$	$2^{n-3}e$	$2^{n-2}e$

Table 3.2. Duality pairing for $V = \varepsilon^{\oplus n-1} \oplus \alpha$, with basis $\{1, u, \ldots, u^{n-1}\}$ for $K^0_{C_2}(\mathbb{C}P(V))$, where $u = 1 - z, v = 1 - \alpha z$ and $e = 1 - \alpha^{-1}$.

	1	u	u^2	u^3	u^4	u^5	u^6	$\cdots$	u^{n-6}	u^{n-5}	u^{n-4}	u^{n-3}	u^{n-2}	$u^{n-2}v$
1	1	1	1	1	1	1	1		1	1	1	1	1	1
u	1	1	1	1	1	1	1		1	1	1	1	1	e
u^2	1	1	1	1	1	1	1		1	1	1	1	0	$2e$
u^3	1	1	1	1	1	1	1		1	1	1	0	$-2e$	$4e$
u^4	1	1	1	1	1	1	1		1	1	0	$-2e$	$-8e$	$8e$
u^5	1	1	1	1	1	1	1		1	0	$-2e$	$-8e$	$-24e$	$16e$
u^6	1	1	1	1	1	1	1		0	$-2e$	$-8e$	$-24e$	$-64e$	$32e$
$\vdots$								$\iddots$						
u^{n-6}	1	1	1	1	1	1	0		$f(12)e$	$f(11)e$	$f(10)e$	$f(9)e$	$f(8)e$	$2^{n-7}e$
u^{n-5}	1	1	1	1	1	0	$-2e$		$f(11)e$	$f(10)e$	$f(9)e$	$f(8)e$	$f(7)e$	$2^{n-6}e$
u^{n-4}	1	1	1	1	0	$-2e$	$-8e$		$f(10)e$	$f(9)e$	$f(8)e$	$f(7)e$	$f(6)e$	$2^{n-5}e$
u^{n-3}	1	1	1	0	$-2e$	$-8e$	$-24e$		$f(9)e$	$f(8)e$	$f(7)e$	$f(6)e$	$f(5)e$	$2^{n-4}e$
u^{n-2}	1	1	0	$-2e$	$-8e$	$-24e$	$-64e$		$f(8)e$	$f(7)e$	$f(6)e$	$f(5)e$	$f(4)e$	$2^{n-3}e$
$u^{n-2}v$	1	e	$2e$	$4e$	$8e$	$16e$	$32e$		$2^{n-7}e$	$2^{n-6}e$	$2^{n-5}e$	$2^{n-4}e$	$2^{n-3}e$	$2^{n-2}e$

Table 3.3. Duality pairing for $V = \varepsilon^{\oplus n-2} \oplus \alpha^{\oplus 2}$, with basis $\{1, u, \ldots, u^{n-2}, u^{n-2}v\}$ for $K^0_{C_2}(\mathbb{C}P(V))$, where $u = 1-z, v = 1-\alpha z, e = 1-\alpha^{-1}$ and $f(r) = -2^{n-r}(n-r)$.

	1	u	u^2	u^3	u^4	u^5	u^6	u^7	u^8	u^9	u^{10}	$\dots$	u^{n-10}	u^{n-9}	u^{n-8}	u^{n-7}	u^{n-6}	u^{n-5}	u^{n-4}	u^{n-3}	$u^{n-3}v$	$u^{n-3}v^2$
1	1	1	1	1	1	1	1	1	1	1	1		1	1	1	1	1	1	1	1	1	1
u	1	1	1	1	1	1	1	1	1	1	1		1	1	1	1	1	1	1	1	1	e
u^2	1	1	1	1	1	1	1	1	1	1	1		1	1	1	1	1	1	1	1	0	$2e$
u^3	1	1	1	1	1	1	1	1	1	1	1		1	1	1	1	1	1	1		$-2e$	$4e$
u^4	1	1	1	1	1	1	1	1	1	1	1		1	1	1	1	1	1			$-8e$	$8e$
u^5	1	1	1	1	1	1	1	1	1	1	1		1	1	1	1	1				$-24e$	$16e$
u^6	1	1	1	1	1	1	1	1	1	1	1		1	1	1	1					$-64e$	$32e$
u^7	1	1	1	1	1	1	1	1	1	1	1		1	1	1						$-160e$	$64e$
u^8	1	1	1	1	1	1	1	1	1	1	1		1	1							$-384e$	$128e$
u^9	1	1	1	1	1	1	1	1	1	1	1		1								$-896e$	$264e$
u^{10}	1	1	1	1	1	1	1	1	1	1	1										$-2048e$	$512e$
$\vdots$																						
u^{n-10}	1	1	1	1	1	1	1	1	1	1											$f(12)e$	$2^{n-11}e$
u^{n-9}	1	1	1	1	1	1	1	1	1												$f(11)e$	$2^{n-10}e$
u^{n-8}	1	1	1	1	1	1	1	1													$f(10)e$	$2^{n-9}e$
u^{n-7}	1	1	1	1	1	1	1														$f(9)e$	$2^{n-8}e$
u^{n-6}	1	1	1	1	1	1															$f(8)e$	$2^{n-7}e$
u^{n-5}	1	1	1	1	1																$f(7)e$	$2^{n-6}e$
u^{n-4}	1	1	1	1																	$f(6)e$	$2^{n-5}e$
u^{n-3}	1	1	1																		$f(5)e$	$2^{n-4}e$
$u^{n-3}v$	1	1	0	$-2e$	$-8e$	$-24e$	$-64e$	$-160e$	$-384e$	$-896e$	$-2048e$		$f(12)e$	$f(11)e$	$f(10)e$	$f(9)e$	$f(8)e$	$f(7)e$	$f(6)e$	$f(5)e$	$2^{n-4}e$	0
$u^{n-3}v^2$	1	e	$2e$	$4e$	$8e$	$16e$	$32e$	$64e$	$128e$	$256e$	$512e$		$2^{n-11}e$	$2^{n-10}e$	$2^{n-9}e$	$2^{n-8}e$	$2^{n-7}e$	$2^{n-6}e$	$2^{n-5}e$	$2^{n-4}e$	0	0

Diagonals: e, $4e$, $16e$, $56e$, …

$\lceil u^i, u^{n-i+q}\rceil = 2^q\left(\binom{q+3}{q+1} - 2(q+1)\right)e$

Table 3.4. Duality pairing for $V = \varepsilon^{\oplus n-3} \oplus \alpha^{\oplus 3}$, with basis $\{1, u, \dots, u^{n-3}, u^{n-3}v, u^{n-3}v^2\}$ for $K^0_{C_2}(\mathbb{C}P(V))$, where $u = 1 - z, v = 1 - \alpha z, e = 1 - \alpha^{-1}$ and $f(r) = -2^{n-r}(n-r)$.

Remark 3.6.9. Table 3.1 is essentially non-equivariant, and the lower right half is zero. In Tables 3.2 – 3.4 there are equivariant phenomena present but in each case if we take n to be large then we might be tempted to slur "V is only *slightly* equivariant". This wild statement is realised by the factor of 2^n appearing in the lower right halves of the tables so that in the 2-adic topology the lower right entries really are coming close to zero for large n.

Chapter 4

The non-abelian world

It is time now to remove the safety net and release our restriction on G being an abelian group. We take G to be a finite group and consider whether the ideas of Chapter 3 are, in the first instance, meaningful, and if so, do they remain true?

The crucial dependence of Chapter 3 upon the group of equivariance being abelian was when we took *bases* (3.3.19) and (3.4.13) coming from the complete *flag*

$$\mathcal{F} = \Big(0 \subset \alpha_1 \subset \alpha_1 \oplus \alpha_2 \subset \cdots \subset \alpha_1 \oplus \cdots \oplus \alpha_n\Big).$$

In the more general case there is no guarantee that such a flag exists. If we can write $V = \alpha_1 \oplus \cdots \oplus \alpha_n$ as a sum of one dimensional representations then we *do* have such a flag and we can proceed just as before. But in the interesting case, when V contains simple representations of dimension bigger than one, we shall need a new approach. In any event, we still have Proposition 3.3.12,

$$K_G^0(\mathbb{C}P(V)) = \frac{R(G)[z]}{(\chi(V \otimes z))},$$

and Preliminary Definition 3.4.1, Corollary 3.4.10 together still tell us

$$K_0^G(\mathbb{C}P(V)) \cong \frac{\frac{1}{\chi(V\otimes z)}R(G)[z]}{R(G)[z]} \cong \mathrm{Hom}_{R(G)}\left(K_G^0(\mathbb{C}P(V)), R(G)\right).$$

We considered in §3.5 the basis $\mathcal{B} = \{(1-z)^i\}_{0\leq i\leq n-1}$ for $K_A^0(\mathbb{C}P(V))$. This does *not* rely on V being a sum of one dimensional representations and so $\mathcal{B}$ is always a basis for $K_G^0(\mathbb{C}P(V))$. Recall that Corollary 3.5.15 told us that, in the abelian case, the dual basis had the property

$$\sum_{i=0}^{n-1} \beta_i^{\mathcal{B}} = \frac{1}{\chi(V \otimes z)}.$$

It is this which inspires our approach in the non-abelian world.

We shall need to do some *equivariant stable homotopy theory* to get at our results. In §4.1 – §4.2 we discuss relevant aspects of equivariant stable homotopy theory. In §4.3 we relate this to *restriction* in equivariant K-theory before returning to considerations about $\mathbb{C}P(V)$ in §4.4.

4.1 Introduction to spectra and the equivariant stable homotopy category

Non-equivariantly, Adams [4 (III)] provides an excellent introduction to $\mathbb{Z}$-graded, or *naive* spectra. We shall need to be more general and work in what is known as the *equivariant stable homotopy category.* An authoritative reference is [41 (I)]: more readable accounts [46 (XII), 3, 29] exist. Here we content ourselves with a brief whistle-stop tour on the journey to the equivariant stable homotopy category.

We begin non-equivariantly.

Definition 4.1.1. **(i)** A *spectrum* E is a sequence of based spaces E_i (either $i \in \mathbb{N}$ or $i \in \mathbb{Z}$) and structure maps $\sigma_i : \Sigma E_i \longrightarrow E_{i+1}$.

(ii) A *map* $f : E \longrightarrow F$ *of spectra* is a sequence of pointed maps $f_i : E_i \longrightarrow F_i$ which commute with the structure maps.

Warning 4.1.2. The reader will notice that our definition of a *spectrum* is what May often calls a *pre*spectrum.

The next step is to consider a more general indexing, which permits an equivariant definition.

Definition 4.1.3. **(i)** A *G-universe* $\mathcal{U}$ is a countable direct sum of finite dimensional real representations of G such that

(a) the trivial representation occurs;

(b) if a representation occurs, its subrepresentations occur infinitely often.

We use the adjectives *trivial* to mean that *only* trivial representations occur and *complete* to mean that *all* simple representations occur. If V is a representation which occurs in $\mathcal{U}$ we say that V is an *indexing representation.*

(ii) A *G-spectrum E indexed on a G-universe $\mathcal{U}$* consists of a based G-space E_V for each indexing representation $V \subset \mathcal{U}$ and a transitive system of based G-maps

$$\sigma_{V,W} : \Sigma^{W-V} E_V \longrightarrow E_W$$

whenever $V \subseteq W$. Here, $W - V$ is the orthogonal complement of V in W and $\Sigma^{W-V} E_V := E_V \wedge S^{W-V}$.

(iii) A *map $f : E \longrightarrow F$ of G-spectra* indexed over the same G-universe $\mathcal{U}$ is a collection of based G-maps $f_V : E_V \longrightarrow F_V$ commuting with the structure maps of (ii).

We now have what is known as the *category of G-spectra indexed on $\mathcal{U}$*. It makes sense to talk about indexing over a complete G-universe and maps of G-spectra indexed over a complete G-universe without mentioning the universe, since all complete G-universes are isomorphic. We may thus talk about the category of G-spectra. Whenever we talk about a G-spectrum without stating its underlying G-universe $\mathcal{U}$ we shall assume that $\mathcal{U}$ is complete.

Example 4.1.4. Given a pointed space X we have the *suspension spectrum* of X whose V^{th} term is $X \wedge S^V$. Equivariantly, if X was a G-space we obtain the G-suspension spectrum of X. If we wish to emphasise strongly that we are thinking of a spectrum we write $\Sigma^\infty X$ but we shall commonly employ "X" as notation for both the space and the spectrum. We shall commonly write $\mathbb{S}$ for the *sphere spectrum*, $\mathbb{S} = \Sigma^\infty S^0$.

We now begin the passage to the *stable* category. The first stop is the *homotopy* category. In defining that we need the notion of the *smash product* of a G-spectrum and a G-space. It seems convenient to introduce here also the *function spectrum* of a G-spectrum and a G-space. Given based *G-spaces* X and Y, $X \wedge Y$ is the usual space-level smash product, and $F(X, Y)$ is the function *space* of based G-maps $X \longrightarrow Y$.

Definition 4.1.5. Let E be a G-spectrum indexed on the G-universe $\mathcal{U}$ with structure maps $\rho_{V,W}$, and let X be a based G-space.

(i) We define the *smash product* $E \wedge X$ to be the G-spectrum with

$$(E \wedge X)_V = E_V \wedge X$$

for $V \subset \mathcal{U}$, and with structure maps

$$\sigma_{V,W} : \Sigma^{W-V}(E_V \wedge X) \cong (\Sigma^{W-V} E_V) \wedge X \overset{\rho_{V,W} \wedge 1}{\longrightarrow} E_W \wedge X$$

for $V \subset W$.

(ii) We define the *function spectrum* $F(X, E)$ to be the G-spectrum with

$$(F(X, E))_V = F(X, E_V)$$

for $V \subset \mathcal{U}$, and with structure maps

$$\sigma_{V,W} : \Sigma^{W-V} F(X, E_V) \overset{(\rho_{V,W})_*}{\longrightarrow} F(X, E_W)$$

for $V \subset W$.

Definition 4.1.6. **(i)** Given maps $f, g : E \longrightarrow F$ of G-spectra E and F, a *homotopy* between f and g is a map $E \wedge I_+ \longrightarrow F$ which begins at f and ends at g. Here, I is the unit interval. We write $[E, F]_G$ for the set of homotopy classes of maps $E \longrightarrow F$ and define the *homotopy groups* of a G-spectrum E to be

$$\pi_n^H(E) = [G/H_+ \wedge S^n, E]_G$$

for $H \leq G$.

(ii) The *homotopy category of G-spectra* has objects homotopy classes of G-spectra and morphisms homotopy classes of maps of G-spectra.

Given the homotopy category of G-spectra we are in position now to approach our destination, the *equivariant stable homotopy category.*

Definition 4.1.7. **(i)** We say that a map $f : E \longrightarrow F$ of G-spectra E, F is a *weak equivalence* if $f_* : \pi_*^H(E) \longrightarrow \pi_*^H(F)$ is an isomorphism for all $H \leq G$.

(ii) The *equivariant stable homotopy category* or the *equivariant stable category* or just the *stable category* is constructed from the homotopy category of G-spectra by formally inverting all weak equivalences.

In view of Definition 4.1.5 (i) we can make sense of generalised suspensions by setting

$$\Sigma^V E = E \wedge S^V$$

for a $\mathbb{C}G$-module V and G-spectrum E. A delightful property of the stable homotopy category is that we can also *desuspend.*

Lemma 4.1.8 (Desuspension). *Given G-spectra E, F, in the equivariant stable homotopy category we have*

$$E \simeq F \iff \Sigma^V E \simeq \Sigma^V F$$

for all $\mathbb{C}G$-modules V. Moreover, for each $\mathbb{C}G$-module V, there is a G-spectrum E' such that

$$E \simeq \Sigma^V E'.$$

Proof. This is [41 (I, Theorem 6.1)]. □

Remark 4.1.9. Our choice of the verb *approach* was deliberate – we have journeyed to the equivariant stable category but we have *not* seen all the sights. If one takes sufficient care, one may construct a *smash product* $\wedge$ in the stable category. It turns out[1] to be associative, commutative and has the G-sphere spectrum $\mathbb{S}$ as a unit. In categorical language, $\wedge$ makes the equivariant stable category into a *symmetric monoidal* category with unit $\mathbb{S}$. Similarly we can extend the notion of *function spectrum* $F(-,-)$ to the stable category so that the stable category is a *closed* category. We shall frequently make use of $\wedge$ and $F(-,-)$: we refer the reader to [41 (II, §3)] for details of the constructions.

4.2 The relevance of spectra and the equivariant stable homotopy category

We find an immediate application of equivariant stable homotopy theory. Recall that T is the circle group with natural representation z.

[1] Modulo canonical isomorphism.

Lemma 4.2.1. *Suppose that V is a $\mathbb{C}G$-module which decomposes as $V \cong_{\mathbb{C}G} W \oplus \alpha$ with α of dimension one. Then, in the stable homotopy category,*

$$S(V \otimes z)/S(W \otimes z) \simeq S(\alpha \otimes z)_+ \wedge S^{W \otimes z}.$$

Proof. We claim that we have a commuting diagram of cofibre sequences as below.

$$\begin{array}{ccccccc}
S(W \otimes z)_+ & \longrightarrow & S^0 & \longrightarrow & S^{W \otimes z} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
S((W \oplus \alpha) \otimes z)_+ & \longrightarrow & S^0 & \longrightarrow & S^{(W \oplus \alpha) \otimes z} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
A & \longrightarrow & \mathrm{pt} & \longrightarrow & B & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

The first two horizontal rows are the cofibre sequence (3.3.3) and the indicated maps are inclusions. Using mapping cylinders we can take *all* of the maps in the top left square to be inclusions, from which commutativity is obvious. We now need to identify A. We have

$$\Sigma A \simeq B/\mathrm{pt} = B$$

so we consider B.

Now (3.3.3) gives the cofibre sequence $S(\alpha \otimes z)_+ \longrightarrow S^0 \longrightarrow S^{\alpha \otimes z} \longrightarrow \cdots$, so that, smashing with $S^{W \otimes z}$, we have a cofibre sequence

$$S(\alpha \otimes z)_+ \wedge S^{W \otimes z} \longrightarrow S^{W \otimes z} \longrightarrow S^{(W \oplus \alpha) \otimes z} \longrightarrow \Sigma(S(\alpha \otimes z)_+ \wedge S^{W \otimes z}) \longrightarrow \cdots .$$

From this we take $B = \Sigma(S(\alpha \otimes z)_+ \wedge S^{W \otimes z})$ and use Lemma 4.1.8 to deduce the stated equivalence. $\square$

As discussed in [46 (XIII, §1-§3)], $RO(G)$-graded equivariant cohomology theories are represented by G-spectra. Given a G-spectrum $E = E_G$ we write E_G^* and E_*^G for the equivariant cohomology and homology theory it represents. In particular we have a G-spectrum $\mathbb{K}$ which represents equivariant K-theory.[2] We make this statement precise in Lemma 4.2.4 but we suppress a proof, referring the interested reader to [46 (XIV, §4)].

[2]We continue to write K_G^*, etc, for K-theory, rather than $\mathbb{K}_G^*$, etc.

In our discussion we often suppress the prefix "G-" so that, unless stated otherwise, the reader may assume that, in the current section, we are always working G-equivariantly.

Definition 4.2.2. Let E be a G-spectrum and X a G-CW-complex. We define the zeroth reduced E-cohomology and E-homology of X by

$$\widetilde{E}^0_G(X_+) = [\Sigma^\infty X_+, E]_G \text{ and } \widetilde{E}^G_0(X_+) = [\mathbb{S}, E \wedge X_+]_G.$$

By taking $E = \mathbb{K}$ (see Lemma 4.2.4 below) we have a definition of equivariant K-cohomology and K-homology. This is the *represented* theory we alluded to earlier. We have given our definition in degree zero only but the Eilenberg-Steenrod axioms tell us everything else. Our immediate priority is now to reconcile this definition with the ideas of Chapter 3. To do that we introduce the notion of a *ring spectrum.*

Definition 4.2.3. A *ring spectrum* E is one that comes with a multiplicative structure $\mu : E \wedge E \longrightarrow E$ and a unit $\eta : \mathbb{S} \longrightarrow E$ such that the diagrams

$$\begin{array}{ccccc} \mathbb{S} \wedge E & \xrightarrow{\eta \wedge 1} & E \wedge E & \xleftarrow{1 \wedge \eta} & E \wedge \mathbb{S} \\ & \searrow^{\simeq} & \downarrow^{\mu} & \swarrow_{\simeq} & \\ & & E & & \end{array} \qquad \begin{array}{ccc} E \wedge E \wedge E & \xrightarrow{1 \wedge \mu} & E \wedge E \\ {\scriptstyle \mu \wedge 1}\downarrow & & \downarrow{\scriptstyle \mu} \\ E \wedge E & \xrightarrow{\mu} & E \end{array}$$

commute in the stable homotopy category.

Lemma 4.2.4. *There is a ring G-spectrum $\mathbb{K}$ which represents G-equivariant K-theory on finite G-CW-complexes X in the sense that taking $E = \mathbb{K}$ in Definition 4.2.2 agrees with Chapter 3.*

Proof. For the construction of $\mathbb{K}$ and its agreement with the G-bundle approach to equivariant K-cohomology, see [46 (XIV, §4)]. We still need to consider our Preliminary Definition 3.4.1, but that will follow from Proposition 4.2.9 below. □

It is now appropriate to prove some earlier results. First, we prove Lemma 3.4.4. For that we shall need the *Adams isomorphism.* This is a deep result: we present it here only at the level of generality which we shall need for the application, and we omit a proof. A suitable reference is [46 (XVI, §5)]. Recall that T is the circle group.

Theorem 4.2.5 (Adams isomorphism). *Let $\mathcal{U}$ be a complete $(G\times T)$-universe. Let A be a T-fixed $(G\times T)$-spectrum indexed on $\mathcal{U}^T$ and let B be a T-free $(G\times T)$-spectrum also indexed on $\mathcal{U}^T$. Then*

$$[A,\Sigma B/T]_G \cong [i_*A, i_*B]_{G\times T},$$

where the left hand side is understood in the universe $\mathcal{U}^T$, the right hand side in $\mathcal{U}$ and i is the inclusion $\mathcal{U}^T \hookrightarrow \mathcal{U}$. □

Proof of Lemma 3.4.4. In the Adams isomorphism, take $A=\mathbb{S}$ and $B=\mathbb{K}\wedge S(V\otimes z)_+$. This gives the isomorphism in

$$\begin{aligned}
\widetilde{K}_0^G(S(V\otimes z)_+/T) &= [\mathbb{S},\mathbb{K}\wedge S(V\otimes z)_+/T]_G\\
&\cong [\mathbb{S},\Sigma^{-1}\mathbb{K}\wedge S(V\otimes z)_+]_{G\times T}\\
&= \widetilde{K}_0^{G\times T}(\Sigma^{-1}S(V\otimes z)_+)\\
&= \widetilde{K}_1^{G\times T}(S(V\otimes z)_+),
\end{aligned}$$

and the lemma now follows from equivariant Bott periodicity. □

In order to continue with our proofs of earlier results, and to make progress elsewhere, we need to consider *duality*.

4.2.1 Duality in the equivariant stable homotopy category

Non-equivariantly, Spanier and Whitehead [62] studied duality on the level of spaces (precisely, finite subpolyhedra X of spheres). Their method was to embed such an X in some S^n and define an *n-dual of X* to be an S-deformation retract of $S^n\setminus X$. Spanier [60] went on to generalise this duality by defining the *functional dual* of a polyhedron X to be the function spectrum $F(X,\mathbb{S})$. Wirthmüller [66] generalised the embedding method for the equivariant case. We follow the functional dual approach of [22, 41].

Definition 4.2.6. We define a contravariant functor D on the equivariant stable homotopy category as follows. For a G-spectrum X we define $D(X)$ to be the function spectrum

$D(X) = F(X, \mathbb{S})$ and given a map $f : X \longrightarrow Y$ we define $D(f)$ to be such that the diagram

$$\begin{array}{ccc} D(Y) \wedge X & \xrightarrow{1\wedge f} & D(Y) \wedge Y \\ {\scriptstyle D(f)\wedge 1}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ D(X) \wedge X & \xrightarrow{\quad\varepsilon\quad} & \mathbb{S} \end{array}$$

commutes. (Here, ε is the obvious evaluation map.)

Dold and Puppe [22] give a categorical account and many details are found in [41, 46 (XVI, §7)]. Of these, we record the properties which shall be of most use to us. Just as spaces with the structure of a CW-complex have good properties (cf *Classical Spanier-Whitehead duality*, Theorem 1.1.2), there is a notion [41 (I, §5)] of a *G-CW spectrum.* We shall only be interested in *finite* G-CW-spectra, and [41 (I, Corollary 8.16)] justifies us in taking a finite G-CW-spectrum to be the suspension spectrum of a finite G-CW-complex.

