Math 248B. K-structures

1. Motivation

Let K be a compact open subgroup of $\mathrm{GL}_2(\widehat{\mathbf{Z}})$, and consider an elliptic curve $f : E \to M$ over a complex manifold M. Pick an integer $N \geq 1$ such that

$$K(N) := \ker(\mathrm{GL}_2(\widehat{\mathbf{Z}}) \twoheadrightarrow \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})) \subseteq K$$

and let $\Gamma = K/K(N) \subseteq \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ (so K is the preimage of Γ in $\mathrm{GL}_2(\widehat{\mathbf{Z}})$). We earlier defined a notion of Γ-*structure* on $E \to M$ (implicitly involving N too), namely a global section of the quotient sheaf $\Gamma\backslash I_{E/M,N}$ where $I_{E/M,N}$ is the sheaf of sets on M defined by assigning to any open $U \subseteq M$ the set

$$I_{E/M,N}(U) = \mathrm{Isom}_U((\mathbf{Z}/N\mathbf{Z})^2 \times U, E_U[N])$$

of full level-N structures on $E \to M$ (using the evident restriction maps). The sheaf $I_{E/M,N}$ is also denoted $\mathscr{I}som((\mathbf{Z}/N\mathbf{Z})^2 \times M, E[N])$, and it is a left $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$-torsor via the action $g.\phi = \phi \circ g^t$.

It was observed in class that if we replace N with a multiple N' and let Γ' denote the preimage of Γ in $\mathrm{GL}_2(\mathbf{Z}/N'\mathbf{Z})$ then $K/K(N') = \Gamma'$ in $\mathrm{GL}_2(\mathbf{Z}/N'\mathbf{Z})$ and the natural quotient map of sheaves $I_{E/M,N'} \twoheadrightarrow I_{E/M,N}$ equivariant for the reduction map $\mathrm{GL}_2(\mathbf{Z}/N'\mathbf{Z}) \to \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ induces an isomorphism of sheaves $\Gamma'\backslash I_{E/M,N'} \simeq \Gamma\backslash I_{E/M,N}$. In this way, Γ'-structures on E are naturally identified with Γ-structures on E. In particular, when considering moduli problems involving Γ-structures, we may always replace the pair (N, Γ) with the pair (N', Γ') to arrange that $N \geq 3$.

In class we used