Lemma 4.2.7. *Let X, Y be finite G-CW-spectra and let E be a G-spectrum. Then the functor D enjoys the following properties.*

- **(i)** *$D(X)$ is a finite G-CW-spectrum (up to equivalence);*
- **(ii)** *$D(D(X)) \simeq X$ in the stable homotopy category;*
- **(iii)** *$D(X \wedge D(X)) \simeq D(X) \wedge X$ in the stable homotopy category;*
- **(iv)** *there is an isomorphism $E^*_G(X) \xrightarrow{SW} E^G_*(D(X))$;*
- **(v)** *a map $f : X \longrightarrow Y$ gives rise to a commutative diagram*

$$\begin{array}{ccc} E^*_G(Y) & \xrightarrow{\quad f^*\quad} & E^*_G(X) \\ {\scriptstyle SW}\downarrow & & \downarrow{\scriptstyle SW} \\ E^G_*(D(Y)) & \xrightarrow{(D(f))_*} & E^G_*(D(X)). \end{array}$$

Proof. Parts (i) – (iv) are, respectively, Corollary 2.6, Proposition 2.8 (i), (iii) and Corollary 2.9 of [41 (III)]. Since (iv) will be of particular use for us, we record an isomorphism

$$E^*_G(X) \xrightarrow{SW} E^G_*(D(X)).$$

Consider the obvious evaluation map $\varepsilon : D(X) \wedge X \longrightarrow \mathbb{S}$. Taking duals and using (iii) gives the coevaluation map $\eta : \mathbb{S} \longrightarrow X \wedge D(X)$. Precisely, η is the composite

$$\eta : \mathbb{S} \simeq D(\mathbb{S}) \xrightarrow{D(\varepsilon)} D(D(X) \wedge X) \simeq X \wedge D(X).$$

We interpret the categorical nonsense of [41 (III, Proposition 1.2)] to see that the required isomorphism is

$$(X \xrightarrow{c} E) \xmapsto{SW} (\mathbb{S} \xrightarrow{\eta} X \wedge D(X) \xrightarrow{c\wedge 1} E \wedge D(X)). \tag{4.2.8}$$

For (v), take $(Y \xrightarrow{c} E) \in E^*(Y)$. Chasing both ways around the diagram we see that we are required to show that the two composites

$$\mathbb{S} \xrightarrow{\eta} X \wedge D(X) \xrightarrow{f\wedge 1} Y \wedge D(X) \xrightarrow{c\wedge 1} E \wedge D(X);$$
$$\mathbb{S} \xrightarrow{\eta} Y \wedge D(Y) \xrightarrow{c\wedge 1} E \wedge D(Y) \xrightarrow{1\wedge D(f)} E \wedge D(X)$$

are equal. Appeal to [41 (III, Proposition 1.5)] to see that there is a commutative diagram

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\eta} & Y \wedge D(Y) \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle 1\wedge D(f)} \\ X \wedge D(X) & \xrightarrow{f\wedge 1} & Y \wedge D(X). \end{array}$$

This permits us to rewrite the first composite as

$$\mathbb{S} \xrightarrow{\eta} Y \wedge D(Y) \xrightarrow{1\wedge D(f)} Y \wedge D(X) \xrightarrow{c\wedge 1} E \wedge D(X),$$

which is obviously the same as the second. □

Proposition 4.2.9. *Let E be a strictly commutative ring G-spectrum and X a G-space. Then*

(i) *if $E_*^G(X)$ is free over E_*^G, there is an isomorphism*

$$E_G^*(X) \xrightarrow{\cong} \mathrm{Hom}_{E_*^G}(E_*^G(X), E_*^G)$$

given by

$$(X \xrightarrow{c} E) \longmapsto \Big((\mathbb{S} \xrightarrow{h} E \wedge X) \longmapsto (\mathbb{S} \xrightarrow{h} E \wedge X \xrightarrow{1\wedge c} E \wedge E \xrightarrow{\mu} E)\Big); \tag{4.2.10}$$

(ii) *if $E_G^*(X)$ is finitely generated and free over E_G^*, there is an isomorphism*

$$E_*^G(X) \xrightarrow{\cong} \mathrm{Hom}_{E_G^*}(E_G^*(X), E_G^*)$$

given by

$$(\mathbb{S} \xrightarrow{h} E \wedge X) \longmapsto \Big((X \xrightarrow{c} E) \longmapsto (\mathbb{S} \xrightarrow{h} E \wedge X \xrightarrow{1\wedge c} E \wedge E \xrightarrow{\mu} E)\Big). \quad (4.2.11)$$

Proof. (i) We shall need the fact [25 (IV, Theorem 4.5)] that given such E and X, there is a right half-plane spectral sequence with

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{E_*^G}(E_*^G(X), E_*^G) \Longrightarrow E_G^*(X)$$

whose edge homomorphism is (4.2.10). Since $E_*^G(X)$ is, by assumption, free over E_*^G, we have $\mathrm{Ext}^{p,q}_{E_*^G}(E_*^G(X), E_*^G) = 0$ for $p > 0$. It follows that the spectral sequence collapses at the E_2-page, (4.2.10) is an isomorphism and

$$E_G^*(X) \cong \mathrm{Ext}^0_{E_*^G}(E_*^G(X), E_*^G) = \mathrm{Hom}_{E_*^G}(E_*^G(X), E_*^G)$$

as required.

(ii) First, let us show that such an isomorphism exists: following that we shall show it has the form (4.2.11). Now X is finite so $D(X)$ is finite by Lemma 4.2.7 (i). So by (i) we have

$$E_G^*(D(X)) \cong \mathrm{Hom}_{E_*^G}(E_*^G(D(X)), E_*^G).$$

Now use Lemma 4.2.7 (iv) to see that

$$E_*^G(D(D(X))) \cong \mathrm{Hom}_{E_G^*}(E_G^*(X), E_G^*),$$

and Lemma 4.2.7 (ii) shows that the required isomorphism exists. Completing the proof now amounts to showing that the following diagram commutes.

$$\begin{array}{ccc} E_G^*(X) & \xrightarrow{(4.2.10)} & \mathrm{Hom}_{E_*^G}(E_*^G(X), E_*^G) \\ \Big\downarrow{\scriptstyle SW} & & \Big\downarrow{\scriptstyle SW_*} \\ E_*^G(D(X)) & \xrightarrow{(4.2.11)} & \mathrm{Hom}_{E_G^*}(E_G^*(D(X)), E_G^*)) \end{array}$$

Recall that $E^*_G(X) \xrightarrow{SW} E^G_*(D(X))$ is (4.2.8). Take $(X \xrightarrow{x} E) \in E^*_G(X)$. Chasing horizontally, then vertically gives

$$(D(X) \xrightarrow{y} E) \longmapsto \left(\mathbb{S} \xrightarrow{\eta} D(X) \wedge D^2(X) \simeq D(X) \wedge X \xrightarrow{y\wedge 1} E \wedge X \xrightarrow{1\wedge x} E \wedge E \xrightarrow{\mu} E\right)$$

whilst chasing vertically, then horizontally gives

$$(D(X) \xrightarrow{y} E) \longmapsto \left(\mathbb{S} \xrightarrow{\eta} X \wedge D(X) \xrightarrow{x\wedge 1} E \wedge D(X) \xrightarrow{1\wedge y} E \wedge E \xrightarrow{\mu} E\right).$$

These are equal in the stable homotopy category because E is a *commutative* ring G-spectrum.

□

Remark 4.2.12. Our Preliminary Definition 3.4.1 is now justified – just take $E = \mathbb{K}$ in Proposition 4.2.9. (Of course one needs to know [35] that $\mathbb{K}$ *is* a strictly commutative ring G-spectrum.)

There is still (3.5.3) to take care of. We now turn our attention to this.

Definition 4.2.13. Let E be a ring G-spectrum and X a G-space. We then define a pairing

$$\wr -, - \int : E^*_G(X) \otimes E^G_*(X) \longrightarrow E^G_*$$

by

$$\wr X \xrightarrow{c} E, \mathbb{S} \xrightarrow{h} E \wedge X \int = \left(\mathbb{S} \xrightarrow{h} E \wedge X \xrightarrow{1\wedge c} E \wedge E \xrightarrow{\mu} E\right).$$

Proposition 4.2.14. *Suppose that we have a G-map $f : X \longrightarrow Y$, either on the space or the spectrum level. Then for $h_X \in E^G_*(X)$ and $c_Y \in E^*_G(Y)$ we have*

$$\wr f^*(c_Y), h_X \int = \wr c_Y, f_*(h_X) \int.$$

Proof.

$$\begin{aligned}
\wr f^*(c_Y), h_X \int &= \wr X \xrightarrow{f} Y \xrightarrow{c_Y} E, \mathbb{S} \xrightarrow{h_X} E \wedge X \int \\
&= (\mathbb{S} \xrightarrow{h_X} E \wedge X \xrightarrow{1\wedge f} E \wedge Y \xrightarrow{1\wedge c_Y} E \wedge E \xrightarrow{\mu} E) \\
&= \wr Y \xrightarrow{c_Y} E, \mathbb{S} \xrightarrow{h_X} E \wedge X \xrightarrow{1\wedge f} E \wedge Y \int \\
&= \wr c_Y, f_*(h_X) \int.
\end{aligned}$$

□

Remark 4.2.15. The reader will recall that we discussed in §3.4.1 the Kronecker pairing $\langle -,-\rangle$ for equivariant K-theory. More generally we have

$$\langle -,-\rangle : E_G^*(X) \otimes \mathrm{Hom}_{E_G^*}(E_G^*(X), E_G^*) \longrightarrow E_G^*$$

given by $\langle c, f\rangle = f(c)$. Now, if E and X satisfy the conditions in Proposition 4.2.9 (ii), we have a commutative diagram

$$\begin{array}{ccc} E_G^*(X) \otimes E_*^G(X) & \xrightarrow{\langle -,-\rangle} & E_*^G \\ \cong \updownarrow & & \updownarrow \cong \\ E_G^*(X) \otimes \mathrm{Hom}_{E_G^*}(E_G^*(X), E_G^*) & \xrightarrow{\langle -,-\rangle} & E_G^*. \end{array}$$

This, and Proposition 4.2.14, justify (3.5.3).

4.2.2 Change of group results

We make use of duality in the equivariant stable homotopy category to pursue results about changing group of equivariance. This will be invaluable later. For $H \leq G$ there is a unique *space*-level map $G/H \longrightarrow G/G$ which gives rise to a based map

$$\pi : G/H_+ \longrightarrow G/G_+. \tag{4.2.16}$$

In the stable category we have the dual, $D(\pi) : D(G/G_+) \longrightarrow D(G/H_+)$. In fact, the *Wirthmüller isomorphism* [66 (Proposition 3.1)] implies that, for finite G, $D(G/H_+) \simeq G/H_+$. Thus we have a stable map

$$D(\pi) : G/G_+ \longrightarrow G/H_+. \tag{4.2.17}$$

This is the moment that our investment in equivariant stable homotopy theory begins to pay off: *at the level of spaces there is no such map*, yet $D(\pi)$ is fundamental in what follows.

The following lemma is well known yet invaluable because it provides a comparison between G- and H-equivariant data.

Lemma 4.2.18. *Suppose that A and B are G-spaces and that $H \leq G$. Then*

(i) $G_+ \wedge_H A \cong_G G/H_+ \wedge A$;

(ii) $\{H\text{-maps } A \longrightarrow B\} \cong \{G\text{-maps } G_+ \wedge_H A \longrightarrow B\} \cong \{G\text{-maps } G/H_+ \wedge A \longrightarrow B\}$.

Remark 4.2.19. Recall that $G_+ \wedge_H A$ is the quotient of $G_+ \wedge A$ by the equivalence relation $g \wedge (ha) = (gh) \wedge a$ for $h \in H$. The G-action on $G_+ \wedge_H A$ is via $g' \cdot (g \wedge_H a) = (g'g) \wedge_H a$. Thus $G_+ \wedge_H A$ is a well-defined G-space, called an *induced space*, even if A was only an H-space.

Proof of Lemma 4.2.18. (i) Define $u : G/H_+ \wedge A \longrightarrow G_+ \wedge_H A$ by

$$gH \wedge a \stackrel{u}{\longmapsto} g \wedge_H (g^{-1}a).$$

One checks this is well defined, G-equivariant and has G-inverse

$$gH \wedge ga \stackrel{u^{-1}}{\longleftarrow\!\shortmid} g \wedge_H a.$$

(ii) Given a map $\longrightarrow$ we may wish to emphasise its equivariance. If so we shall write $\underset{G}{\longrightarrow}$. We define a map $\{A \underset{H}{\longrightarrow} B\} \longrightarrow \{G_+ \wedge_H A \underset{G}{\longrightarrow} B\}$ by

$$\Big(A \underset{H}{\stackrel{f}{\longrightarrow}} B\Big) \longmapsto \Big(g \wedge_H a \longmapsto gf(a)\Big).$$

One checks that this is a well defined bijection with inverse

$$\Big(a \longmapsto f(e \wedge_H a)\Big) \longleftarrow\!\shortmid \Big(G_+ \wedge_H A \underset{G}{\stackrel{f}{\longrightarrow}} B\Big).$$

That takes care of the first isomorphism in the lemma. For the second, just precompose with the u or u^{-1} of (i).

□

Lemma 4.2.18, which takes place at the space-level, generalises in the sense of Lemma 4.2.20. In that lemma, we have a similar result which has no space-level counterpart but which shall be invaluable to us. These are standard results [3 (§5)], but we record the details since we shall need them later.

Lemma 4.2.20. *Let $H \leq G$ and suppose that A is an H-spectrum and B is a G-spectrum. Then there are bijections*

(i) $\theta : [A, B]_H \stackrel{\cong}{\longrightarrow} [G_+ \wedge_H A, B]_G$;

(ii) $\phi : [B, A]_H \xrightarrow{\cong} [B, G_+ \wedge_H A]_G$.

Proof. For (i) observe that Lemma 4.2.18 is the space-level analogue and, since everything in Lemma 4.2.18 commutes with suspension, we deduce the spectrum-level result.

For (ii) we do not have the luxury of a space-level counterpart. But there *is* [3 (Theorem 5.2)] a counterpart if we work in the equivariant S-category (in the sense of [66]). From this we see that, stably, the appropriate constructions are

$$\left(B \xrightarrow[H]{f} A\right) \longmapsto \left(B \xrightarrow[G]{D(\pi)} G_+ \wedge_H B \xrightarrow[G]{1\wedge_H f} G_+ \wedge_H A\right);$$
$$\left(B \xrightarrow[H]{f} G_+ \wedge_H A \xrightarrow[H]{p} A\right) \longleftarrow \left(B \xrightarrow[G]{f} G_+ \wedge_H A\right),$$

where p is induced by the space-level map $p : G_+ \wedge_H B \longrightarrow B$ in which

$$p(g \wedge_H b) = \begin{cases} gb & g \in H \\ * & g \notin H \end{cases}.$$

□

Corollary 4.2.21. *For based G-CW-complexes X we have isomorphisms*

$$\theta : \widetilde{K}^0_H(X) \xrightarrow{\cong} \widetilde{K}^0_G(G/H_+ \wedge X) \text{ and } \phi : \widetilde{K}^H_0(X) \xrightarrow{\cong} \widetilde{K}^G_0(G/H_+ \wedge X).$$

Proof. For θ take $A = X$ and $B = \mathbb{K}$ in Lemma 4.2.20 (i) to see that

$$\widetilde{K}^0_H(X) = [X, \mathbb{K}]_H \cong [G_+ \wedge_H X, \mathbb{K}]_G = \widetilde{K}^0_G(G/H_+ \wedge X),$$

and for ϕ take $B = \mathbb{S}$ and $A = \mathbb{K} \wedge X$ in Lemma 4.2.20 (ii) to see that

$$\widetilde{K}^H_0(X) = [\mathbb{S}, \mathbb{K} \wedge X]_H \cong [\mathbb{S}, G_+ \wedge_H \mathbb{K} \wedge X]_G \cong \widetilde{K}^G_0(G/H_+ \wedge X).$$

□

4.2.3 Duality and change of group

We conclude this section by combining *duality* and *change of group* to give the following two results. The first involves the use of *restriction* maps which are discussed in §4.3. Here the reader may view restriction intuitively – we delay both the definition of restriction and the proof of Lemma 4.2.22 until §4.3 below.

Lemma 4.2.22. *Let ϕ be as in Lemma 4.2.20 (ii) and let $\wr -,- \int$ be the pairing of Definition 4.2.13. Then the diagram*

$$\begin{array}{ccc} [X,\mathbb{K}]_G \times [\mathbb{S},\mathbb{K}\wedge X]_G & \xrightarrow{\langle -,-\rangle_G} & [\mathbb{S},\mathbb{K}]_G \\ \Big\downarrow{\scriptstyle (D\pi)_* \times 1} & & \Big\downarrow{\scriptstyle (D\pi)_*} \\ [X,\mathbb{K}\wedge G/H_+]_G \times [\mathbb{S},\mathbb{K}\wedge X]_G & \xrightarrow{\wr -,- \int} & [\mathbb{S},\mathbb{K}\wedge G/H_+]_G \\ \Big\downarrow{\scriptstyle \phi^{-1}\times \mathrm{Res}^G_H} & & \Big\downarrow{\scriptstyle \phi^{-1}} \\ [X,\mathbb{K}]_H \times [\mathbb{S},\mathbb{K}\wedge X]_H & \xrightarrow{\langle -,-\rangle_H} & [\mathbb{S},\mathbb{K}]_H \end{array}$$

commutes.

Proof. Delayed until §4.3. □

Proposition 4.2.23. *Let θ and ϕ be as in Lemma 4.2.20, and suppose that A, B are G-spectra. Then, for $H \leq G$, the diagram*

$$\begin{array}{ccc} [A,B]_G & \xrightarrow{\pi^*} & [A\wedge G/H_+, B]_G \\ \Big\downarrow{\scriptstyle (D(\pi))_*} & & \Big\downarrow{\scriptstyle \theta^{-1}} \\ [A, B\wedge G/H_+]_G & \xrightarrow{\phi^{-1}} & [A,B]_H \end{array}$$

commutes.

Proof. Take $f \in [A,B]_G$. Since both π and θ^{-1} have space-level versions, we can apply an elemental argument to see that

$$(\theta^{-1}\pi^*(f))(a) = f\pi(a\wedge eH) = f(a),$$

or in other words $\theta^{-1}\pi(f)$ is just f viewed as an H-map. Moving the other way around the diagram, $\phi^{-1}D(\pi)_*(f)$ is the composite

$$A \xrightarrow{f} B \xrightarrow{D(\pi)} B\wedge G/H_+ \xrightarrow{p} B$$

which again is just f viewed as an H-map. □

4.3 Restriction

In Chapter 3 we described $K^0_G(\mathbb{C}P(V))$ and $K^G_0(\mathbb{C}P(V))$ in terms of (virtual) representations. We now define *restriction* maps in equivariant K-theory which are a generalisation (in the sense of Lemma 4.3.3) of the usual restriction of representations. We comment that one can make analogous definitions and then prove results for *induction* maps but it turns out that we shall not need to make use of those.

Definition 4.3.1. Take a based G-CW-complex X. Let π and $D(\pi)$ be as in equations (4.2.16) and (4.2.17). Let $\theta : \widetilde{K}^0_H(X) \xrightarrow{\cong} \widetilde{K}^0_G(G/H_+ \wedge X)$ and $\phi : \widetilde{K}^H_0(X) \xrightarrow{\cong} \widetilde{K}^G_0(G/H_+ \wedge X)$ be the isomorphisms of Corollary 4.2.21.

(i) Define the *equivariant K-cohomology restriction* $\mathrm{Res}^G_H(-) : \widetilde{K}^0_G(X) \longrightarrow \widetilde{K}^0_H(X)$ to be such that the diagram

$$\begin{array}{ccc} \widetilde{K}^0_G(X) & \xrightarrow{\mathrm{Res}^G_H} & \widetilde{K}^0_H(X) \\ \| & & \cong \downarrow \theta \\ \widetilde{K}^0_G(G/G_+ \wedge X) & \xrightarrow{(\pi\wedge 1)^*} & \widetilde{K}^0_G(G/H_+ \wedge X) \end{array}$$

commutes;

(ii) Define the *equivariant K-homology restriction* $\mathrm{Res}^G_H(-) : \widetilde{K}^G_0(X) \longrightarrow \widetilde{K}^H_0(X)$ to be such that the diagram

$$\begin{array}{ccc} \widetilde{K}^G_0(X) & \xrightarrow{\mathrm{Res}^G_H} & \widetilde{K}^H_0(X) \\ \| & & \cong \downarrow \phi \\ \widetilde{K}^G_0(G/G_+ \wedge X) & \xrightarrow{(D(\pi)\wedge 1)_*} & \widetilde{K}^G_0(G/H_+ \wedge X) \end{array}$$

commutes.

We use the same notation for the restriction maps in both homology and cohomology, leaving the context to clarify the meaning. We also write Res^G_H for the usual restriction map $R(G) \longrightarrow R(H)$ for $H \leq G$ – Lemma 4.3.3 below shows that this is not dangerous.

Since both θ and π have space level versions, it is easy to check that our cohomology restriction map is a ring map. It then follows from Proposition 4.2.23 that our homology restriction map is also a ring map. We shall shortly examine some further properties, but we first pause to prove an earlier result.

Proof of Lemma 4.2.22. Take $c \in [X, \mathbb{K}]_G$ and $h \in [\mathbb{S}, \mathbb{K} \wedge X]_G$. Then $(D\pi)_*(\langle c, h\rangle_G)$ is the composite

$$\mathbb{S} \xrightarrow{h} \mathbb{K} \wedge X \xrightarrow{1\wedge c} \mathbb{K} \wedge \mathbb{K} \xrightarrow{\mu} \mathbb{K} \xrightarrow{D\pi} \mathbb{K} \wedge G/H_+,$$

whilst $\wr(D\pi)_*(c), 1(h)\int$ is the composite

$$\mathbb{S} \xrightarrow{h} \mathbb{K} \wedge X \xrightarrow{1\wedge c} \mathbb{K} \wedge \mathbb{K} \xrightarrow{1\wedge D\pi} \mathbb{K} \wedge \mathbb{K} \wedge G/H_+ \xrightarrow{\mu\wedge 1} \mathbb{K} \wedge G/H_+.$$

These are equal, and this demonstrates commutativity of the upper square.

Now for the lower square, take $c \in [X, \mathbb{K}\wedge G/H_+]_G$ and $h \in [\mathbb{S}, \mathbb{K}\wedge X]_G$. Then $\phi^{-1}\wr c, h\int$ is the composite

$$\mathbb{S} \xrightarrow{h} \mathbb{K} \wedge X \xrightarrow{1\wedge c} \mathbb{K} \wedge \mathbb{K} \wedge G/H_+ \xrightarrow{\mu\wedge 1} \mathbb{K} \wedge G/H_+ \xrightarrow{p} \mathbb{K},$$

whilst $\langle \phi^{-1}(c), \mathrm{Res}^G_H(h)\rangle_H$ is the composite

$$\mathbb{S} \xrightarrow{h} \mathbb{K} \wedge X \xrightarrow{1\wedge c} \mathbb{K} \wedge \mathbb{K} \wedge G/H_+ \xrightarrow{1\wedge p} \mathbb{K} \wedge \mathbb{K} \xrightarrow{\mu} \mathbb{K}.$$

These are equal, and this demonstrates the commutativity of the lower square. □

One might expect any reasonable restriction to behave well on subspaces, and this is the case.

Lemma 4.3.2. *Suppose that $j : Y \hookrightarrow X$ is a G-inclusion. Then we have*

$$\mathrm{Res}^G_H\, j^* = j^*\mathrm{Res}^G_H \text{ and } j_*\mathrm{Res}^G_H = \mathrm{Res}^G_H\, j_*.$$

Proof. The lemma is immediate from the definitions. □

In the application of Lemma 4.3.2 we shall often omit the j^* or j_*, leaving the context to make the meaning clear.

Lemma 4.3.3. *The maps $\mathrm{Res}^G_H : \widetilde{K}^0_G \longrightarrow \widetilde{K}^0_H$ and $\mathrm{Res}^G_H : \widetilde{K}^G_0 \longrightarrow \widetilde{K}^H_0$ both agree with the usual restriction map $\mathrm{Res}^G_H : R(G) \longrightarrow R(H)$.*

Proof. For the cohomological map we need to prove that the diagram

$$\begin{array}{ccc} \widetilde{K}^0_G(S^0) & \xrightarrow{\mathrm{Res}^G_H} & \widetilde{K}^0_H(S^0) \\ \| & & \cong \downarrow \theta \\ \widetilde{K}^0_G(G/G_+ \wedge S^0) & \xrightarrow{(\pi\wedge 1)^*} & \widetilde{K}^0_G(G/H_+ \wedge S^0), \end{array}$$

in which we understand Res^G_H for the usual restriction, commutes. Take $c \in [\mathbb{S}, \mathbb{K}]_G = \widetilde{K}^0_G(S^0)$. Then since both π and θ^{-1} have space-level versions, we can apply an elemental argument to see that

$$(\theta^{-1}(\pi \wedge 1)^*(c))(s) = (c(\pi \wedge 1))(eH \wedge s) = c(s),$$

or in other words $\theta^{-1}(\pi \wedge 1)^*(c)$ is just c regarded as an H-map. That is precisely the same as $\mathrm{Res}^G_H(c)$.

For the homological map, observe that we have a commutative diagram

$$\begin{array}{ccc}
\widetilde{K}^0_G(S^0) & \xrightarrow{\mathrm{Res}^G_H} & \widetilde{K}^0_H(S^0) \\
\| & & \downarrow \cong \\
\widetilde{K}^0_G(G/G_+ \wedge S^0) & \xrightarrow{(\pi \wedge 1)^*} & \widetilde{K}^0_G(G/H_+ \wedge S^0) \\
\cong \downarrow & & \downarrow \cong \\
\widetilde{K}^G_0(G/G_+ \wedge S^0) & \xrightarrow{(D(\pi) \wedge 1)_*} & \widetilde{K}^G_0(G/H_+ \wedge S^0) \\
\| & & \downarrow \cong \\
\widetilde{K}^G_0(S^0) & \xrightarrow{\mathrm{Res}^G_H} & \widetilde{K}^H_0(S^0).
\end{array}$$

The top and bottom squares are just the definitions of the K-theory restriction maps and the middle square commutes by Lemma 4.2.7 (v). The lemma follows. □

Next, we consider the compatibility of restriction with the module structure in equivariant K-theory. Let us agree to write m for the module structure $K^0_G \times K^0_G(X) \longrightarrow K^0_G(X)$ described in Proposition 3.1.13.

Proposition 4.3.4. *The diagram*

$$\begin{array}{ccccc}
\widetilde{K}^0_G \times \widetilde{K}^0_G(G/H_+ \wedge X) & & \xrightarrow{m} & & \widetilde{K}^0_G(G/H_+ \wedge X) \\
\| & & & & \| \\
\widetilde{K}^0_G \times \widetilde{K}^0_G(G/H_+ \wedge X) & \xrightarrow{\mathrm{pr}^* \times 1} & \widetilde{K}^0_G(G/H_+ \wedge X) \times \widetilde{K}^0_G(G/H_+ \wedge X) & \xrightarrow{\otimes} & \widetilde{K}^0_G(G/H_+ \wedge X) \\
\mathrm{Res}^G_H \times \theta^{-1} \downarrow & & & & \cong \downarrow \theta^{-1} \\
\widetilde{K}^0_H \times \widetilde{K}^0_H(X) & \xrightarrow{\mathrm{pr}^* \times 1} & \widetilde{K}^0_H(X) \times \widetilde{K}^0_H(X) & \xrightarrow{\otimes} & \widetilde{K}^0_H(X) \\
\| & & & & \| \\
\widetilde{K}^0_H \times \widetilde{K}^0_H(X) & & \xrightarrow{m} & & \widetilde{K}^0_H(X)
\end{array}$$

commutes, where θ is as in Corollary 4.2.21.

Proof. The top and bottom squares are just the definition of m so we need only consider the middle square. Take $\rho \in \widetilde{K}^0_G$ and $\xi \in \widetilde{K}^0_G(G/H_+ \wedge X)$. Then the proposition amounts to proving that

$$\theta^{-1}(\mathrm{pr}^*(\rho) \otimes \xi) = (\mathrm{pr}^*\mathrm{Res}^G_H(\rho)) \otimes \theta^{-1}(\xi).$$

It suffices to show that $\theta^{-1}\mathrm{pr}^*(\rho) = \mathrm{pr}^*\mathrm{Res}^G_H(\rho)$. Here it is convenient to work with the G-bundle description of K-theory. Recalling Proposition 3.1.13 we see that we are required to prove that

$$\theta^{-1}((G/H_+ \wedge X) \times \rho) = X \times \mathrm{Res}^G_H(\rho),$$

but this is evident from [55 (§2, Example (iii))]. $\square$

We shall want to consider $X = \mathbb{C}P(V)$. The following lemma turns out to be invaluable. Recall that T is the circle group.

Lemma 4.3.5. *We have a commutative diagram*

$$\begin{array}{ccc}
\widetilde{K}^0_G(\mathbb{C}P(V)_+) & \xrightarrow{\mathrm{Res}^G_H} & \widetilde{K}^0_H(\mathbb{C}P(V)_+) \\
\cong \downarrow & & \downarrow \cong \\
\widetilde{K}^0_{G\times T}(S(V\otimes z)_+) & \xrightarrow{\mathrm{Res}^{G\times T}_{H\times T}} & \widetilde{K}^0_{H\times T}(S(V\otimes z)_+)
\end{array}$$

in which the isomorphisms are given by Corollary 3.1.12. Similarly in homology.

Proof. Simply observe that Res^G_H and $\mathrm{Res}^{G\times T}_{H\times T}$ are both induced by $\pi : G/H_+ \longrightarrow G/G_+$. In more detail, we have the commutative diagram below.

$$\begin{array}{ccc}
\widetilde{K}^0_G(\mathbb{C}P(V)_+) & \xrightarrow{\mathrm{Res}^G_H} & \widetilde{K}^0_H(\mathbb{C}P(V)_+) \\
\| & & \theta \downarrow \cong \\
\widetilde{K}^0_G(G/G_+ \wedge \mathbb{C}P(V)_+) & \xrightarrow{(\pi\wedge 1)^*} & \widetilde{K}^0_G(G/H_+ \wedge \mathbb{C}P(V)_+) \\
\cong \downarrow & & \downarrow \cong \\
\widetilde{K}^0_{G\times T}(G/G_+ \wedge S(V\otimes z)_+) & \xrightarrow{(\pi\wedge 1)^*} & \widetilde{K}^0_{G\times T}(G/H_+ \wedge S(V\otimes z)_+) \\
\| & & \theta \uparrow \cong \\
\widetilde{K}^0_{G\times T}(S(V\otimes z)_+) & \xrightarrow{\mathrm{Res}^{G\times T}_{H\times T}} & \widetilde{K}^0_{H\times T}(S(V\otimes z)_+)
\end{array}$$

(The unlabelled isomorphisms are given by Corollary 3.1.12.)

The homological part is similar. □

Remark 4.3.6. In the usual notation, observe that $\mathrm{Res}_H^G(z_G) = z_H$. This follows from Lemmas 4.3.3 and 4.3.5.

4.3.1 Injectivity of restriction

The map Res_H^G is *not* injective – taking notation as in Appendix A, one easily checks that $\mathrm{Res}_{C_2}^{C_4}(\varepsilon) = \mathrm{Res}_{C_2}^{C_4}(\alpha^2)$. However, we can prove an injectivity result which we shall need shortly.

Notation 4.3.7. Write $\mathfrak{Res}_*^G$ for the product

$$K_G^0(\mathbb{C}P(V)) \longrightarrow \prod_{\substack{H\leq G \\ H \text{ cyclic}}} K_H^0(\mathbb{C}P(V))$$

in which

$$x \longmapsto \prod_{\substack{H\leq G \\ H \text{ cyclic}}} \mathrm{Res}_H^G(x),$$

and write the same thing, $\mathfrak{Res}_*^G$, for the similar product

$$K_0^G(\mathbb{C}P(V)) \longrightarrow \prod_{\substack{H\leq G \\ H \text{ cyclic}}} K_0^H(\mathbb{C}P(V)).$$

McClure [48] has shown that the obvious generalisation of $\mathfrak{Res}_*^G$ is injective for finite G-CW-complexes X if we take the product over *all* finite subgroups H of G. That result is not quite adequate for our situation, because we want to restrict only to *abelian* subgroups. At the cost of losing a little generality, we take $X = \mathbb{C}P(V)$ in order to proceed.

Proposition 4.3.8. *The maps*

(i) $\mathfrak{Res}_*^G : K_G^0(\mathbb{C}P(V)) \longrightarrow \prod_{\substack{H\leq G \\ H \text{ cyclic}}} K_H^0(\mathbb{C}P(V))$;

(ii) $\mathfrak{Res}_*^G : K_0^G(\mathbb{C}P(V)) \longrightarrow \prod_{\substack{H\leq G \\ H \text{ cyclic}}} K_0^H(\mathbb{C}P(V))$

are injective.

Proof. For (i) we show that $\operatorname{Ker}(\mathfrak{Res}_*^G)$ is trivial. Take x in the kernel. First suppose that $x \in R(G) \subseteq K_G^0(\mathbb{C}P(V))$. Write $\langle g \rangle$ for the cyclic subgroup generated by $g \in G$ and view x as a character of G. Then

$$\begin{aligned} x(g) &= \left(\operatorname{Res}_{\langle g \rangle}^G x\right)(g) \\ &= \varepsilon_{\langle g \rangle}(g) \text{ since } x \in \operatorname{Ker}(\mathfrak{Res}_*^G) \\ &= 1. \end{aligned}$$

Thus $x = \varepsilon_G$. A general element $x \in K_G^0(\mathbb{C}P(V))$ may be uniquely written as

$$x = c_0 + c_1 z_G + \cdots + c_{n-1} z_G^{n-1} \qquad (c_i \in R(G)).$$

If also $x \in \operatorname{Ker}(\mathfrak{Res}_*^G)$, it follows that $c_i = 0$ for $i > 0$ since $\operatorname{Res}_H^G(z_G^i) = z_H^i$ for all $H \leq G$.

For (ii) suppose we have $x, y \in K_0^G(\mathbb{C}P(V))$ with

$$\mathfrak{Res}_*^G(x) = \mathfrak{Res}_*^G(y).$$

Write $x = \sum_{i=0}^{n-1} x_i \beta_i^G$ $(x_i \in R(G))$ and similarly with y. Fix a cyclic subgroup H of G. Then, appealing to Theorem 4.4.1 below, we have

$$\sum_{i=0}^{n-1} \operatorname{Res}_H^G(x_i) \beta_i^H = \sum_{i=0}^{n-1} \operatorname{Res}_H^G(y_i) \beta_i^H.$$

Equating coefficients allows us to deduce

$$\{\operatorname{Res}_H^G(x_i)\}_{H \text{ cyclic}} = \{\operatorname{Res}_H^G(y_i)\}_{H \text{ cyclic}}$$

for $i = 0, \ldots, n-1$. Then we argue as for (i) to see $x_i = y_i$ for all i and so $x = y$. □

4.4 The behaviour of $\frac{1}{\chi(V \otimes z)}$ under restriction of groups

Suppose that we have a representation V of a *non-abelian* group G. There is always an abelian subgroup H of G. Then we can view V as a $\mathbb{C}H$-module and the techniques of Chapter 3 apply. Our task is to use this H-equivariant information to make G-equivariant deductions. Specifically, we consider the question *for $H \leq G$ what can we say about $\frac{1}{\chi(V \otimes z)} \in K_0^G(\mathbb{C}P(V))$ in terms of $\frac{1}{\chi(V \otimes z)} \in K_0^H(\mathbb{C}P(V))$?* If we wish to be pedantic we write V_H for V viewed as a $\mathbb{C}H$-module but usually the context will ensure that a simple "V" suffices. Our strategy is to consider how both $\frac{1}{\chi(V\otimes z)}$ and $\sum_{i=0}^{n-1} \beta_i$ behave under restriction, and then to make deductions based on §4.3.1.

Theorem 4.4.1. *Let G be a finite group, H a subgroup of G, and V a complex representation of G of dimension n. For $L = G, H$, choose the basis $\{(1 - z_L)^j\}_{j=0}^{n-1}$ for $K_L^0(\mathbb{C}P(V))$ and let the dual basis for $K_0^L(\mathbb{C}P(V))$ be $\{\beta_j^L\}_{j=0}^{n-1}$. Then for $i = 0, 1, \ldots, n-1$ we have*

$$\mathrm{Res}_H^G\,(\beta_i^G) = \beta_i^H.$$

In particular,

$$\mathrm{Res}_H^G\left(\sum_{i=0}^{n-1} \beta_i^G\right) = \sum_{i=0}^{n-1} \beta_i^H.$$

Proof. We need to show that $\left\langle (1 - z_H)^j, \mathrm{Res}_H^G\,(\beta_i^G)\right\rangle_H = \delta_{ij}$, or in other words that

$$\left\langle \mathrm{Res}_H^G\,((1 - z_G)^j), \mathrm{Res}_H^G\,(\beta_i^G)\right\rangle_H = \delta_{ij}.$$

Now, by Lemma 4.2.22 we have

$$\phi^{-1}D(\pi)_*\langle(1 - z_G)^i, \beta_j^G\rangle_G = \langle\phi^{-1}D(\pi)_*((1 - z_G)^i), \mathrm{Res}_H^G\,(\beta_j^G)\rangle_H. \tag{4.4.2}$$

On the left hand side of equation (4.4.2), view $\langle(1 - z_G)^i, \beta_j^G\rangle_G = \delta_{ij} \in R(G)$ so that $\phi^{-1}D(\pi)_*$ behaves as Res_H^G and the left hand side is $\delta_{ij} \in R(H)$. To complete the proof it now suffices to show that $\phi^{-1}D(\pi)_* = \theta^{-1}\pi^*$, but that was Proposition 4.2.23, so we are done. ∎

Theorem 4.4.1 tells us how $\sum_{i=0}^{n-1} \beta_i$ behaves under restriction. That behaviour is encouraging but we do not yet understand how $\frac{1}{\chi(V\otimes z)}$ behaves because Theorem 3.5.10 tells us only in the abelian case that $\sum_{i=0}^{n-1} \beta_i = \frac{1}{\chi(V\otimes z)}$.

Lemma 4.4.3. *Suppose V is a representation of G and that $\lambda^j V$ is the j^{th} exterior power of V. Then for subgroups H of G,*

$$\lambda^j \mathrm{Res}_H^G\,(V) = \mathrm{Res}_H^G\,(\lambda^j V)$$

as $\mathbb{C}H$-modules.

Proof. Recall that if V has basis $\{v_1, \ldots, v_n\}$ then $\lambda^j V$ has basis

$$\{v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_j} \mid 1 \leq i_1 < i_2 < \cdots < i_j \leq n\}$$

with G acting via $g \cdot (v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_j}) = (g \cdot v_{i_1}) \wedge v_{i_2} \wedge \cdots \wedge v_{i_j}$. As vector spaces,

$$(\mathfrak{B} \text{ is a basis for } V) \iff (\mathfrak{B} \text{ is a basis for } \mathrm{Res}_H^G V) \tag{4.4.4}$$

and thus

$$(\mathfrak{B} \text{ is a basis for } \lambda^j V) \iff (\mathfrak{B} \text{ is a basis for } \lambda^j \mathrm{Res}_H^G V). \tag{4.4.5}$$

Replacing V with $\lambda^j V$ in (4.4.4) and comparing with (4.4.5) gives the lemma on the level of vector spaces. The action of H on $\lambda^j \mathrm{Res}_H^G (V)$ is clearly the same as on $\mathrm{Res}_H^G (\lambda^j V)$ and the lemma follows. □

Corollary 4.4.6. *We have* $\mathrm{Res}_{H\times T}^{G\times T} (\chi(V \otimes z)) = \chi(V_H \otimes z) \in K_H^0(\mathbb{C}P(V))$.

Proof. Given equivariant Bott periodicity (Theorem 3.2.4), Remark 4.3.6 and Lemma 4.4.3 this is clear. □

Proposition 4.4.7. *Given $H \leq G$ the following diagram has exact rows and commutes.*

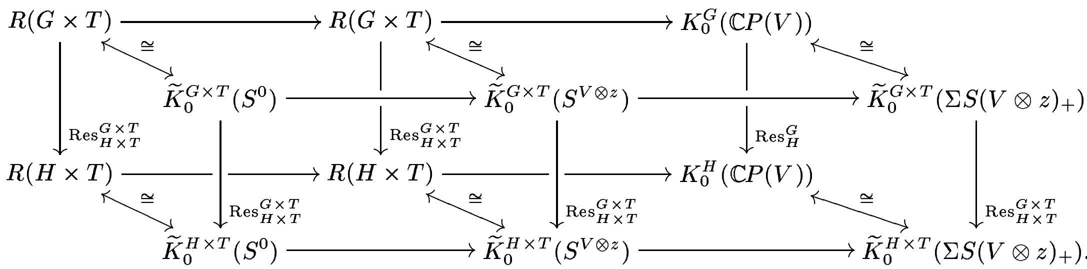
$$\begin{array}{ccccccccc}
0 \longrightarrow & R(G\times T) & \xrightarrow{\chi(V\otimes z)} & R(G\times T) & \longrightarrow & K_0^G(\mathbb{C}P(V)) & \longrightarrow 0 \\
& \Big\downarrow{\scriptstyle \mathrm{Res}_{H\times T}^{G\times T}} & & \Big\downarrow{\scriptstyle \mathrm{Res}_{H\times T}^{G\times T}} & & \Big\downarrow{\scriptstyle \mathrm{Res}_H^G} & \\
0 \longrightarrow & R(H\times T) & \xrightarrow{\chi(V_H\otimes z)} & R(H\times T) & \longrightarrow & K_0^H(\mathbb{C}P(V)) & \longrightarrow 0
\end{array} \tag{4.4.8}$$

Proof. There is no question about exactness since the rows are precisely the short exact sequence (3.4.7) of §3.4. Given $x \in R(G \times T)$ the first square is

$$\begin{array}{ccc}
x & \longmapsto & \chi(V\otimes z)x \\
\Big\downarrow & & \Big\downarrow \\
& & \mathrm{Res}_{H\times T}^{G\times T}(\chi(V\otimes z))\mathrm{Res}_{H\times T}^{G\times T}(x) \\
\mathrm{Res}_{H\times T}^{G\times T}(x) & \longmapsto & \| \\
& & \chi(V_H\otimes z)\mathrm{Res}_{H\times T}^{G\times T}(x).
\end{array}$$

The equality follows from Corollary 4.4.6.

Consider now the second square. Recall that the short exact sequence (3.4.7) arose from the cofibre sequence (3.3.2) of the pair $(D(V\otimes z)_+, S(V\otimes z)_+)$. Thus our diagram (4.4.8) fits into the larger diagram



In the front face, Lemma 4.3.2 tells us that the second square commutes because the horizontal maps are induced by the $(\Gamma\times T)$-inclusions

$$S^{V\otimes z} \hookrightarrow \Sigma S(V\otimes z)_+$$

for $\Gamma = G, H$. Hence the front face, and so the back face commutes. □

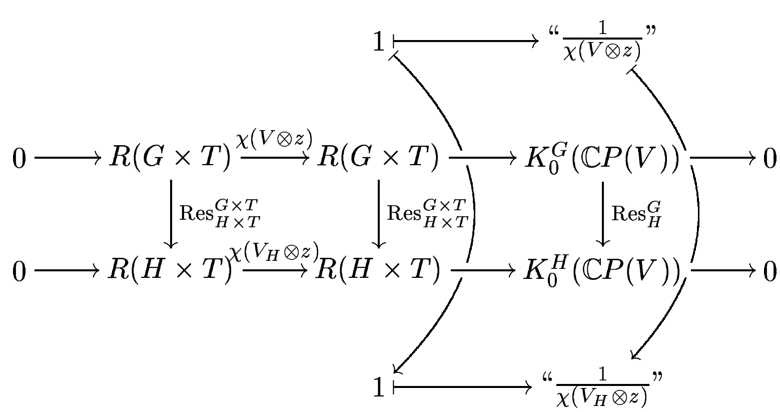
Theorem 4.4.9. *If $H \le G$ then we have*

$$\mathrm{Res}^G_H\left(\frac{1}{\chi(V\otimes z)}\right) = \frac{1}{\chi(V_H\otimes z)}.$$

Proof. Recall from §3.4 that $\frac{1}{\chi(V\otimes z)}$ is notation for the image in $K_0^G(\mathbb{C}P(V))$ of 1 in the short exact sequence (3.4.7), viz

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R(G\times T) & \xrightarrow{\chi(V\otimes z)} & R(G\times T) & \longrightarrow & K_0^G(\mathbb{C}P(V)) & \longrightarrow & 0 \\ & & & & 1 & \longmapsto & \text{“}\frac{1}{\chi(V\otimes z)}\text{”}. & & \end{array}$$

Chasing $1 \in R(G\times T)$ in the diagram (4.4.8),



we conclude the result.

□

Theorem 4.4.1 and Theorem 4.4.9 tell us that $\sum_{i=0}^{n-1} \beta_i^G$ and $\frac{1}{\chi(V \otimes z)}$ behave well with respect to Res_H^G. For a fixed, abelian $H = A \leq G$, Corollary 3.5.15 applies and we have

$$\begin{array}{ccc} \frac{1}{\chi(V\otimes z)} & \stackrel{?}{\longleftrightarrow} & \sum_{i=0}^{n-1} \beta_i^G \\ {\scriptstyle \mathrm{Res}_A^G}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_A^G} \\ \frac{1}{\chi(V_A\otimes z)} & = & \sum_{i=0}^{n-1} \beta_i^A. \end{array} \tag{4.4.10}$$

Our task is now to prove that the dotted arrow in (4.4.10) is in fact equality.

Theorem 4.4.11. *Let V be a complex representation,* $\dim_{\mathbb{C}} V = n$, *of the finite group G. Take $\{(1-z)^i\}_{i=0}^{n-1}$ as a basis for $K_G^0(\mathbb{C}P(V))$ and let the dual basis for $K_0^G(\mathbb{C}P(V))$ be $\{\beta_i\}_{i=0}^{n-1}$. Then*

$$\sum_{i=0}^{n-1} \beta_i = \frac{1}{\chi(V \otimes z)}.$$

Proof. We replace the diagram (4.4.10) with

$$\begin{array}{ccc} \frac{1}{\chi(V\otimes z)} & = & \sum_{i=0}^{n-1} \beta_i^G \\ {\scriptstyle \mathfrak{Res}_*^G}\downarrow & & \downarrow{\scriptstyle \mathfrak{Res}_*^G} \\ \prod_{\substack{H \leq G \\ H \text{ cyclic}}} \frac{1}{\chi(V_H\otimes z)} & = & \prod_{\substack{H \leq G \\ H \text{ cyclic}}} \sum_{i=0}^{n-1} \beta_i^H. \end{array}$$

The upper equality follows from the injectivity of $\mathfrak{Res}_*^G$ given in Proposition 4.3.8. □

4.5 Two dimensional (simple) examples

To illustrate the preceding theory in use, we consider some two dimensional simple representations of the dihedral groups D_6, D_8 and the quaternion group Q_8. The ethos of this section is as in §3.6.1. There we took the group of equivariance to be the smallest non-trivial group possible, viz C_2. Here we consider groups with a simple representation of dimension bigger than one, and though the jump from order two to order six is large, it is the smallest possible.

4.5.1 The dihedral group of order six

The character table of D_6 is shown on page 171 and we use the notation from it. We shall consider two dimensional representations V of D_6 and for these we always use the basis $\{1, 1-z\}$ for $K^0_{D_6}(\mathbb{C}P(V))$. For completeness we shall consider non-simple examples as well as the case of interest.

Example 4.5.1 (Non-simple examples). Consider a two dimensional representation V of D_6 which is the sum of one dimensionals, say $V = \rho \oplus \sigma$ for $\rho, \sigma \in \{\varepsilon, \alpha\}$. Then by equivariant Bott periodicity and Proposition 3.2.6, $\chi(V \otimes z) = (1-\rho z)(1-\sigma z)$ so one finds that the duality pairing is given by

	1	$1-z$
1	1	1
$1-z$	1	$1-\rho\sigma$

.

Example 4.5.2 (A simple example). This time take V to be the two dimensional simple representation γ of D_6. We know from Theorem 3.2.4 that

$$\chi(\gamma \otimes z) = 1 - \lambda(\gamma \otimes z) + \lambda^2(\gamma \otimes z)$$

where λ^i represents the i^{th} exterior power. Of course, λ^1 is the identity and if V is of dimension n, recall from equation (3.3.17) that $\lambda^n(V)$ is the determinant $\det(V)$. It is easy to see that γ is isomorphic to D_6 acting on $\mathbb{C}^2$ via

$$r \longleftrightarrow \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}; \qquad s \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Computing determinants one finds

Conjugacy class	$\{e\}$	$\{s, rs, r^2s\}$	$\{r, r^2\}$
$\det(\gamma)$	1	-1	1

,

so that $\det(\gamma) = \alpha$ and $\chi(\gamma \otimes z) = 1 - \gamma z + \alpha z^2$. Thus our duality pairing is given by

	1	$1-z$
1	1	1
$1-z$	1	$1-\alpha$

.

4.5.2 The dihedral group of order eight

The character table of D_8 is shown on page 171 and we use the notation from it. We shall consider two dimensional representations V of D_8 and for these we always use the basis $\{1, 1-z\}$ for $K^0_{D_8}(\mathbb{C}P(V))$.

Example 4.5.3 (Non-simple examples). Consider a two dimensional representation V of D_8 which is the sum of one dimensionals, say $V = \rho \oplus \sigma$ for $\rho, \sigma \in \{\varepsilon, \alpha, \beta, \alpha\beta\}$. Then $\chi(V \otimes z) = (1 - \rho z)(1 - \sigma z)$ and one finds that the duality pairing is given by

	1	$1-z$
1	1	1
$1-z$	1	$1-\rho\sigma$

.

Example 4.5.4 (A simple example). This time take V to be the two dimensional simple representation γ of D_8. It is easy to see that γ is isomorphic to D_8 acting on $\mathbb{C}^2$ via

$$r \longleftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \qquad s \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Computing determinants one finds

Conjugacy class	$\{e\}$	$\{r, r^3\}$	$\{r^2\}$	$\{s, r^2 s\}$	$\{rs, r^3 s\}$
$\det(\gamma)$	1	1	1	-1	-1

,

so that $\det(\gamma) = \alpha$. We now find, as before, that $\chi(\gamma \otimes z) = 1 - \gamma z + \alpha z^2$ and our duality pairing is given by

	1	$1-z$
1	1	1
$1-z$	1	$1-\alpha$

.

4.5.3 The quaternion group of order eight

The character table of Q_8 is shown on page 172 and we use the notation from it. We shall consider two dimensional representations V of Q_8 and for these we always use the basis $\{1, 1-z\}$ for $K^0_{Q_8}(\mathbb{C}P(V))$.

Example 4.5.5 (Non-simple examples). Two dimensional representations V which are not simple offer no new interest – the duality pairing turns out to be the same as in Example 4.5.3. This happens because not only do D_8 and Q_8 share the same character tables, but also their one dimensional representations share the same Euler class.

Example 4.5.6 (A simple example). This time take V to be the two dimensional simple representation γ of Q_8. As before,

$$\chi(\gamma \otimes z) = 1 - \gamma z + \det(V) z^2.$$

It is easy to see that γ is isomorphic to Q_8 acting on $\mathbb{C}^2$ via

$$i \longleftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \qquad j \longleftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad k \longleftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Computing determinants one finds

Rep. of conjugacy class	e	-1	i	j	k
$\det(\gamma)$	1	1	1	1	1

so that $\det(\gamma) = \varepsilon$ and $\chi(\gamma \otimes z) = 1 - \gamma z + z^2$. It is now easy to compute that the duality pairing is given by

	1	$1-z$
1	1	1
$1-z$	1	0

Comparing with Example 4.5.4, we see that despite the two characters γ taking the same values, the difference in their determinants produces a difference in their duality pairings.

Chapter 5

The Atiyah-Hirzebruch spectral sequence and $\mathbb{C}\boldsymbol{P}(\boldsymbol{V})$

5.1 Another real interlude

In this section we revisit the real example of §2.1. In doing so we highlight the value of the G-cellular structures of Chapter 2 and illustrate the techniques which will follow.

Bredon [13] developed *equivariant ordinary cohomology theory*, meaning that he generalised singular cohomology to G-spaces for finite groups G and G-CW-complexes. A concise survey is to be found in [40].

In the non-equivariant scene, CW-complexes are built from *cells*, which are contractible. This is why the *coefficients* $H^0(\text{pt})$ play such an important role in the study of a cohomology theory H. In the equivariant setting, the orbit spaces G/H for H a subgroup of G take on the role of pt and so, in this case, we should ask that a coefficient system take account of all G/H as well as the equivariant maps between them.

Definition 5.1.1 (Bredon). Let G be a finite group. A *(generic) coefficient system M for G* is a contravariant functor $\boldsymbol{\theta_G} \longrightarrow \mathbf{Ab}$. (Here, $\mathbf{Ab}$ is the category of abelian groups and homomorphisms and $\boldsymbol{\theta_G}$ is the *category of canonical orbits of G*, with objects the (left) coset spaces G/H and morphisms the equivariant[1] maps $G/H_1 \longrightarrow G/H_2$.)

[1]With respect to left translation.

Aside 5.1.2. Bredon [13 (I, §3)] classifies the morphisms of $\boldsymbol{\theta_G}$: they are precisely the composites of maps induced by inclusions of subgroups and by the right translations

$$gH \overset{\gamma}{\longmapsto} gH\gamma = g\gamma(\gamma^{-1}H\gamma).$$

This classification shall prove useful later.

Now given a coefficient system M for G, Bredon goes on to construct the *equivariant cochain complex* $C^*_G(X; M)$ and defines the *equivariant cohomology groups*

$$H^q_G(X; M) = H^q(C^*_G(X; M)).$$

Rather than repeat the construction of $C^*_G(X; M)$, which Bredon supplies in detail ([13 (I, §6)]) we shall proceed by studying an illuminating example. Singular homology may be more geometrically clear than cohomology, so we replay the game with a *covariant* functor $M : \boldsymbol{\theta_G} \longrightarrow \mathbf{Ab}$. If we wish to be horrifically explicit we may write *(generic) covariant coefficient system for G* or *(generic) contravariant coefficient system for G* depending on context.

Now let us return to §2.1. Recall that $G = A_5$ and we are considering $V_{\mathbb{R}}$, a three dimensional real representation that arises by viewing A_5 as the direct symmetry group of an icosahedron. Recall from (2.1.4) on page 37 that $\mathbb{R}P(V_{\mathbb{R}})/A_5$ is as in Figure 5.1.

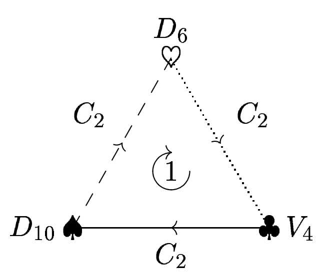


Figure 5.1. $\mathbb{R}P(V_{\mathbb{R}})/A_5$, with an arbitrary choice of orientation on the A_5-simplices (A_5-vertices are taken to have positive orientation). A_5-simplices are labelled with their isotropy groups.

Let M be a covariant coefficient system for A_5, that is a covariant functor $\boldsymbol{\theta}_{\boldsymbol{A_5}} \longrightarrow \mathbf{Ab}$. The construction of the equivariant chain complex $C_*^{A_5}(\mathbb{R}P(V_{\mathbb{R}}); M)$ is as follows. We take

$$C_q^{A_5}(\mathbb{R}P(V_{\mathbb{R}}); M) = \bigoplus_{\sigma \text{ an } A_5\text{-}q\text{-cell}} M(A_5/H_\sigma),$$

where H_σ is the isotropy group of σ. If there are no A_5-q-cells then of course we interpret $C_q^{A_5}(\mathbb{R}P(V_{\mathbb{R}}); M)$ as zero. We define the boundary

$$\delta^{A_5} : C_q^{A_5}(\mathbb{R}P(V_{\mathbb{R}}); M) \longrightarrow C_{q-1}^{A_5}(\mathbb{R}P(V_{\mathbb{R}}); M)$$

by

$$\delta^{A_5}(\sigma) = \bigoplus_{\tau \in \partial\sigma} \varepsilon_{\sigma,\tau} M(A_5/H_\sigma \longrightarrow A_5/H_\tau)(M(A_5/H_\sigma))$$

where $\varepsilon_{\sigma,\tau} = \pm 1$ according to the relative orientation of $\delta\sigma \cap \tau$ and τ.

Let us specify our coefficient system for A_5: fix an abelian group A and take $M = \mathbf{A}$ where $\mathbf{A}(A_5/H) = A$ for all $H \leq A_5$, and set $\mathbf{A}(A_5/H_1 \longrightarrow A_5/H_2)$ to be multiplication by $|H_2 : H_1|$.

Figure 5.1 allows us to write down the chain complex $C_*^{A_5}(\mathbb{R}P(V_{\mathbb{R}}); \mathbf{A})$, viz

$$\cdots \longrightarrow 0 \longrightarrow A \xrightarrow{\delta_2^{A_5}} A \oplus A \oplus A \xrightarrow{\delta_1^{A_5}} A \oplus A \oplus A \longrightarrow 0 \longrightarrow \cdots$$

Degree	2	1	0

where $\delta_2^{A_5}$ has matrix $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ and $\delta_1^{A_5}$ has matrix $\begin{pmatrix} -2 & 0 & 2 \\ 5 & -5 & 0 \\ 0 & 3 & -3 \end{pmatrix}$, whose row reduced echelon form over $\mathbb{Z}$ is $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$.

In the standard notation, we find that

$$Z_q(C_*^{A_5}(\mathbb{R}P(V_{\mathbb{R}});\mathbf{A})) = \begin{cases} 0 & \text{otherwise} \\ \operatorname{Ker}(\delta_2^{A_5}) = {}_2A & q=2 \\ \operatorname{Ker}(\delta_1^{A_5}) = \operatorname{Span}_A\{(1,1,1)\} & q=1 \\ A\oplus A\oplus A & q=0 \end{cases};$$

$$B_q(C_*^{A_5}(\mathbb{R}P(V_{\mathbb{R}});\mathbf{A})) = \begin{cases} 0 & \text{otherwise} \\ \operatorname{Im}(\delta_2^{A_5}) = \operatorname{Span}_A\{(2,2,2)\} & q=1 \\ \operatorname{Im}(\delta_1^{A_5}) = \operatorname{Span}_A\{(1,0,0),(0,1,0)\} & q=0 \end{cases}.$$

(Here, for $k\in\mathbb{N}$, ${}_kA = \{a\in A \mid ka=0\}$.)

We conclude that

$$H_q^{A_5}(\mathbb{R}P(V_{\mathbb{R}});\mathbf{A}) \cong \begin{cases} 0 & \text{otherwise} \\ {}_2A & q=2 \\ A/2A & q=1 \\ A & q=0 \end{cases}.$$

In summary, our equivariant cellular structure gave us an easy method of computing homology. In the rest of this chapter we show how to do the same thing for equivariant K-theory, and we look at some examples.

5.2 Mackey functors

Recall that in the non-equivariant scene, both ordinary homology and cohomology have a *coefficient group* – this is a simple algebraic object which is a crucial ingredient, but the *same* ingredient, for both homology and cohomology. In §5.1, we discussed the importance of *coefficient systems* for the equivariant case. Equivariant *cohomology* involves a *contravariant* coefficient system whilst equivariant *homology* requires a *covariant* coefficient system. This motivates us to generalise, and replace both sorts of coefficient system with a single entity. The appropriate concept is the *Mackey functor*. Since we are interested in generalised theories, which may have non-zero coefficients in non-zero degrees, we must consider *graded* Mackey functors. We shall often write *Mackey functor* for *graded Mackey functor*.

Dress [23] is a standard reference for Mackey functors, which makes much use of Green [27]. But the author finds the notation used by tom Dieck [63 (§6)] most intuitive, so we follow that approach.

Definition 5.2.1. Let G be a finite group, let **G-Set** be the category of G-sets and G-maps and let $\mathbf{Ab}_*$ be the category of graded abelian groups and graded homomorphisms.

(i) A *graded bi-functor* $M = (M^*, M_*) : \mathbf{G\text{-}Set} \longrightarrow \mathbf{Ab}_*$ consists of a contravariant functor $M^* : \mathbf{G\text{-}Set} \longrightarrow \mathbf{Ab}_*$ and a covariant functor $M_* : \mathbf{G\text{-}Set} \longrightarrow \mathbf{Ab}_*$ which coincide on objects. We write $M(S) = M_*(S) = M^*(S)$ for $S \in \mathbf{G\text{-}Set}$. For a morphism $f : S \longrightarrow T$ we write $M_*(f) = f_*$ and $M^*(f) = f^*$.

(ii) A graded bi-functor $M = (M^*, M_*)$ is called a *graded Mackey functor* if it has the following two properties.

(a) For any pullback diagram

$$\begin{array}{ccc} U & \xrightarrow{F} & S \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle h} \\ T & \xrightarrow[f]{} & V \end{array}$$

in **G-Set**, the diagram

$$\begin{array}{ccc} M(U) & \xrightarrow{F_*} & M(S) \\ {\scriptstyle H^*}\uparrow & & \uparrow{\scriptstyle h^*} \\ M(T) & \xrightarrow[f_*]{} & M(V) \end{array}$$

in $\mathbf{Ab}_*$ is commutative.

(b) The two embeddings $S \longrightarrow S \coprod T \longleftarrow T$ into the disjoint union define an isomorphism $M^*(S \coprod T) \longrightarrow M^*(S) \oplus M^*(T)$.

Remark 5.2.2. Given a Mackey functor one can always recover the underlying coefficient systems, but there is no reason to expect that *any* given coefficient system should extend to a Mackey functor. But Mackey functors are relevant to us because [29, 40] a cohomology theory $H^*_G(-, M)$ extends to an $RO(G)$-graded theory precisely when the coefficient system M extends to a Mackey functor.

5.2.1 The representation ring Mackey functor

Recall that a Mackey functor is a bi-functor **G-Set** $\longrightarrow$ $\mathbf{Ab}_*$, but since a G-set can be written as a (possibly infinite) union $\bigcup_I G/H \times \mathrm{pt}$ we see that to give a Mackey functor it suffices to write down a bi-functor $\boldsymbol{\theta_G} \longrightarrow \mathbf{Ab}_*$. In view of Aside 5.1.2 we see that Definition 5.2.3 is therefore a candidate to be a graded Mackey functor.

Definition 5.2.3. We define the *representation ring graded Mackey functor* $\mathbf{R}$ by writing down a bi-functor $\mathbf{R} : \boldsymbol{\theta_G} \longrightarrow \mathbf{Ab}_*$ as follows. Given $G/H \in \boldsymbol{\theta_G}$, we set

(i) $\mathbf{R}(G/H)_q = \begin{cases} R(H) & q \text{ even} \\ 0 & q \text{ odd} \end{cases}$, where $R(H)$ is the *representation ring* of H;

and, if $K \leq H \leq G$ so that we have $\pi : G/K \longrightarrow G/H$, we define

(ii) $\mathbf{R}^*(\pi) = \pi^* = \mathrm{Res}^H_K : R(K) \longleftarrow R(H)$ to be restriction of representations;

(iii) $\mathbf{R}_*(\pi) = \pi_* = \mathrm{Ind}^H_K : R(K) \longrightarrow R(H)$ to be induction of representations.

Remark 5.2.4. One needs to know that Definition 5.2.3 really does give a graded Mackey functor. To demonstrate this, the only thing of substance is to verify that, given $H, K \leq G$, the canonical pullback diagram

$$\begin{array}{ccc} G/K \times G/H & \longrightarrow & G/K \\ \downarrow & & \downarrow \\ G/H & \longrightarrow & G/G \end{array}$$

in **G-Set** induces the correct diagram in $\mathbf{Ab}_*$. Write $[K\backslash G/H]$ for a set of representatives of the double cosets $K\backslash G/H$. Given $(gK, g'H) \in G/K \times G/H$, one checks that $K(g^{-1}g')H \in K\backslash G/H$ is a well-defined double coset, independent of the choice of coset representatives for gK, $g'H$. So we may take $\gamma := g^{-1}g' \in [K\backslash G/H]$ and one checks that $(gK, g'H) \longmapsto g(K \cap (\gamma H \gamma^{-1}))$ gives an isomorphism

$$G/K \times G/H \cong \coprod_{\gamma \in [K\backslash G/H]} G/(K \cap (\gamma H \gamma^{-1})).$$

So we must show that the diagram

$$\begin{CD}
\bigoplus_{\gamma\in[K\backslash G/H]} R(K\cap(\gamma H\gamma^{-1})) @>{\mathrm{Ind}^{K}_{K\cap(\gamma H\gamma^{-1})}}>> R(K)\\
@A{\mathrm{Res}^{\gamma H\gamma^{-1}}_{K\cap(\gamma H\gamma^{-1})}\circ c_\gamma}AA @AA{\mathrm{Res}^G_K}A\\
R(H) @>{\mathrm{Ind}^G_H}>> R(G),
\end{CD}$$

where c_γ is conjugation $H \longrightarrow \gamma H\gamma^{-1}$, commutes. This is immediate from [38 (XVIII, Theorem 7.6)].

Lemma 5.2.5. *Let G be a finite group. Then the representation ring Mackey functor $\mathbf{R}$ specifies the covariant coefficient system of G-equivariant K-homology and the contravariant coefficient system of G-equivariant K-cohomology.*

Proof. It suffices to show that $\mathbf{R}$ coincides with K^*_G and K^G_* in degrees 0 and 1. Degree 1 is obvious, so we consider degree 0. Suppose that we have subgroups $K \leq H \leq G$. We need to show that $K^0_G(G/H \times \mathrm{pt}) = \mathbf{R}(G/H) = R(H) = K^G_0(G/H \times \mathrm{pt})$ (up to isomorphism). We have

$$\begin{aligned}
K^0_G(G/H \times \mathrm{pt}) &\cong K^0_H(\mathrm{pt}) \text{ by Corollary 4.2.21}\\
&\cong R(H) \text{ by Example 3.1.4,}
\end{aligned}$$

and homology is dealt with similarly by using Corollaries 4.2.21 and 3.4.2.

Now suppose that we have a G-map $G/K \xrightarrow{\pi} G/H$. We need to know that the map $\pi^* = \mathrm{Res}^H_K$ induced by $\mathbf{R}$ is the same as the map π^* induced by K^0_G – but that was done in Lemma 4.3.3.

This is sufficient to complete the proof, because we can appeal to [41 (V, Proposition 9.9)] to see that, for finite G, a Mackey functor determines and is determined by an additive contravariant functor $\boldsymbol{\theta_G} \longrightarrow \mathbf{Ab}$. ☐

5.3 The Atiyah-Hirzebruch spectral sequence

"there is a spectral sequence..."

J. McCleary [47]

Atiyah and Hirzebruch [11] first published results on what is now known as the *Atiyah-Hirzebruch spectral sequence*, though it is generally accepted that others (Whitehead and Lima) knew of its existence before its publication. McCleary [47 (11.16)] claims its existence without supplying a proof, [25 (IV, §3)] gives few details and Adams [4 (III, §7)] writes about what he calls "a folk-theorem". Thus we are in good company if we follow McCleary's spirit above: by this we mean we shall write down a spectral sequence and use it for computations with barely a pause for comment about convergence.

Suppose that X is a G-space and we have a filtration by G-spaces

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \text{ with } \bigcup_{k\geq 0} X^k = X. \tag{5.3.1}$$

If F^G is a G-equivariant homology theory, each G-pair (X^p, X^{p-1}) gives us a long exact sequence

$$\cdots \longrightarrow F^G_n(X^{p-1}) \xrightarrow{i} F^G_n(X^p) \xrightarrow{j} F^G_n(X^p, X^{p-1}) \xrightarrow{k} F^G_{n-1}(X^{p-1}) \longrightarrow \cdots . \tag{5.3.2}$$

Arranging these in a convenient labyrinth we achieve the diagram below.

$$\begin{array}{ccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & \\
\cdots \longrightarrow & F^G_n(X^{p-1}) & \xrightarrow{i} & F^G_n(X^p) & \xrightarrow{j} & F^G_n(X^p, X^{p-1}) & \xrightarrow{k} & F^G_{n-1}(X^{p-1}) & \longrightarrow \cdots \\
 & & & \| & & & & & \\
\cdots \rightarrow & F^G_{n+1}(X^{p+1}, X^p) & \xrightarrow{k} & F^G_n(X^p) & \xrightarrow{i} & F^G_n(X^{p+1}) & \xrightarrow{j} & F^G_n(X^{p+1}, X^p) & \rightarrow \cdots \\
 & & & & & \| & & & \\
\cdots \longrightarrow & F^G_{n+1}(X^{p+2}) & \xrightarrow{j} & F^G_{n+1}(X^{p+2}, X^{p+1}) & \xrightarrow{k} & F^G_n(X^{p+1}) & \xrightarrow{i} & F^G_n(X^{p+2}) & \longrightarrow \cdots \\
 & \vdots & & \vdots & & \vdots & & \vdots &
\end{array} \tag{5.3.3}$$

Writing $E^1_{p,q} = F^G_{p+q}(X^p, X^{p-1})$ and $D^1_{p,q} = F^G_{p+q}(X^p)$ we obtain an *exact couple*

$$\begin{array}{ccc}
D^1_{*,*} & \xrightarrow{\quad i \quad} & D^1_{*,*} \\
 {}_{k}\nwarrow & & \swarrow_{j} \\
 & E^1_{*,*} &
\end{array}$$

in the sense of Massey [44]. Given such an exact couple, Massey constructs the associated *derived exact couple*

$$\begin{array}{ccc} D^2_{*,*} & \longrightarrow & D^2_{*,*} \\ \nwarrow & & \swarrow \\ & E^2_{*,*} & \end{array}$$

and iterates the process. In doing so, one obtains the pages $E^1_{*,*}, E^2_{*,*}, E^3_{*,*}, \ldots$ of a spectral sequence. We summarise (see Adams [4 (p216-217)]):

Summary 5.3.4. With notation as above, one achieves a spectral sequence with

$$E^1_{p,q} = F^G_{p+q}(X^p, X^{p-1}) \Longrightarrow F^G_{p+q}(X).$$

The d^1 differential $d^1 : E^1_{p,q} = F^G_{p+q}(X^p, X^{p-1}) \longrightarrow F^G_{p+q-1}(X^{p-1}, X^{p-2}) = E^1_{p-1,q+1-1}$ is the composite $j \circ k$: one of the d^1 differentials is shown by a dotted box in the labyrinth (5.3.3) above.

Now let us specialise to the Atiyah-Hirzebruch spectral sequence. Take X to be a finite G-CW-complex and write $X^{(p)}$ for the G-p-skeleton. Then for the filtration (5.3.1) we take the skeletal filtration in which $X^p = X^{(p)}$. (When $p > \dim(X)$ we take $X^p = X$.) We now have

$$\begin{aligned} E^1_{p,q} &= F^G_{p+q}(X^p, X^{p-1}) \\ &= \widetilde{F}^G_{p+q}(X^p/X^{p-1}) \\ &= \widetilde{F}^G_{p+q}\left(\bigvee_{\sigma\in\Sigma_p} G/H_{\sigma+} \wedge S^p\right) \quad \text{(where } \Sigma_p \text{ indexes the } G\text{-}p\text{-cells)} \\ &= \bigoplus_{\sigma\in\Sigma_p} \widetilde{F}^G_{p+q}\left(G/H_{\sigma+} \wedge S^p\right) \\ &\cong \bigoplus_{\sigma\in\Sigma_p} \widetilde{F}^G_q(G/H_{\sigma+}) \text{ by the suspension isomorphisms} \\ &= \bigoplus_{\sigma\in\Sigma_p} F^G_q(G/H_\sigma) \\ &= C^G_p(X; \mathbf{F^G_q}); \end{aligned}$$

in words the E^1 page is the G-cellular chains with coefficient system, or graded Mackey functor, $\mathbf{F^G_*}$. In order to progress to the E^2 page we now must consider the d^1 differentials.

The long exact sequence (5.3.2) is induced by the cofibre sequence

$$X^{p-1} \longrightarrow X^p \longrightarrow X^p/X^{p-1} \longrightarrow \Sigma X^{p-1} \longrightarrow \Sigma X^p \longrightarrow \cdots$$

and so, concentrating on the boxed d^1 in the labyrinth (5.3.3) we observe that k is induced by applying F^G_{n+1} to $X^{p+1}/X^p \longrightarrow \Sigma X^p$, whilst j is induced by applying F^G_n to $X^p \longrightarrow X^p/X^{p-1}$. Using the suspensions to take care of the dimension shift, we gather that

$$d^1 \text{ is induced by } X^{p+1}/X^p \longrightarrow \Sigma X^p \longrightarrow \Sigma X^p/X^{p-1}. \tag{5.3.5}$$

Now in the G-cellular chain complex, δ^G is also induced by (5.3.5) and it follows that

$$E^2_{p,q} = H_*(E^1_{p,q}; d^1) \cong H_*(C^G_p(X; \mathbf{F}^{\mathbf{G}}_q); \delta^G) = H^G_p(X; \mathbf{F}^{\mathbf{G}}_q).$$

We summarise:

Theorem 5.3.6 (The Atiyah-Hirzebruch spectral sequence).

(i) *Let X be a finite G-CW-complex of dimension n. Let F^G be a G-equivariant homology theory with underlying graded Mackey functor $\mathbf{F}^{\mathbf{G}}_*$. Then there exists a spectral sequence $E^r_{p,q}$ $(r \geq 1, -\infty \leq p, q \leq \infty)$ of F^G_*-modules with*

(a) $E^1_{p,q} \cong C^G_p(X; \mathbf{F}^{\mathbf{G}}_q)$, *$d^1$ being the boundary δ^G;*

(b) $E^2_{p,q} \cong H^G_p(X; \mathbf{F}^{\mathbf{G}}_q)$.

The spectral sequence converges strongly to $F^G_{p+q}(X)$ with respect to the skeletal filtration

$$0 \subseteq \phi_1 \subseteq \phi_2 \subseteq \cdots \subseteq \phi_n = F^G_{p+q}(X) \tag{5.3.7}$$

of $F^G_{p+q}(X)$ in which $\phi_i = \operatorname{Im}(F^G_{p+q}(X^{(i)}) \hookrightarrow F^G_{p+q}(X))$ for $0 \leq i \leq n$. (Here, $X^{(i)}$ is the G-i-skeleton of X.)

(ii) *Similarly in cohomology, except that the convergence is conditional. If further, for each fixed (p, q), there are only finitely many r such that d^r is non-zero on $E^{p,q}_r$, then the spectral sequence converges strongly.*

□

5.4 Examples

At last we are in position to produce some examples of the Atiyah-Hirzebruch spectral sequence in action. It is here that we capitalise on the G-cellular structures of §2.3.

5.4.1 The dihedral group of order six

As in §2.3.1, take G to be the dihedral group D_6 of order six and let $V = \gamma$ be the unique two dimensional simple representation of G. Recall from Proposition 2.3.1 that there is a D_6-CW-complex structure for $\mathbb{C}P(\gamma)$ given by

$$\Big((D_6/C_3 \amalg D_6/C_2 \amalg D_6/C_2') \times e^0\Big) \bigcup_{f_1} \Big((D_6/1 \amalg D_6/1') \times e^1\Big) \bigcup_{f_2} \Big(D_6/1 \times e^2\Big)$$

where the attaching maps are as shown in Figures 2.8 and 2.9. There is only one D_6-2-cell which means we need consider only one of the panels shown in Figure 2.8, say $e \times e^2$, as in Figure 5.2.

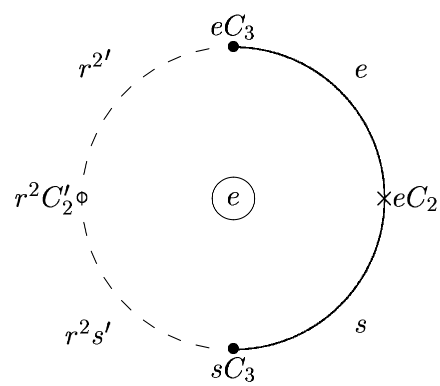


Figure 5.2. The panel $e \times e^2$ of Figure 2.8.

Thus we see that the quotient $\mathbb{C}P(\gamma)/D_6$ is as in Figure 5.3.

Proposition 5.4.1. *Take $G = D_6$ and let $V = \gamma$ be as above. Then there is an $R(D_6)$-module A of rank five (as an abelian group) such that*

$$K_0^G(\mathbb{C}P(\gamma))/A \cong R(1)$$

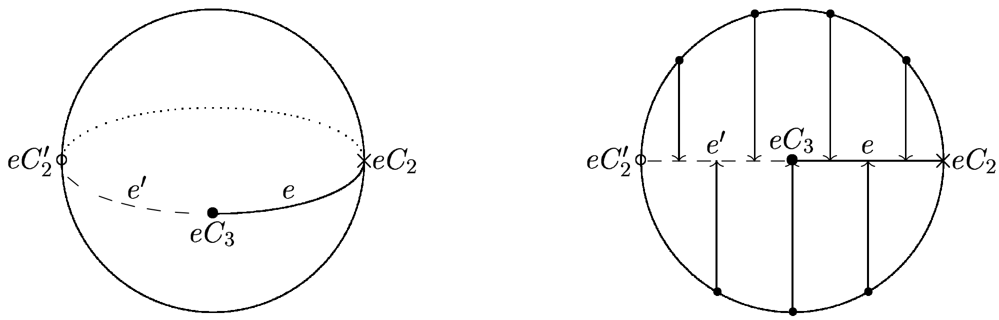


Figure 5.3. The quotient $\mathbb{C}P(\gamma)/D_6$ (left) and the attaching map $\partial e^2 \longrightarrow (\mathbb{C}P(\gamma))^{(1)}$ (right). The dots are only to emphasise the shape of the sphere and are *not* part of the CW-structure.

as $R(D_6)$-modules. Moreover, there is a short exact sequence

$$0 \longrightarrow R(1) \oplus R(1) \longrightarrow R(C_3) \oplus R(C_2) \oplus R(C_2) \longrightarrow A \longrightarrow 0$$

of $R(D_6)$-modules.

Proof. Looking at Figure 5.3 one sees that the chain complex for ordinary singular homology of $\mathbb{C}P(\gamma)/D_6$ (with integer coefficients) is

$$\mathbb{Z} \xrightarrow{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}. \tag{5.4.2}$$

Degree 2 1 0

Now in Theorem 5.3.6 take $X = \mathbb{C}P(\gamma)$ and F^G to be G-equivariant K-theory. By Lemma 5.2.5 we know that $\mathbf{F}_*^{\mathbf{G}} = \mathbf{R}$ and (5.4.2) allows us to see that $C_*^G(X; \mathbf{R})$ is

$$R(1) \xrightarrow{0} R(1) \oplus R(1) \xrightarrow{a} R(C_3) \oplus R(C_2) \oplus R(C_2), \tag{5.4.3}$$

Degree 2 1 0

where $a = \begin{pmatrix} \mathrm{Ind}_1^{C_3} & \mathrm{Ind}_1^{C_3} \\ -\mathrm{Ind}_1^{C_2} & 0 \\ 0 & -\mathrm{Ind}_1^{C_2} \end{pmatrix}$. The theorem gives the E^1- and E^2-pages of the Atiyah-Hirzebruch spectral sequence as in Figure 5.4.

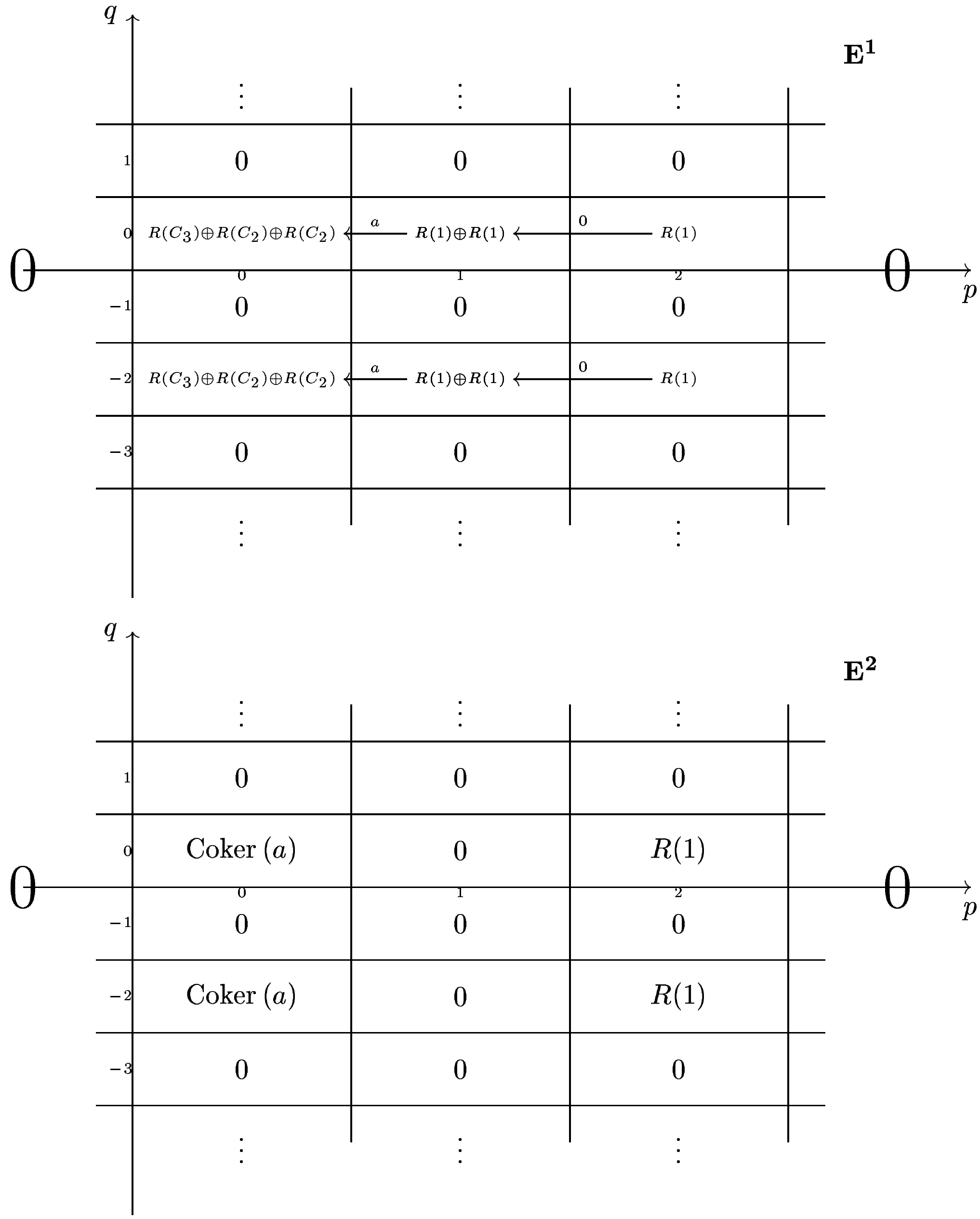


Figure 5.4. The E^1- and E^2-pages of the Atiyah-Hirzebruch spectral sequence for $K_0^{D_6}(\mathbb{C}P(\gamma))$.

From Figure 5.4 it is obvious that the spectral sequence collapses at the E^2-page. Thus reading off the diagonal $p+q=0$ gives the successive quotients in the filtration (5.3.7). This means, in interesting places, we have

$$E^2_{0,0} \cong \phi_0/0; \qquad E^2_{1,-1} \cong \phi_1/\phi_0; \qquad E^2_{2,-2} \cong \phi_2/\phi_1$$

so that $\phi_0 = \mathrm{Coker}\,(a)$, $\phi_1 = \mathrm{Coker}\,(a)$, and if we put $A = \mathrm{Coker}\,(a)$ then

$$K^G_0(\mathbb{C}P(\gamma))/A \cong R(1)$$

as claimed.

Finally, the short exact sequence in the proposition is obvious from the chain complex (5.4.3) once we recall that $A = \mathrm{Coker}\,(a)$. Then additive considerations show that the rank of A is five. □

Cohomology

With the underlying interest of *duality* one wants to know what happens in cohomology.

Proposition 5.4.4. *There is an $R(D_6)$-module B of rank five (as an abelian group) such that*

$$K^0_{D_6}(\mathbb{C}P(\gamma))/B \cong R(1)$$

as $R(D_6)$-modules. Moreover, there is a short exact sequence

$$0 \longrightarrow B \longrightarrow R(C_3) \oplus R(C_2) \oplus R(C_2) \longrightarrow R(1) \oplus R(1) \longrightarrow 0$$

of $R(D_6)$-modules.

Proof. Looking at Figure 5.3 one sees that the chain complex for ordinary singular cohomology of $\mathbb{C}P(\gamma)/D_6$ (with integer coefficients) is

$$\mathbb{Z} \xleftarrow{\begin{pmatrix} 0 & 0 \end{pmatrix}} \mathbb{Z}\oplus\mathbb{Z} \xleftarrow{\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}} \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}. \tag{5.4.5}$$

Codegree 2 1 0

Thus the cochain complex $C^*_G(\mathbb{C}P(\gamma);\mathbf{R})$ is

$$R(1) \xleftarrow{\ 0\ } R(1)\oplus R(1) \xleftarrow{\ b\ } R(C_3)\oplus R(C_2)\oplus R(C_2), \tag{5.4.6}$$

Codegree 2 1 0

where $b = \begin{pmatrix} \mathrm{Res}_1^{C_3} & -\mathrm{Res}_1^{C_2} & 0 \\ \mathrm{Res}_1^{C_3} & 0 & -\mathrm{Res}_1^{C_2} \end{pmatrix}$.

Theorem 5.3.6 gives the E_1- and E_2-pages of the Atiyah-Hirzebruch spectral sequence as in Figure 5.5.

From Figure 5.5 it is obvious that the spectral sequence collapses at the E^2-page. Of course, we need the dual of the filtration (5.3.7), which is the filtration

$$0 \subseteq \phi_n \subseteq \phi_{n-1} \subseteq \cdots \subseteq \phi_0 = K^0_G(X) \tag{5.4.7}$$

of $K^0_G(X)$ in which $\phi_i = \mathrm{Ker}\,(K^0_G(X) \longrightarrow K^0_G(X^{(i)}))$ for $0 \le i \le n$.

Remembering that $X = \mathbb{C}P(\gamma)$ and so $n = 2$ we read from the diagonal and see that

$$\phi_2 \cong E_2^{0,0}; \qquad \phi_1/\phi_2 \cong E_2^{1,-1}; \qquad \phi_0/\phi_1 \cong E_2^{2,-2}.$$

Thus $\phi_2 = \phi_1 = \mathrm{Ker}\,(b)$ and, if we put $B = \mathrm{Ker}\,(b)$ then

$$K^0_{D_6}(\mathbb{C}P(\gamma))/B \cong R(1)$$

as claimed.

Finally, the short exact sequence in the proposition is obvious from the chain complex (5.4.6) once we recall that $B = \mathrm{Ker}\,(b)$. Then additive considerations show that the rank of B is five. □

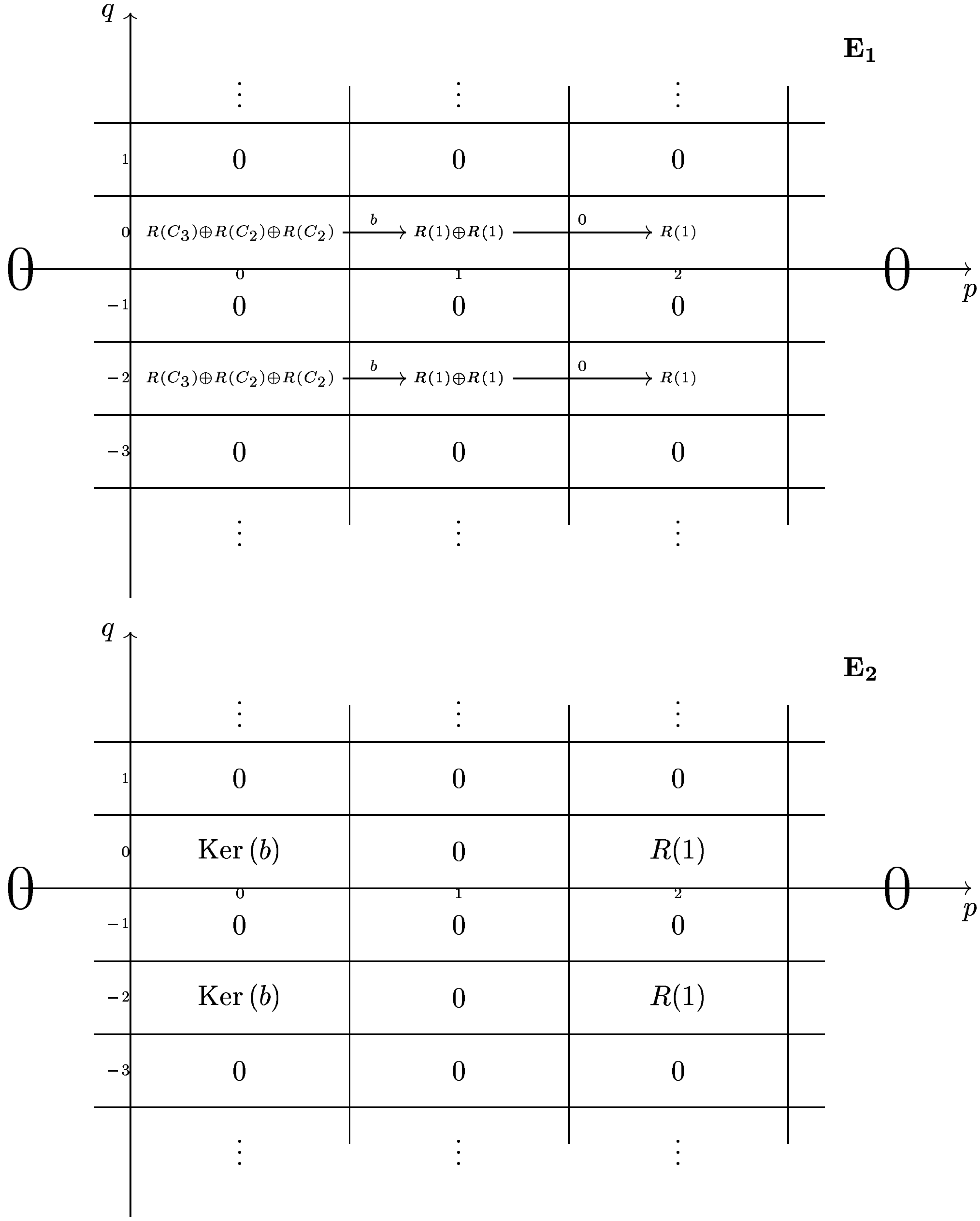


Figure 5.5. The E_1- and E_2-pages of the Atiyah-Hirzebruch spectral sequence for $K^0_{D_6}(\mathbb{C}P(\gamma))$.

5.4.2 The dihedral group of order eight

As in §2.3.2, take G to be the dihedral group D_8 of order eight and let $V = \gamma$ be the unique two dimensional simple representation of G. Recall from Proposition 2.3.2 that there is a D_8-CW-complex structure for $\mathbb{C}P(\gamma)$ given by

$$\Big((D_8/V_4 \amalg D_8/V_4' \amalg D_8/C_4) \times e^0\Big) \bigcup_{f_1} \Big((D_8/H \amalg D_8/H') \times e^1\Big) \bigcup_{f_2} \Big(D_8/H \times e^2\Big)$$

where the attaching maps are as shown in Figures 2.10 and 2.11. There is only one D_8-2-cell which means we need consider only one of the panels shown in Figure 2.10, say $eC_2 \times e^2$, as in Figure 5.6.

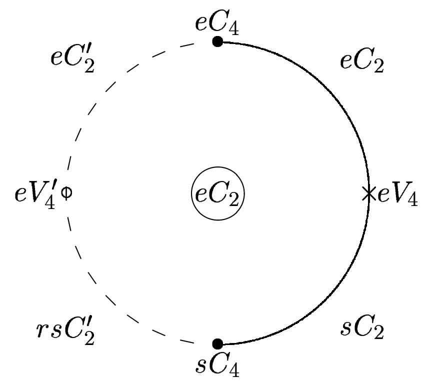


Figure 5.6. The panel $eC_2 \times e^2$ of Figure 2.10.

Thus we see that the quotient $\mathbb{C}P(\gamma)/D_8$ is as in Figure 5.7.

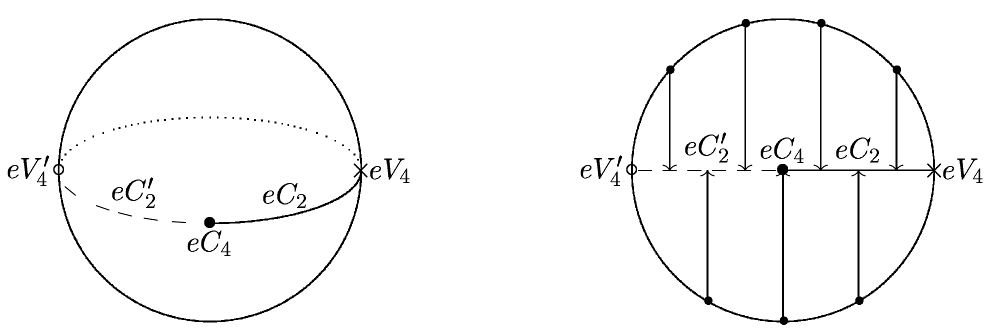


Figure 5.7. The quotient $\mathbb{C}P(\gamma)/D_8$ (left) and the attaching map $\partial e^2 \longrightarrow (\mathbb{C}P(\gamma))^{(1)}$ (right). The dots are only to emphasise the shape of the sphere and are *not* part of the CW-structure.

Observe that the picture of $\mathbb{C}P(\gamma)/D_8$ in Figure 5.7 is of precisely the same form as the picture of $\mathbb{C}P(\gamma)/D_6$ in Figure 5.3 – the only difference is the labelling with isotropy groups. The upshot of this is that the singular chain and cochain complexes are precisely as in (5.4.2) and (5.4.5). Accordingly, $C_*^G(\mathbb{C}P(\gamma);\mathbf{R})$ is the chain complex

$$\begin{array}{lccc} & R(C_2) \xrightarrow{\;0\;} & R(C_2)\oplus R(C_2) \xrightarrow{\;a\;} & R(C_4)\oplus R(V_4)\oplus R(V_4) \\ \textit{Degree} & 2 & 1 & 0 \end{array} \tag{5.4.8}$$

where $a = \begin{pmatrix} \mathrm{Ind}_{C_2}^{C_4} & \mathrm{Ind}_{C_2}^{C_4} \\ -\mathrm{Ind}_{C_2}^{V_4} & 0 \\ 0 & -\mathrm{Ind}_{C_2}^{V_4} \end{pmatrix}$, and $C^*_G(\mathbb{C}P(\gamma);\mathbf{R})$ is the cochain complex

$$\begin{array}{lccc} & R(C_2) \xleftarrow{\;0\;} & R(C_2)\oplus R(C_2) \xleftarrow{\;b\;} & R(C_4)\oplus R(V_4)\oplus R(V_4) \\ \textit{Codegree} & 2 & 1 & 0 \end{array} \tag{5.4.9}$$

in which $b = \begin{pmatrix} \mathrm{Res}_{C_2}^{C_4} & -\mathrm{Res}_{C_2}^{V_4} & 0 \\ \mathrm{Res}_{C_2}^{C_4} & 0 & -\mathrm{Res}_{C_2}^{V_4} \end{pmatrix}$.

Proposition 5.4.10. *Take $G = D_8$ and let $V = \gamma$ be as above. Then there are $R(D_8)$-modules A and B of rank eight (as abelian groups) such that*

$$K_0^G(\mathbb{C}P(\gamma))/A \cong R(C_2) \text{ and } K_G^0(\mathbb{C}P(\gamma))/B \cong R(C_2)$$

as $R(D_8)$-modules. Moreover, there are short exact sequences

$$0 \longrightarrow R(C_2)\oplus R(C_2) \longrightarrow R(C_4)\oplus R(V_4)\oplus V_4 \longrightarrow A \longrightarrow 0;$$
$$0 \longrightarrow B \longrightarrow R(C_4)\oplus R(V_4)\oplus R(V_4) \longrightarrow R(C_2)\oplus R(C_2) \longrightarrow 0$$

of $R(D_8)$-modules.

Proof. The (co)chain complexes (5.4.8) and (5.4.9) and Theorem 5.3.6 give the E^1- and E^2-pages of the Atiyah-Hirzebruch (homology) spectral sequence as in Figure 5.8, and the E_1- and E_2-pages of the Atiyah-Hirzebruch (cohomology) spectral sequence as in Figure 5.9.

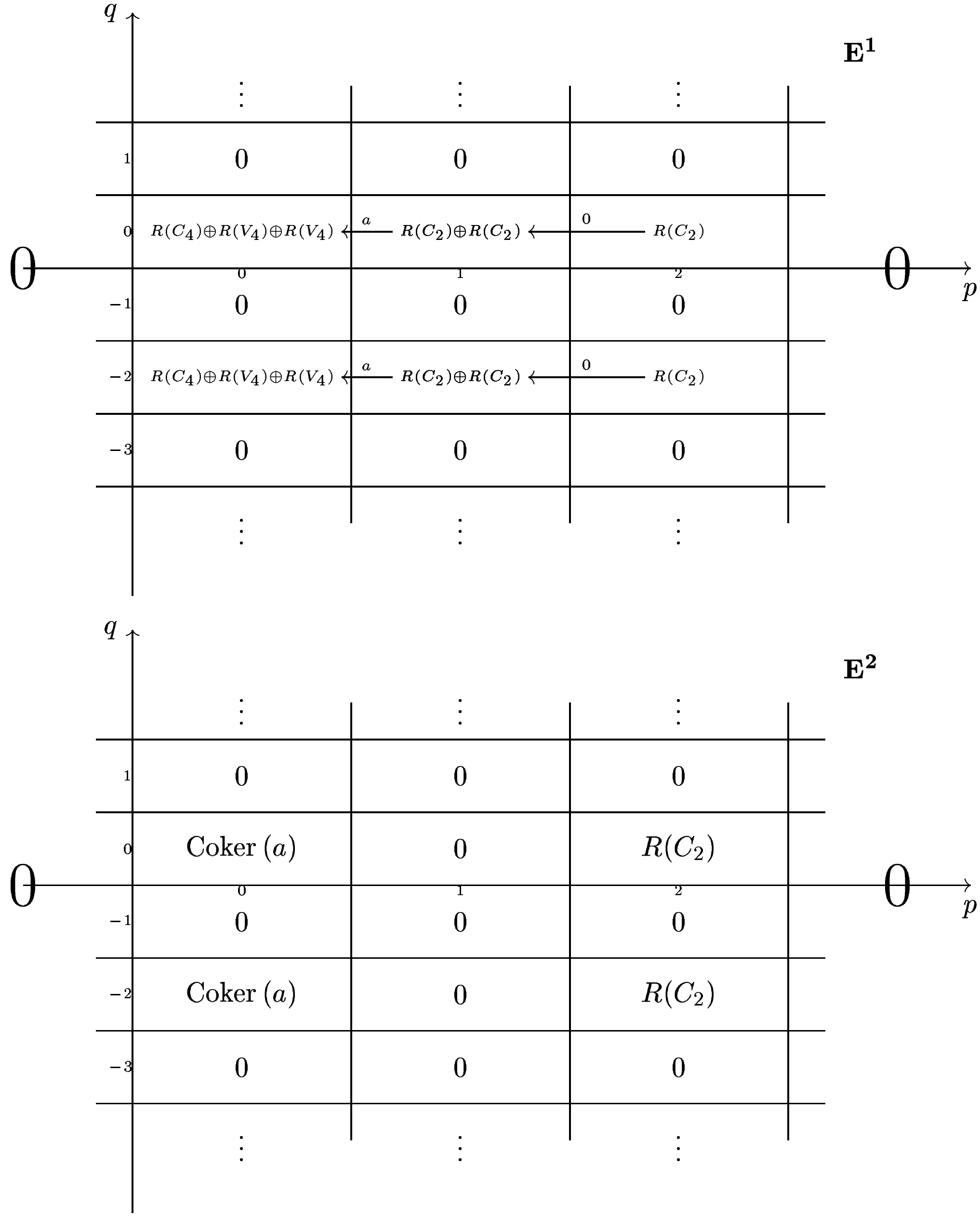


Figure 5.8. The E^1- and E^2-pages of the Atiyah-Hirzebruch spectral sequence for $K_0^{D_8}(\mathbb{C}P(\gamma))$.

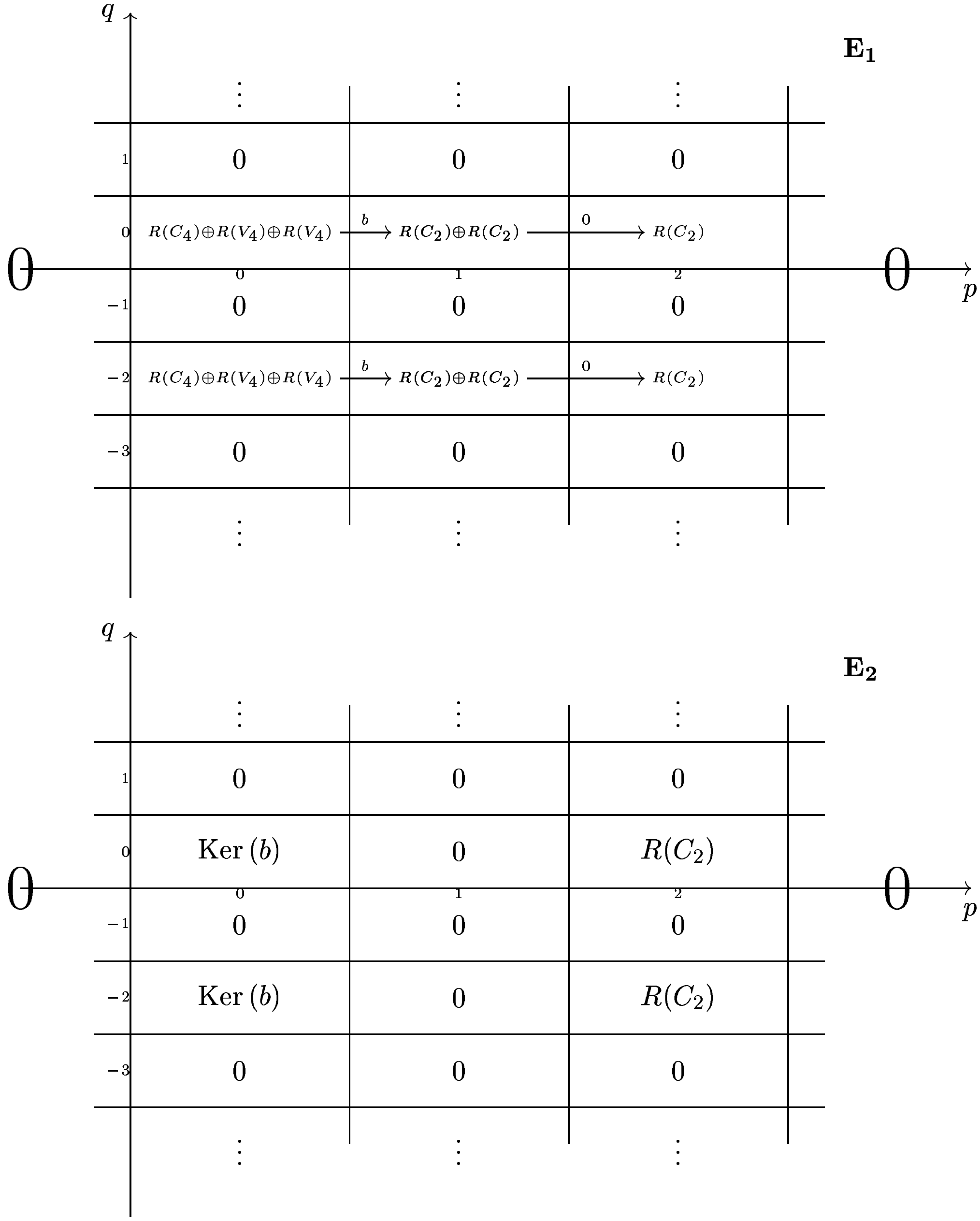


Figure 5.9. The E_1- and E_2-pages of the Atiyah-Hirzebruch spectral sequence for $K^0_{D_8}(\mathbb{C}P(\gamma))$.

The spectral sequences collapse at their second pages. Write $A = \mathrm{Coker}\,(a)$ and $B = \mathrm{Ker}\,(b)$. Using the filtration (5.3.7) for $K_0^{D_8}(\mathbb{C}P(\gamma))$ one reads from the relevant diagonal that

$$\phi_0 \cong E^2_{0,0} = A; \qquad \phi_1/\phi_0 \cong E^2_{1,-1} = 0; \qquad \phi_2/\phi_1 \cong E^2_{2,-2} = R(C_2).$$

Thus $\phi_1 \cong A$ whence $K_0^{D_8}(\mathbb{C}P(\gamma))/A \cong R(C_2)$. Similarly, using the filtration (5.4.7) for $K^0_{D_8}(\mathbb{C}P(\gamma))$ one reads from the relevant diagonal that

$$\phi_2 \cong E_2^{0,0} = B; \qquad \phi_1/\phi_2 \cong E_2^{1,-1} = 0; \qquad \phi_0/\phi_1 \cong E_2^{2,-2} = R(C_2).$$

Thus $\phi_1 \cong B$ whence $K^0_{D_8}(\mathbb{C}P(\gamma))/B \cong R(C_2)$.

The short exact sequences in the proposition are then immediate from the (co)chain complexes (5.4.8) and (5.4.9), and additive considerations show that A, B have rank eight.

□

5.4.3 The quaternion group of order eight

As in §2.3.3, take G to be the quaternion group Q_8 of order eight and let $V = \gamma$ be the unique two dimensional simple representation of G. Recall from Proposition 2.3.3 that there is a Q_8-CW-complex structure for $\mathbb{C}P(\gamma)$ given by

$$\left((Q_8/C_4^i \amalg Q_8/C_4^j \amalg Q_8/C_4^k) \times e^0\right) \bigcup_{f_1} \left((Q_8/C_2 \amalg Q_8/C_2') \times e^1\right) \bigcup_{f_2} \left(Q_8/C_2 \times e^2\right)$$

where the attaching maps are as shown in Figures 2.12 and 2.13. There is only one Q_8-2-cell which means we need consider only one of the panels shown in Figure 2.12, say $eC_2 \times e^2$, as in Figure 5.10.

Thus we see that the quotient $\mathbb{C}P(\gamma)/Q_8$ is as in Figure 5.11.

Proposition 5.4.11. *Take $G = Q_8$ and let $V = \gamma$ be as above. Then there are $R(Q_8)$-modules A and B of rank eight (as abelian groups) such that*

$$K_0^G(\mathbb{C}P(\gamma))/A \cong R(C_2) \text{ and } K^0_G(\mathbb{C}P(\gamma))/B \cong R(C_2)$$

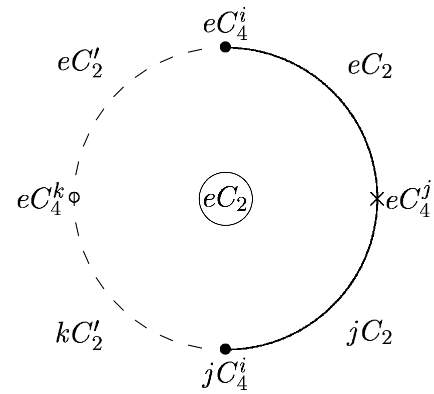


Figure 5.10. The panel $eC_2 \times e^2$ of Figure 2.12.

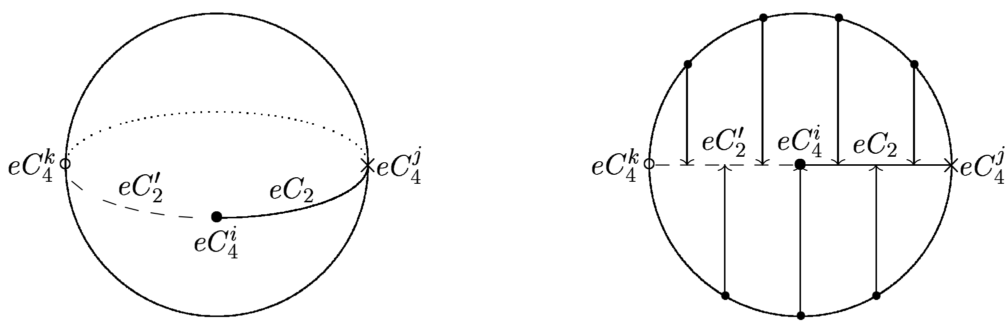


Figure 5.11. The quotient $\mathbb{C}P(\gamma)/Q_8$ (left) and the attaching map $\partial e^2 \longrightarrow (\mathbb{C}P(\gamma))^{(1)}$ (right). The dots are only to emphasise the shape of the sphere and are *not* part of the CW-structure.

as $R(Q_8)$-modules. Moreover, there are short exact sequences

$$0 \longrightarrow R(C_2) \oplus R(C_2) \longrightarrow R(C_4) \oplus R(C_4) \oplus R(C_4) \longrightarrow A \longrightarrow 0;$$

$$0 \longrightarrow B \longrightarrow R(C_4) \oplus R(C_4) \oplus R(C_4) \longrightarrow R(C_2) \oplus R(C_2) \longrightarrow 0$$

of $R(Q_8)$-modules.

Proof. Once again the picture of the quotient $\mathbb{C}P(\gamma)/Q_8$ looks just like the picture of the quotient $\mathbb{C}P(\gamma)/D_6$. So we use the singular (co)chain complexes (5.4.2) and (5.4.5) to see that $C_*^G(\mathbb{C}P(\gamma); \mathbf{R})$ is the chain complex

$$R(C_2) \xrightarrow{\;0\;} R(C_2) \oplus R(C_2) \xrightarrow{\;a\;} R(C_4) \oplus R(C_4) \oplus R(C_4), \tag{5.4.12}$$

Degree 2 1 0

in which $a = \begin{pmatrix} \mathrm{Ind}_{C_2}^{C_4} & \mathrm{Ind}_{C_2}^{C_4} \\ -\mathrm{Ind}_{C_2}^{C_4} & 0 \\ 0 & -\mathrm{Ind}_{C_2}^{C_4} \end{pmatrix}$, and that $C_G^*(\mathbb{C}P(\gamma); \mathbf{R})$ is the cochain complex

$$R(C_2) \xleftarrow{\ 0\ } R(C_2) \oplus R(C_2) \xleftarrow{\ b\ } R(C_4) \oplus R(C_4) \oplus R(C_4), \tag{5.4.13}$$

Codegree 2 1 0

in which $b = \begin{pmatrix} \mathrm{Res}_{C_2}^{C_4} & -\mathrm{Res}_{C_2}^{C_4} & 0 \\ \mathrm{Res}_{C_2}^{C_4} & 0 & -\mathrm{Res}_{C_2}^{C_4} \end{pmatrix}$.

Thus the Atiyah-Hirzebruch spectral sequences are as in Figure 5.12 (homology) and Figure 5.13 (cohomology).

Take $A = \mathrm{Coker}\,(a)$ and $B = \mathrm{Ker}\,(b)$ and proceed as in the proof of Proposition 5.4.10 to deduce the proposition. □

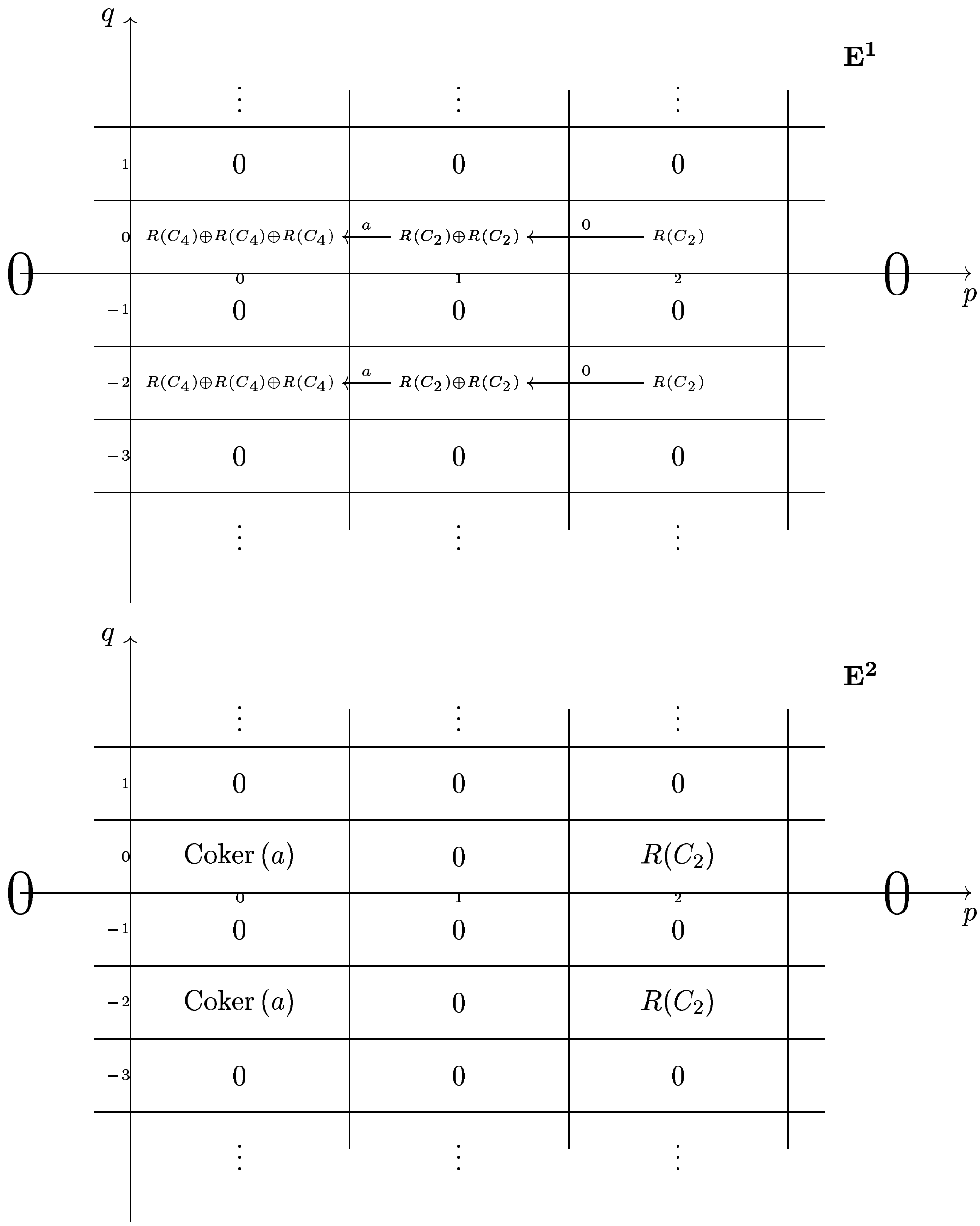


Figure 5.12. The E^1- and E^2-pages of the Atiyah-Hirzebruch spectral sequence for $K_0^{Q_8}(\mathbb{C}P(\gamma))$.

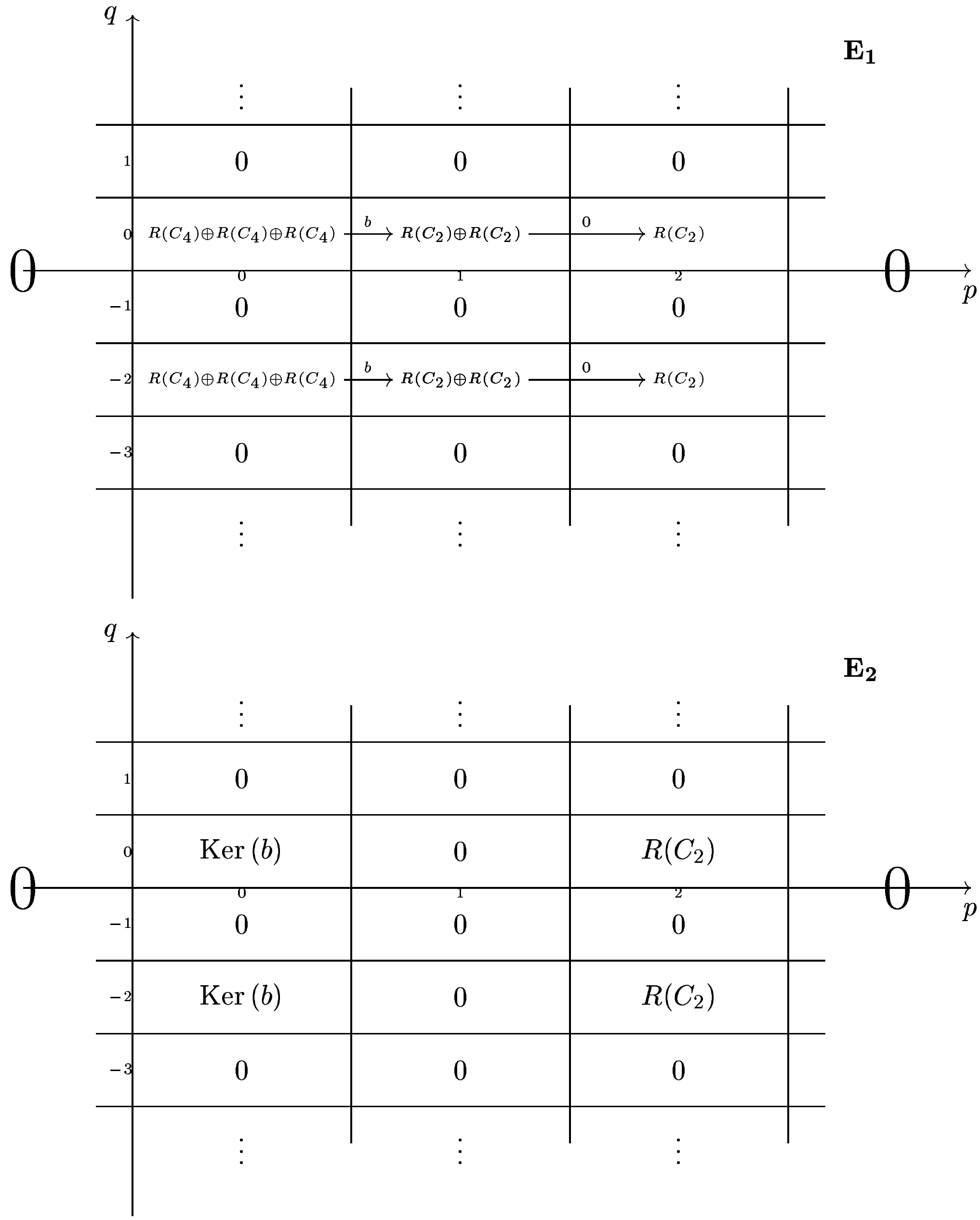


Figure 5.13. The E_1- and E_2-pages of the Atiyah-Hirzebruch spectral sequence for $K^0_{Q_8}(\mathbb{C}P(\gamma))$.

Chapter 6

Conclusions

6.1 Review

At page 153 we should have *done* something. It is now that we review what we have done and ask *what have we achieved?* We began in Chapter 1 by studying $\mathbb{C}P(V)$, explaining why this is a suitable equivariant generalisation of $\mathbb{C}P^n$. Our next step was breaking $\mathbb{C}P(V)$ into pieces – in other words devising the G-cellular structures of Chapter 2.

To some extent, Chapter 2 is at risk of being described as nothing more than a collection of examples. The first defence is that the examples are *useful*, as we found in Chapter 5. Often in mathematics, one needs a platform to deduce conclusions which are independent of the choice of platform. Our G-cell structures are a typical example: of course there is a plethora of G-cellular structures on $\mathbb{C}P(V)$ (even for finite G and finite dimensional V) but any one would do for the spectral sequence calculations of Chapter 5. It so happens that we managed to pick convenient structures so that in every example of §5.4 we could fall back to the same underlying ordinary (co)chain complex. This was not by good fortune alone: indeed the author spent many hours in laborious computation with complicated cellular structures before finding economic structures which gave the same results. The second defence is that the cellular structures are *pretty*. The author is jubilant to discover mathematical ideas leading to relevant pictures such as Figures 2.3, 2.15 and 2.17.

Chapter 3 is a splendid example of using a platform to gain an independent result.

Recall that we used the flag

$$\mathcal{F} = \Big(0 \subset \alpha_1 \subset \alpha_1 \oplus \alpha_2 \subset \cdots \subset \alpha_1 \oplus \cdots \oplus \alpha_n = V\Big) \tag{3.3.21}$$

to construct our favourite homology class $\sum_{i=0}^{n-1} \beta_i^{\mathcal{F}}$. In §3.5 we showed that $\sum_{i=0}^{n-1} \beta_i^{\mathcal{F}}$ is independent of $\mathcal{F}$, justifying the notation $\sum_{i=0}^{n-1} \beta_i$. Precisely we had Theorem 3.5.10. Of course, in Chapter 3, G was everywhere abelian but in Chapter 4 we addressed that problem.

Chapter 5 seems at first sight to be a little out of place. We computed $K_G^0(\mathbb{C}P(V))$ and $K_0^G(\mathbb{C}P(V))$ in Chapter 3 so why bother using a spectral sequence to achieve a similar result? The answer is the same as the reasoning behind Chapter 2: we want to break things into small pieces in order to gain a better understanding. Whilst we know that

$$K_G^0(\mathbb{C}P(V)) = \frac{R(G)[z]}{(\chi(V \otimes z))}, \tag{3.3.13}$$

the task of understanding $R(G)$ for a given G may be non-trivial. At least the methods of Chapter 5 permit us to break things into pieces which are easier to understand.

So far so good: we have done plenty. But *what was the purpose?* Recall from §1.2.3 that we hoped to understand *Poincaré duality in equivariant K-theory for $\mathbb{C}P(V)$*. Back in §3.6 we explained how to use the *perfect pairing* (3.6.5) to construct an isomorphism

$$K_A^0(\mathbb{C}P(V)) \cong K_0^A(\mathbb{C}P(V)). \tag{3.6.2}$$

Could we call this Poincaré duality*?*

6.2 Fundamental classes and Poincaré duality

In the present section we formulate what we mean by *Poincaré duality in equivariant K-theory for $\mathbb{C}P(V)$*. We afford every effort to develop a treatment parallel to the non-equivariant case of §1.1.3. As was promised in that section, we now discuss in greater generality what we mean by a *fundamental class*. We begin by recalling the non-equivariant treatment of May [45 (Chapter 20)] before making our equivariant generalisations.

6.2.1 Fundamental classes and Poincaré duality in E-theory

This section should be viewed as a preamble to §6.2.1.G. That is, we present here the non-equivariant case for the purpose of drawing parallels when we come to the equivariant material.

Let E be a homology theory. Take $M = M^n$ to be a compact manifold of dimension n. Take $x \in M$ and let U be an open neighbourhood of x in M with U homeomorphic to a neighbourhood of 0 in $\mathbb{R}^n$ via a homeomorphism taking x to $0 \in \mathbb{R}^n$. Consider the pair $(M, M \setminus \{x\})$.

Lemma 6.2.1. *For each i, there are isomorphisms*

(i) $E_i(M, M \setminus \{x\}) \cong E_i(U, U \setminus \{x\})$;

(ii) $E_i(U, U \setminus \{x\}) \cong E_i(S^n, S^n \setminus \{x\})$;

(iii) $E_i(S^n, S^n \setminus \{x\}) \cong \widetilde{E}_i(S^n)$.

Proof. Recall [24] the excision axiom: *given a pair (X, A) and an open $V \subseteq X$ with $\bar{V} \subseteq \operatorname{Int}(A)$ there are isomorphisms $E_i(X \setminus V, A \setminus V) \cong E_i(X, A)$.* Take $X = M$, $A = M \setminus \{x\}$ and $V = M \setminus U$ to deduce (i), and for (ii) take $X = S^n$, $A = S^n \setminus \{x\}$ and $V = S^n \setminus U$. Finally, (iii) follows because $S^n \setminus \{x\}$ is homeomorphic to $\mathbb{R}^n$, which is contractible, so that

$$E_i(S^n, S^n \setminus \{x\}) \cong E_i(S^n, \mathrm{pt}) \cong \widetilde{E}_i(S^n).$$

□

Composing the three isomorphisms of Lemma 6.2.1, the outcome is that

$$E_*(M, M \setminus \{x\}) \cong \widetilde{E}_*(S^n)$$

and we can view $E_*(M, M \setminus \{x\})$ as a free $\widetilde{E}_*(S^n)$-module on one generator in degree n.

Definition 6.2.2 (Fundamental classes in E-theory). Given $x \in M$, consider the composite ϕ_x below.

$$\begin{array}{ccc} E_*(M) & \xrightarrow{\phi_x} & \widetilde{E}_*(S^n) \\ {\scriptstyle i_*^x}\big\downarrow & & \big\uparrow {\scriptstyle \cong\,(iii)} \\ E_*(M, M \setminus \{x\}) & & \\ {\scriptstyle (i)\,\cong}\big\downarrow & & \\ E_*(U, U \setminus \{x\}) & \xrightarrow[(ii)]{\cong} & E_*(S^n, S^n \setminus \{x\}) \end{array} \tag{6.2.3}$$

The maps labelled (i), (ii), (iii) are the corresponding isomorphisms of Lemma 6.2.1 and

$$i_*^x : E_*(M) \cong E_*(M, \emptyset) \longrightarrow E_*(M, M \setminus \{x\})$$

is the map induced by inclusion of the pairs $(M, \emptyset) \xhookrightarrow{i^x} (M, M \setminus \{x\})$. An element $\mu \in E_n(M)$ is an *E-fundamental class* for M if, for all $x \in M$, the image $\phi_x(\mu)$ is an ($\widetilde{E}_*(S^n)$-module) generator for $\widetilde{E}_*(S^n)$. In this case one writes $[M]$ for such a μ.

Once one has a fundamental class $[M]$ for M, one obtains what one really wants, a *Poincaré duality* isomorphism $E^*(M) \xrightarrow{\cong} E_*(M)$. The final ingredient is the *cap product*. We present the definition as in [41 (III, §3)]. Observe the parallel with Definition 1.1.6.

Definition 6.2.4 (Cap products in E-theory). Let E be a commutative ring spectrum and X a CW-complex. We define the *cap product*

$$(-) \cap (-) : E^*(X) \otimes E_*(X) \longrightarrow E_*(X)$$

by taking $(X \xrightarrow{c} E) \cap (\mathbb{S} \xrightarrow{h} E \wedge X)$ to be the composite

$$\mathbb{S} \xrightarrow{h} E \wedge X \xrightarrow{1 \wedge \Delta} E \wedge X \wedge X \xrightarrow{1 \wedge c \wedge 1} E \wedge E \wedge X \xrightarrow{\mu \wedge 1} E \wedge X.$$

Recalling the classical Poincaré duality theorem (Theorem 1.1.7) we state here its counterpart in E-theory.

Theorem 6.2.5 (Poincaré duality in E-theory). *Let E be a generalised (co)homology theory. If M is a manifold with E-fundamental class $[M]$ then there is an isomorphism*

$$E^*(M) \xrightarrow{\cong} E_*(M)$$

given by capping with the fundamental class, *precisely $a \longmapsto a \cap [M]$ for $a \in E^*(M)$.* □

We shall comment on the proof in Remark 6.2.12 on page 159.

6.2.1.G Fundamental classes and Poincaré duality in equivariant complex stable theories

Our aim here is to develop an equivariant analogue to §6.2.1. We proceed with a discussion punctuated by questions and answers. Let E^G be a complex stable G-equivariant homology theory and take a smooth G-manifold M.

Question 1. *Why do we need a* smooth *G-manifold?*

Answer 1. Non-equivariantly there was nothing to choose: every point of a manifold is locally like $\mathbb{R}^n$. Equivariantly, one might hope to replace $\mathbb{R}^n$ with some representation but it is not clear which to choose. Taking a smooth manifold allows us to apply the *Slice Theorem* [14 (II, Theorem 5.4)]. This asserts that given $x \in M$ with isotropy group H_x, there is a neighbourhood U of the orbit $\{Gx\}$ which is G-homeomorphic to $G \times_{H_x} A$ for some H_x-space A. Since M was a smooth manifold we can take A to be a $\mathbb{C}H_x$-module V_x. (We appeal to the uniqueness result [14 (VI, Theorem 2.6)] to see that there is no choice for A.)

Question 2. *Why do we need a* complex stable *theory?*

Answer 2. Non-equivariantly, $[M]$ was a class in $E_n(M)$. Since we replace (the representation) n with V_x we should seek a class in $E^G_{V_x}(M)$. The complex stability means that our theory is sufficiently nice that morally we still want a class in $E^G_n(M)$.

To proceed we need a G-version of Lemma 6.2.1.

Question 3. *Can we replace the "E" with "E^G" in Lemma 6.2.1?*

Answer 3. No. Given $x \in M$ the pair $(M, M \setminus \{x\})$ might not be a G-pair. (Even if x is G-fixed, we shall eventually want a clause "for all $x \in M$" and assuming *all* points of M to be G-fixed would lead to an unsatisfactory equivariant treatment.)

But of course replacing x with its G-orbit, $(M, M \setminus \{Gx\})$ *does* give a G-pair. Then we can generalise Lemma 6.2.1. Take $x \in M$. As in Answer 1 above, let U be a neighbourhood of $\{Gx\}$ in M, homeomorphic to a neighbourhood of $G \times_{H_x} \{0\}$ in $G \times_{H_x} V_x$. Consider the G-pair $(M, M \setminus \{Gx\})$.

Lemma 6.2.6. *For each i, there are isomorphisms*

(i) $E_i^G(M, M \setminus \{Gx\}) \cong E_i^G(U, U \setminus \{Gx\})$;

(ii) $E_i^G(U, U \setminus \{Gx\}) \cong E_i^G(G \times_{H_x} S^{V_x}, (G \times_{H_x} S^{V_x}) \setminus \{Gx\})$;

(iii) $E_i^G(G \times_{H_x} S^{V_x}, (G \times_{H_x} S^{V_x}) \setminus \{Gx\}) \cong \widetilde{E}_i^G(G_+ \wedge_{H_x} S^{V_x})$.

Proof. For (i) and (ii), proceed just as in Lemma 6.2.1 using the G-excision axiom [31]. Corollary 4.2.21[1] means that (iii) is equivalent to showing that

$$E_i^{H_x}(S^{V_x}, S^{V_x} \setminus \{0\}) \cong \widetilde{E}_i^{H_x}(S^{V_x}),$$

and this follows since $S^{V_x} \setminus \{0\} \cong_G V_x$ which is contractible. □

Composing the three isomorphisms of Lemma 6.2.6, the outcome is that

$$E_*^G(M, M \setminus \{Gx\}) \cong \widetilde{E}_*^{H_x}(S^{V_x}), \tag{6.2.7}$$

and we can view $E_*^G(M, M \setminus \{Gx\})$ as a free $\widetilde{E}_*^{H_x}(S^{V_x})$-module on one generator.

Definition 6.2.8 (Fundamental classes in $\mathbf{E_G}$-theory). Let M be a smooth n-manifold with G-action and let E^G be a complex stable G-equivariant homology theory. Write H_x for the isotropy group of $x \in M$ and let V_x be as in Lemma 6.2.6. Consider the composite ϕ_{Gx} below.

$$\begin{array}{ccc}
E_*^G(M) & \xrightarrow{\quad\phi_{Gx}\quad} & \widetilde{E}_*^{H_x}(S^{V_x}) \\
{\scriptstyle i_*^{Gx}}\downarrow & & \uparrow{\scriptstyle \cong\ \phi^{-1}} \\
E_*^G(M, M \setminus \{Gx\}) & & \widetilde{E}_*^G(G_+ \wedge_{H_x} S^{V_x}) \\
{\scriptstyle (i)\ \cong}\downarrow & & \uparrow{\scriptstyle \cong\ (iii)} \\
E_*^G(U, U \setminus \{Gx\}) & \xrightarrow[(ii)]{\cong} & E_*^G(G \times_{H_x} S^{V_x}, (G \times_{H_x} S^{V_x}) \setminus \{Gx\})
\end{array} \tag{6.2.9}$$

The maps labelled (i), (ii), (iii) are the corresponding isomorphisms of Lemma 6.2.6, ϕ is the isomorphism of Corollary 4.2.21 and

$$i_*^{Gx} : E_*^G(M) \cong E_*^G(M, \emptyset) \longrightarrow E_*^G(M, M \setminus \{Gx\})$$

[1] Corollary 4.2.21 was for equivariant K-theory but its proof can easily be modified to equivariant E-theory.

is the map induced by G-inclusion of the G-pairs $(M, \emptyset) \xhookrightarrow{i^{Gx}} (M, M \setminus \{Gx\})$. An element $\mu \in E_n^G(M)$ is an E^G*-fundamental class* for M if the image $\phi_{Gx}(\mu)$ is an $(\widetilde{E}_*^{H_x}(S^{V_x})$-module) generator for $\widetilde{E}_*^{H_x}(S^{V_x})$ for all $x \in M$, in which case one writes $[M]$ for such a μ.

Definition 6.2.10 (Cap products in $\mathbf{E_G}$-theory). Adapt Definition 6.2.4 by working with a commutative ring G-spectrum and a G-CW-complex. Everything else goes through unchanged.

Theorem 6.2.11 (Poincaré duality in $\mathbf{E_G}$-theory). *Let E_G be a complex stable theory. If M is a smooth G-manifold with E_G-fundamental class $[M]$ then there is an isomorphism*

$$E_G^*(M) \xrightarrow{\cong} E_*^G(M)$$

given by capping with the fundamental class, *precisely $a \longmapsto a \cap [M]$ for $a \in E_G^*(M)$.* □

Remark 6.2.12. Both May [45 (Chapter 20)] and Greenberg and Harper [28 (§26)] supply an explicit proof when E is singular (co)homology and $G = 1$. The proof proceeds by showing that $(-) \cap [M]$ induces an isomorphism on larger and larger subsets of M, starting from a point, and using Mayer-Vietoris sequences together with excision. The only difference in the equivariant case is that we must start with a G-point, in other words the orbit Gx for $x \in M$. But that is taken care of in Definition 6.2.8. Thus the proof can be adapted for Theorems 6.2.5 and 6.2.11.

6.2.2 Fundamental classes in equivariant K-theory for $\mathbb{C}P(V)$

We finally make a satisfactory statement.

Theorem 6.2.13. *Let G be a finite group and V a finite dimensional representation of G, $\dim_{\mathbb{C}}(V) = n$. Take basis $\{(1-z)^j\}_{j=0}^{n-1}$ for $K_G^0(\mathbb{C}P(V))$ and let $\{\beta_i\}_{i=0}^{n-1}$ be the dual basis for $K_0^G(\mathbb{C}P(V))$. Then*

$$\frac{1}{\chi(V \otimes z)} = \sum_{i=0}^{n-1} \beta_i \in K_0^G(\mathbb{C}P(V))$$

is a fundamental class in equivariant K-theory for $\mathbb{C}P(V)$.

In order to prove Theorem 6.2.13 we proceed with a series of intermediate results. For brevity, we shall often write V_z for $V \otimes z$, etc.

Lemma 6.2.14. *Take $x \in \mathbb{C}P(V)$ and assume that x is G-fixed, i.e. that $Gx = \{x\}$. Then $i_*^{Gx}\left(\frac{1}{\chi(V\otimes z)}\right)$ is an $R(G\times T)$-generator for $K_0^G(\mathbb{C}P(V), \mathbb{C}P(V)\setminus\{Gx\})$.*

Proof. Write α for x viewed as a 1-dimensional $\mathbb{C}G$-module, so $V \cong_{\mathbb{C}G} W \oplus \alpha$ for some $\mathbb{C}G$-module W. Then

$$\mathbb{C}P(V)\setminus\{Gx\} = \mathbb{C}P(V)\setminus\mathbb{C}P(\alpha) \simeq \mathbb{C}P(W),$$

where the homotopy equivalence is Proposition 1.3.7. Thus we are required to prove that $K_0^G(\mathbb{C}P(V), \mathbb{C}P(W))$ is $R(G\times T)$-generated by $i_*^{Gx}\left(\frac{1}{\chi(V\otimes z)}\right)$. Noting that

$$\begin{aligned}
K_0^G(\mathbb{C}P(V), \mathbb{C}P(W)) &\cong \widetilde{K}_0^G(\mathbb{C}P(V)/\mathbb{C}P(W))\\
&= [\mathbb{S}, \mathbb{K}\wedge(\mathbb{C}P(V)/\mathbb{C}P(W))]_G\\
&= [\mathbb{S}, \mathbb{K}\wedge((S(V_z)/S(W_z))/T)]_G\\
&\overset{\text{Adams}}{\cong} [\mathbb{S}, \Sigma^{-1}\mathbb{K}\wedge(S(V_z)/S(W_z))]_{G\times T}\\
&= \widetilde{K}_0^{G\times T}(\Sigma^{-1}S(V_z)/S(W_z))\\
&\cong K_1^{G\times T}(S(V_z), S(W_z)),
\end{aligned}$$

where we have indicated use of the Adams isomorphism, equivariant Bott periodicity permits us to work with $K_{-1}^{G\times T}(S(V_z), S(W_z))$.

Now, we claim the existence of a short exact sequence

$$0 \longrightarrow \widetilde{K}_0^{G\times T}(S^{W_z}) \overset{\chi(\alpha_z)}{\longrightarrow} \widetilde{K}_0^{G\times T}(S^{V_z}) \overset{b}{\longrightarrow} K_{-1}^{G\times T}(S(V_z), S(W_z)) \longrightarrow 0. \tag{6.2.15}$$

This follows from the homology sequence [58 (p201)] of the G-triple $(D(V_z), S(V_z), S(W_z))$, equivariant Bott periodicity, Lemmas 3.3.5 and 3.4.3 and the fact[2] that

$$D(V_z)/S(W_z) \simeq D(W_z)/S(W_z) = S^{W_z}.$$

[2] Just contract the α_z-factor in $D(V_z)$.

Next we claim that we have a commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \widetilde{K}_0^{G\times T}(S^0) & \xrightarrow{\chi(W_z)} & \widetilde{K}_0^{G\times T}(S^{W_z}) & \longrightarrow & \widetilde{K}_{-1}^{G\times T}(S(W_z)_+) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \chi(\alpha_z)} & & \downarrow & & \\
0 & \longrightarrow & \widetilde{K}_0^{G\times T}(S^0) & \xrightarrow{\chi(V_z)} & \widetilde{K}_0^{G\times T}(S^{V_z}) & \xrightarrow{a} & \widetilde{K}_{-1}^{G\times T}(S(V_z)_+) & \longrightarrow & 0 \\
 & & & & {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} & & \\
 & & & & K_{-1}^{G\times T}(S(V_z),S(W_z)) & = & K_{-1}^{G\times T}(S(V_z),S(W_z)) & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

in which the rows and column are exact. This follows since the rows are just the short exact sequence (3.4.7), the centre column is (6.2.15), the right column comes from the pair $(S(V_z), S(W_z))$ and the commutativity is trivial since everything is induced by inclusion.

Recall from §3.2 that we have the Bott class $\tau(1) \in \widetilde{K}^0_{G\times T}(S^{V_z}) \cong R(G\times T) \cong \widetilde{K}_0^{G\times T}(S^{V_z})$. Write $\mathfrak{b}$ for the Bott class in $\widetilde{K}_0^{G\times T}(S^{V_z})$. Then showing that $i_*^{Gx}\left(\frac{1}{\chi(V\otimes z)}\right)$ gives a generator of $K_{-1}^{G\times T}(S(V_z), S(W_z))$ is, in the notation of the diagram, the same thing as showing that $ca(\mathfrak{b})$ is a generator, or equivalently that $b(\mathfrak{b})$ is an $R(G\times T)$-generator. Thus we work with (6.2.15). Clearly $\widetilde{K}_0^{G\times T}(S^{V_z})$ is $R(G\times T)$-generated by $\mathfrak{b}$, and therefore $K_{-1}^{G\times T}(S(V_z), S(W_z))$ is $R(G\times T)$-generated by $b(\mathfrak{b})$ as required. □

Proposition 6.2.16. *Under the hypotheses of Lemma 6.2.14, $i_*^{Gx}\left(\frac{1}{\chi(V\otimes z)}\right)$ is an $R(G)$-generator for $K_0^G(\mathbb{C}P(V), \mathbb{C}P(V)\setminus\{Gx\})$.*

For the proof of Proposition 6.2.16 we shall need the following result.

Corollary 6.2.17. *Regard $\alpha\otimes z$ as a homomorphism $G\times T \longrightarrow \mathbb{C}^\times$, and write κ for* Ker $(\alpha\otimes z)$. *Then we have*

$$K^0_{G\times T}(S(V_z), S(W_z)) \cong \widetilde{K}^0_{G\times T}((G\times T)/\kappa_+ \wedge S^{W_z}).$$

Proof. This is immediate from Lemma 4.2.1 and Proposition 1.3.8. □

Proof of Proposition 6.2.16. Using the same notation as in the proof of Lemma 6.2.14 we shall show that $K_{-1}^{G\times T}(S(V_z), S(W_z))$ is $R(G)$-generated by $b(\mathfrak{b})$.

We claim that the action of $z \in R(G \times T)$ is the same as the action of $\alpha^{-1} \in R(G)$. We choose to work with cohomology, where the module structure is more transparent. The required case will then follow as a formal consequence of duality. Taking dimension shift (arising from the Adams isomorphism) into account, we wish to show that the action of $z \in R(G \times T)$ on $K^0_{G\times T}(S(V_z), S(W_z))$ is the same as that of α^{-1}. Corollary 6.2.17 permits us instead to consider actions on $\widetilde{K}^0_{G\times T}((G \times T)/\kappa_+ \wedge S^{W_z})$, and by Proposition 4.3.4 these fit into the diagram

$$\begin{array}{ccc} \widetilde{K}^0_{G\times T} \times \widetilde{K}^0_{G\times T}((G\times T)/\kappa_+ \wedge S^{W_z}) & \xrightarrow{m} & \widetilde{K}^0_{G\times T}((G\times T)/\kappa_+ \wedge S^{W_z}) \\ \downarrow {\scriptstyle \mathrm{Res}^{G\times T}_{\kappa} \times \theta^{-1}} & & \cong \downarrow {\scriptstyle \theta^{-1}} \\ \widetilde{K}^0_{\kappa} \times \widetilde{K}^0_{\kappa}(S^{W_z}) & \xrightarrow{m} & \widetilde{K}^0_{\kappa}(S^{W_z}), \end{array} \tag{6.2.18}$$

where θ is as in Corollary 4.2.21.

Consider the κ-space $W_z = W \otimes z$. Recall from Remark 1.3.9 that we have a group isomorphism $F : \kappa \xrightarrow{\cong} G$ given by $F(g, \lambda) = g$ with inverse $F^{-1}(g) = (g, \alpha(g)^{-1})$. This allows us to make $W \otimes z$ into a G-space via

$$g \cdot (w \otimes 1) = F^{-1}(g)(w \otimes 1) = (g, \alpha(g)^{-1})(w \otimes 1).$$

But this is precisely the G-space $W \otimes \alpha^{-1}$. Thus we may trivially extend the diagram (6.2.18) to

$$\begin{array}{ccc} \widetilde{K}^0_{G\times T} \times \widetilde{K}^0_{G\times T}((G\times T)/\kappa_+ \wedge S^{W_z}) & \xrightarrow{m} & \widetilde{K}^0_{G\times T}((G\times T)/\kappa_+ \wedge S^{W_z}) \\ \downarrow {\scriptstyle \mathrm{Res}^{G\times T}_{\kappa} \times \theta^{-1}} & & \cong \downarrow {\scriptstyle \theta^{-1}} \\ \widetilde{K}^0_{\kappa} \times \widetilde{K}^0_{\kappa}(S^{W_z}) & \xrightarrow{m} & \widetilde{K}^0_{\kappa}(S^{W_z}) \\ {\scriptstyle (F^{-1})^*} \downarrow \cong & & \cong \downarrow {\scriptstyle (F^{-1})^*} \\ \widetilde{K}^0_{G} \times \widetilde{K}^0_{G}(S^{W\otimes\alpha^{-1}}) & \xrightarrow{m} & \widetilde{K}^0_{G}(S^{W\otimes\alpha^{-1}}). \end{array}$$

All we need do now is verify that $z, \alpha^{-1} \in R(G\times T)$ are both sent to $\alpha^{-1} \in R(G)$ under the composite $(F^{-1})^*\mathrm{Res}^{G\times T}_{\kappa}$. Now $(g, \lambda) \in G \times T$ acts as the scalar λ on $z \in R(G \times T)$, and therefore if $(g, \lambda) \in \kappa$ it acts as the scalar λ on $\mathrm{Res}^{G\times T}_{\kappa}(z)$. Thus $g \in G$ acts on $(F^{-1})^*\mathrm{Res}^{G\times T}_{\kappa}(z)$ as λ where $F^{-1}(g) = (g, \lambda)$. Recalling that

$$\kappa = \mathrm{Ker}\,(\alpha \otimes z) = \{(g, \lambda) \in G \times T \mid \alpha(g)\lambda = 1\}$$

we find that $g \in G$ acts as $\alpha(g)^{-1}$ on $(F^{-1})^*\mathrm{Res}_{\kappa}^{G\times T}(z)$ so that $(F^{-1})^*\mathrm{Res}_{\kappa}^{G\times T}(z) = \alpha^{-1}$ as required. Finally, the action of $(g,\lambda) \in G \times T$ on $\alpha^{-1} \in R(G\times T)$ is as the scalar $\alpha(g)^{-1}$ and so $(g,\lambda) \in \kappa$ acts as $\alpha(g)^{-1}$ on $\mathrm{Res}_{\kappa}^{G\times T}(\alpha^{-1})$. Thus the action of $\gamma \in G$ on $(F^{-1})^*\mathrm{Res}_{\kappa}^{G\times T}(\alpha^{-1})$ is as $\alpha(g)^{-1}$ where $F^{-1}(\gamma) = (g,\lambda)$. But $F^{-1}(\gamma) = (\gamma, \alpha(\gamma)^{-1})$ so we see that $(F^{-1})^*\mathrm{Res}_{\kappa}^{G\times T}(\alpha^{-1})$ is α^{-1} as claimed. □

We now relax our hypothesis and assume that $x \in \mathbb{C}P(V)$ has isotropy $H < G$. (Strictly, we should write H_x but the clutter is not helpful here.) Given G-inclusions $B \subseteq A \subseteq \mathbb{C}P(V)$, write i_B^A for the inclusion of pairs

$$i_B^A : (\mathbb{C}P(V), \mathbb{C}P(V)\setminus A) \hookrightarrow (\mathbb{C}P(V), \mathbb{C}P(V)\setminus B),$$

and to match the notation of Definition 6.2.8 we write i^B for $i_B^{\mathbb{C}P(V)}$.

Proposition 6.2.19. *We have a commutative diagram*

$$\begin{array}{ccccc}
K_0^G(\mathbb{C}P(V)) & \xrightarrow{(i^{Gx})_*} & K_0^G(\mathbb{C}P(V), \mathbb{C}P(V)\setminus\{Gx\}) & \xrightarrow[\text{(6.2.7)}]{\cong} & K_0^H \\
\downarrow{\scriptstyle \mathrm{Res}_H^G} & & \downarrow{\scriptstyle \mathrm{Res}_H^G} & & \| \\
K_0^H(\mathbb{C}P(V)) & \xrightarrow{(i^{Gx})_*} & K_0^H(\mathbb{C}P(V), \mathbb{C}P(V)\setminus\{Gx\}) & & \| \\
\| & & \downarrow{\scriptstyle (i_{Hx}^{Gx})_*} & & \| \\
K_0^H(\mathbb{C}P(V)) & \xrightarrow{(i^{Hx})_*} & K_0^H(\mathbb{C}P(V), \mathbb{C}P(V)\setminus\{Hx\}) & \xrightarrow[\text{(6.2.7)}]{\cong} & K_0^H.
\end{array} \qquad (6.2.20)$$

Proof. The two squares on the left are obvious from the naturality of inclusion. Let us draw the square on the right in a little more detail. The labellings (i), (ii), (iii) correspond to the three isomorphisms of Lemma 6.2.6.

$$\begin{array}{ccccccccc}
K_0^G(\mathbb{C}P(V), \mathbb{C}P(V)\setminus\{Gx\}) & \xrightarrow[\cong]{(ii)\circ(i)} & K_0^G(G\times_H S^{V_x}, (G\times_H S^{V_x})\setminus\{Gx\}) & \xrightarrow[\cong]{(iii)} & \tilde{K}_0^G(G_+ \wedge_H S^{V_x}) & \xrightarrow[\cong]{\phi^{-1}} & \tilde{K}_0^H(S^{V_x}) & \xrightarrow[\cong]{\mathrm{Bott}} & K_0^H \\
\downarrow{\scriptstyle \mathrm{Res}_H^G} & & \downarrow{\scriptstyle \mathrm{Res}_H^G} & & \downarrow{\scriptstyle \mathrm{Res}_H^G} & & \| & & \| \\
K_0^H(\mathbb{C}P(V), \mathbb{C}P(V)\setminus\{Gx\}) & \xrightarrow[\cong]{(ii)\circ(i)} & K_0^H(G\times_H S^{V_x}, (G\times_H S^{V_x})\setminus\{Gx\}) & \xrightarrow[\cong]{(iii)} & \tilde{K}_0^H(G_+ \wedge_H S^{V_x}) & & \| & & \| \\
\downarrow{\scriptstyle (i_{Hx}^{Gx})_*} & & \downarrow{\scriptstyle (i_{Hx}^{Gx})_*} & & \downarrow{\scriptstyle (i_{Hx}^{Gx})_*} & & \| & & \| \\
K_0^H(\mathbb{C}P(V), \mathbb{C}P(V)\setminus\{Hx\}) & \xrightarrow[\cong]{(ii)\circ(i)} & K_0^H(G\times_H S^{V_x}, (G\times_H S^{V_x})\setminus\{Hx\}) & \xrightarrow[\cong]{(iii)} & \tilde{K}_0^H(H_+ \wedge_H S^{V_x}) & = & \tilde{K}_0^H(S^{V_x}) & \xrightarrow[\cong]{\mathrm{Bott}} & K_0^H
\end{array}$$

It is obvious, by naturality, that all of the squares except the one containing ϕ^{-1} commute. To show that this final square commutes, consider the inclusion of pairs

$$i^{Gx}_{Hx} : (G \times_H S^{V_x}, (G \times_H S^{V_x}) \setminus \{Gx\}) \longrightarrow (G \times_H S^{V_x}, (G \times_H S^{V_x}) \setminus \{Hx\}).$$

In the quotient we obtain a map

$$i^{Gx}_{Hx} : G_+ \wedge_H S^{V_x} \longrightarrow H_+ \wedge_H S^{V_x},$$

and after a moment's thought we see that $g \wedge_H v \longmapsto \begin{cases} gv & g \in H \\ * & g \notin H \end{cases}$. But that is just the map p from the proof of Lemma 4.2.20. So it now suffices to show that the diagram

$$\begin{array}{ccc} \widetilde{K}^G_0(G_+ \wedge_H S^{V_x}) & \xrightarrow[\cong]{\phi^{-1}} & \widetilde{K}^H_0(S^{V_x}) \\ \downarrow{\scriptstyle \mathrm{Res}^G_H} & & \| \\ \widetilde{K}^H_0(G_+ \wedge_H S^{V_x}) & & \| \\ \downarrow{\scriptstyle p_*} & & \| \\ \widetilde{K}^H_0(H_+ \wedge_H S^{V_x}) & = & \widetilde{K}^H_0(S^{V_x}) \end{array}$$

commutes. Take $(\mathbb{S} \xrightarrow{h} G_+ \wedge_H S^{V_x} \wedge \mathbb{K}) \in K^G_0(G_+ \wedge_H S^{V_x})$. Then $\phi^{-1}(h)$ is the composite

$$\mathbb{S} \xrightarrow{h} G_+ \wedge_H S^{V_x} \wedge \mathbb{K} \xrightarrow{p} S^{V_x} \wedge \mathbb{K}$$

whilst $p_* \mathrm{Res}^G_H(h)$ is the composite

$$\mathbb{S} \xrightarrow{h} G_+ \wedge_H S^{V_x} \wedge \mathbb{K} \xrightarrow{D(\pi)} G/H_+ \wedge G_+ \wedge_H S^{V_x} \wedge \mathbb{K} \xrightarrow{p} G_+ \wedge_H S^{V_x} \wedge \mathbb{K} \xrightarrow{p} S^{V_x} \wedge \mathbb{K}.$$

These two composites are equal and this proves that (6.2.20) commutes. □

Proof of Theorem 6.2.13. Equivariant Bott periodicity means that in the diagram 6.2.9 we can work in degree zero. Theorem 4.4.11 tells us that $\frac{1}{\chi(V \otimes z)} = \sum_{i=0}^{n-1} \beta_i \in K^G_0(\mathbb{C}P(V))$ so it now suffices to prove that $\frac{1}{\chi(V \otimes z)} \in K^G_0(\mathbb{C}P(V))$ maps to a generator under

$$i^{Gx}_* : K^G_0(\mathbb{C}P(V)) \longrightarrow K^G_0(\mathbb{C}P(V), \mathbb{C}P(V) \setminus \{Gx\})$$

for all $x \in \mathbb{C}P(V)$. If x is G-fixed, we apply Proposition 6.2.16 and are done. If x has isotropy $H < G$, then by Proposition 6.2.19 it suffices to show that

$$(i^{Hx})_* \mathrm{Res}^G_H \left(\frac{1}{\chi(V \otimes z)} \right)$$

is a generator. But Theorem 4.4.9 tells us that $\mathrm{Res}^G_H\left(\frac{1}{\chi(V\otimes z)}\right) = \frac{1}{\chi(V\otimes z)}$, and by Proposition 6.2.16, $(i^{Hx})_*$ maps this to an $R(G)$-generator. □

Remark 6.2.21. We have shown that $[\mathbb{C}P(V)] = \frac{1}{\chi(V\otimes z)} = \sum_{i=0}^{n-1} \beta_i \in K_0^G(\mathbb{C}P(V))$. This result generalises Adams' non-equivariant result [4 (III, Theorem 11.15)].

6.2.3 Poincaré duality in equivariant K-theory for $\mathbb{C}P(V)$

We summarise our work:

Theorem 6.2.22. *With the usual notation, we have a* Poincaré duality *isomorphism*

$$K^0_G(\mathbb{C}P(V)) \xrightarrow{\cong} K_0^G(\mathbb{C}P(V))$$

given by taking cap products with $\frac{1}{\chi(V\otimes z)} \in K_0^G(\mathbb{C}P(V))$*, precisely*

$$a \longmapsto a \cap \frac{1}{\chi(V\otimes z)} \qquad (a \in K^0_G(\mathbb{C}P(V))).$$

Proof. This is obvious in view of Theorems 6.2.11 and 6.2.13. □

In §1.2 we outlined that we hoped to provide an explicit description of *Poincaré duality in equivariant K-theory for* $\mathbb{C}P(V)$. Theorem 6.2.22 achieves that aim. Though we made much use of equivariant stable homotopy theory as machinery, the statement of Theorem 6.2.22 is as down-to-earth as one could hope. Given this, one might expect to see some examples, but in fact we already provided those in §3.6. All we need to know is that our earlier approach of using the pairing

$$\lceil -,- \rceil : K^0_G(\mathbb{C}P(V)) \otimes K^0_G(\mathbb{C}P(V)) \longrightarrow K^0_G(\mathbb{C}P(V)) \longrightarrow R(G) \qquad (3.6.5')$$

to construct a map $K^0_G(\mathbb{C}P(V)) \longrightarrow K_0^G(\mathbb{C}P(V))$ as in Definition 3.6.1 matches the approach of Theorem 6.2.22.

Remark 6.2.23. The reader will notice that (3.6.5′) is obtained from (3.6.5) by replacing A with G. Back in §3.6 we were ignorant as to how the pairing $\lceil -,- \rceil$ should work in the non-abelian case, and were forced to take $G = A$ abelian. After Chapter 4 we know how to make sense of (3.6.5′).

Proposition 6.2.24. *The map*

$$K^0_G(\mathbb{C}P(V)) \xrightarrow{p} \mathrm{Hom}_{K^0_G}(K^0_G(\mathbb{C}P(V)), K^0_G) \cong K_0^G(\mathbb{C}P(V))$$

in which

$$x \overset{p}{\longmapsto} \left(y \longmapsto \lceil x, y\rceil = \langle xy, \frac{1}{\chi(V\otimes z)}\rangle\right)$$

corresponds to taking cap products with $\frac{1}{\chi(V\otimes z)}$ *as in Theorem 6.2.22.*

Proof. Take $x, y \in K^0_G(\mathbb{C}P(V))$. Then, by definition, $x \cap \frac{1}{\chi(V\otimes z)}$ is the composite

$$\mathbb{S} \xrightarrow{\frac{1}{\chi(V\otimes z)}} \mathbb{K}\wedge\mathbb{C}P(V) \xrightarrow{1\wedge\Delta} \mathbb{K}\wedge\mathbb{C}P(V)\wedge\mathbb{C}P(V) \xrightarrow{1\wedge x\wedge 1} \mathbb{K}\wedge\mathbb{K}\wedge\mathbb{C}P(V) \xrightarrow{\mu\wedge 1} \mathbb{K}\wedge\mathbb{C}P(V). \quad (6.2.25)$$

We wish to know how to view (6.2.25) as an element of $\mathrm{Hom}_{K^0_G}(K^0_G(X), K^0_G)$. Appealing to Proposition 4.2.9 we see that, in $\mathrm{Hom}_{K^0_G}(K^0_G(X), K^0_G)$, $x \cap \frac{1}{\chi(V\otimes z)}$ corresponds to the homomorphism

$$y \longmapsto \left(\mathbb{S} \xrightarrow{(6.2.25)} \mathbb{K}\wedge\mathbb{C}P(V) \xrightarrow{1\wedge y} \mathbb{K}\wedge\mathbb{K} \xrightarrow{\mu} \mathbb{K}\right). \quad (6.2.26)$$

On the other hand, $p(x)$ is the homomorphism $y \longmapsto \langle xy, \frac{1}{\chi(V\otimes z)}\rangle = \wr xy, \frac{1}{\chi(V\otimes z)}\int$. Writing this out in full, $p(x)$ is the homomorphism

$$y \longmapsto \left(\mathbb{S} \xrightarrow{\frac{1}{\chi(V\otimes z)}} \mathbb{K}\wedge\mathbb{C}P(V) \xrightarrow{1\wedge xy} \mathbb{K}\wedge\mathbb{K} \xrightarrow{\mu} \mathbb{K}\right)$$

which is obviously the same as (6.2.26). □

Remark 6.2.27. In Proposition 3.6.4 we showed that $\lceil -, -\rceil$ gave a perfect pairing in the abelian case. Proposition 6.2.24 allows us to deduce that $\lceil -, -\rceil$ is perfect in the non-abelian case. Moreover, it allows us to deduce that the resulting isomorphism really is Poincaré duality and the answer to the emphasised question on page 154 is *yes.* Thus we have produced the space-level understanding of *Poincaré duality in equivariant K-theory for* $\mathbb{C}P(V)$ that we hoped for in §1.2.

6.3 Where next?

The story is never complete: there is much work still to be done. Of course, one would like to be able to generalise. The obvious thing to do is to ask if $\mathbb{C}P(V)$ can be replaced with more general G-spaces, and if equivariant K-theory can be replaced with other generalised equivariant cohomology theories. Naturally, one must walk before one can run, so we should consider easier generalisations before harder ones: obvious candidates for spaces are equivariant Grassmanians. As for other cohomology theories, equivariant complex bordism would seem to be a likely candidate – not only is it complex stable but [18] it is *universal*.

Nonetheless, this volume of the story must come to an end at some point. In response to Greenlees' comment on page xv, the author has obtained a skull sufficiently thick that this seems an appropriate place to stop.

Appendix A

Character tables

The character tables [33, 19] used throughout are recorded here. For non-abelian groups we either record conjugacy classes in full, or present a representative and the number of elements in any given conjugacy class.

A.1 Abelian groups

A.1.1 Cyclic groups

C_2	1	-1
ε	1	1
α	1	-1

Table A.1. Character table for C_2.

C_3	1	g	g^2
ε	1	1	1
α	1	ω	ω^2
α^2	1	ω^2	ω

Table A.2. Character table for C_3, in which $\omega = e^{\frac{2\pi i}{3}}$.

C_4	1	i	-1	$-i$
ε	1	1	1	1
α	1	i	-1	$-i$
α^2	1	-1	1	-1
α^3	1	$-i$	-1	i

Table A.3. Character table for C_4.

C_5	1	ξ	ξ^2	ξ^3	ξ^4
ε	1	1	1	1	1
α	1	ξ	ξ^2	ξ^3	ξ^4
α^2	1	ξ^2	ξ^4	ξ	ξ^3
α^3	1	ξ^3	ξ	ξ^4	ξ^2
α^4	1	ξ^4	ξ^3	ξ^2	ξ

Table A.4. Character table for C_5, in which $\xi = e^{\frac{2\pi i}{5}}$.

A.1.2 Other abelian groups

V_4	1	x	y	xy
ε	1	1	1	1
α	1	1	-1	-1
β	1	-1	1	-1
$\alpha\beta$	1	-1	-1	1

Table A.5. Character table for the Klein 4-group, V_4.

A.2 Non-abelian groups

A.2.1 Dihedral groups

We take the convention that D_{2n}, the *dihedral group of order* $2n$, is given by

$$D_{2n} = \langle r, s \mid r^n = s^2 = 1, srs = r^{-1} \rangle.$$

D_6	$\{e\}$	$\{s, rs, r^2s\}$	$\{r, r^2\}$
ε	1	1	1
α	1	-1	1
γ	2	0	-1

Table A.6. Character table for D_6.

D_8	$\{e\}$	$\{r^2\}$	$\{r, r^3\}$	$\{s, r^2s\}$	$\{rs, r^3s\}$
ε	1	1	1	1	1
α	1	1	1	-1	-1
β	1	1	-1	1	-1
$\alpha\beta$	1	1	-1	-1	1
γ	2	-2	0	0	0

Table A.7. Character table for D_8.

D_{10}	$\{e\}$	$\{s, rs, r^2s, r^3s, r^4s\}$	$\{r, r^4\}$	$\{r^2, r^3\}$
ε	1	1	1	1
α	1	-1	1	1
γ	2	0	$-\mu$	$-\lambda$
γ'	2	0	$-\lambda$	$-\mu$

Table A.8. Character table for D_{10}, in which $\lambda = \frac{1+\sqrt{5}}{2}$ and $\mu = \frac{1-\sqrt{5}}{2}$.

A.2.2 Quaternion groups

Q_8	$\{e\}$	$\{-1\}$	$\{i, -i\}$	$\{j, -j\}$	$\{k, -k\}$
ε	1	1	1	1	1
α	1	1	1	-1	-1
β	1	1	-1	1	-1
$\alpha\beta$	1	1	-1	-1	1
γ	2	-2	0	0	0

Table A.9. Character table for Q_8.

A.2.3 Alternating groups

A_4	e	$(1\ 2)(3\ 4)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$
Size of conjugacy class	1	3	4	4
ε	1	1	1	1
α	1	1	ω	ω^2
β	1	1	ω^2	ω
γ	3	-1	0	0

Table A.10. Character table for A_4, in which $\omega = e^{\frac{2\pi i}{3}}$.

A_5	e	$(1\ 2\ 3)$	$(1\ 2)(3\ 4)$	$(1\ 2\ 3\ 4\ 5)$	$(1\ 2\ 3\ 5\ 4)$
Size of conjugacy class	1	20	15	12	12
ε	1	1	1	1	1
γ	3	0	-1	λ	μ
γ'	3	0	-1	μ	λ
χ	4	1	0	-1	-1
ψ	5	-1	1	0	0

Table A.11. Character table for A_5, in which $\lambda = \frac{1+\sqrt{5}}{2}$ and $\mu = \frac{1-\sqrt{5}}{2}$.

Appendix B

Subgroup lattices and degeneration lattices

The subgroup lattices (modulo conjugacy) and degeneration lattices used throughout are recorded here. The notation used for representations is consistent with the character tables of Appendix A. If $H < G$ then the notation (H) is used to mean "H and its conjugates in G". If $g^{-1}Hg = H$ for all $g \in G$ we write simply H. Much of the information about conjugacy can be read from the character table. For other cases, the well known [57 (Theorem 3.44)] Sylow theorem is useful.

Theorem B.1 (Sylow). *Let G be a finite group of order $p^a n$ where the prime p does not divide n. Then*

(i) *there is a subgroup of G of order p^a;*

(ii) *if $Q \leq G$ is a p-group then there is $T \leq G$ of order p^a so that $Q \leq T$;*

(iii) *all subgroups of order p^a are conjugate in G;*

(iv) *the number of subgroups of G of order p^a divides n and is congruent to* $1 \bmod p$.

□

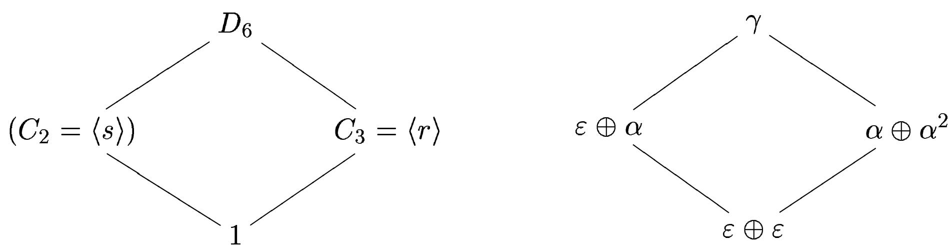


Figure B.1. Subgroup lattice (modulo conjugacy) for D_6 and degeneration lattice for the two-dimensional simple representation γ of D_6

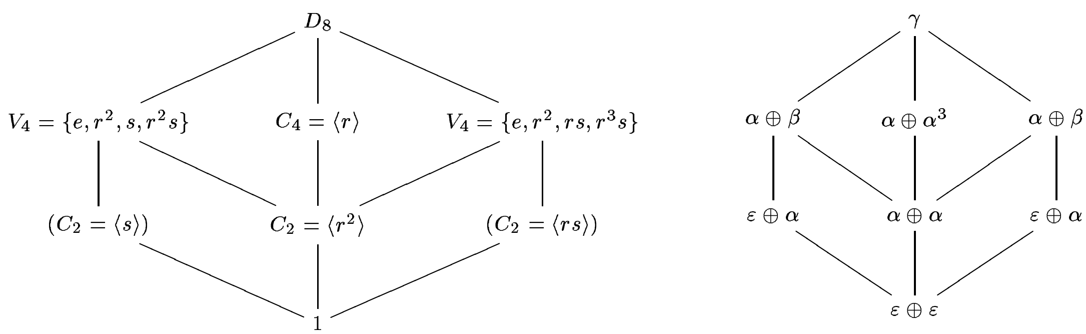


Figure B.2. Subgroup lattice (modulo conjugacy) for D_8 and degeneration lattice for the two-dimensional simple representation γ of D_8.

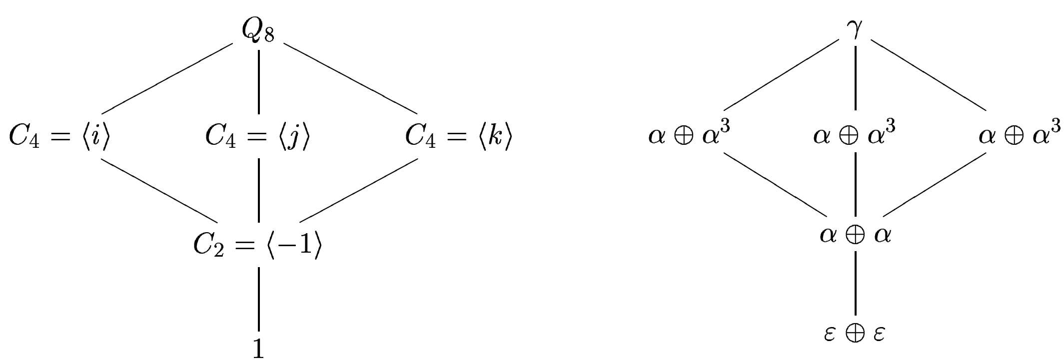


Figure B.3. Subgroup lattice (modulo conjugacy) for Q_8 and degeneration lattice for the two-dimensional simple representation γ of Q_8.

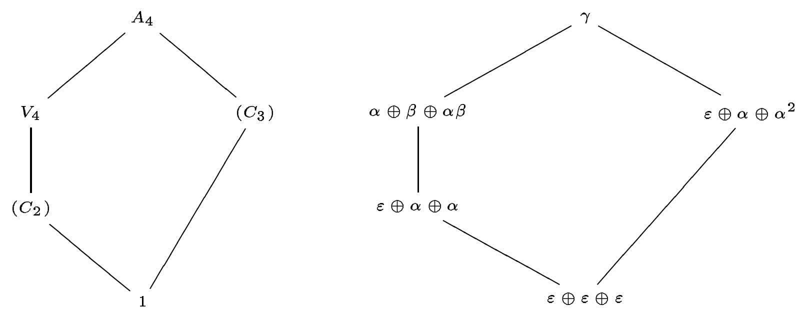


Figure B.4. Subgroup lattice (modulo conjugacy) for A_4 and degeneration lattice for the three-dimensional simple representation γ of A_4. The four copies of C_3 are conjugate in A_4 since each is determined by which of $1, 2, 3, 4$ it fixes and we can conjugate the three cycle fixing i to another not fixing i, e.g. $(1\ 2\ 4)(1\ 2\ 3)(1\ 4\ 2) = (2\ 4\ 3)$. One also finds the three copies of C_2 are conjugate in A_4.

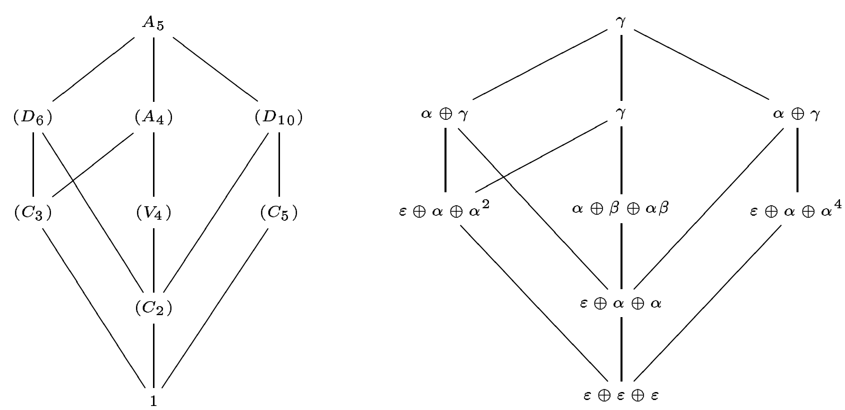


Figure B.5. Subgroup lattice (modulo conjugacy) for A_5 and degeneration lattice for the three-dimensional simple representation γ of A_5. The Sylow theorem (Theorem B.1) tells us that all copies of H are conjugate in A_5 for $H = C_3, C_5, V_4$ and the character table gives the same result for $H = C_2$. It is then not too difficult to see that all copies of H are conjugate in A_5 for $H = D_6, D_{10}, A_4$, provided one knows that A_5 is simple.

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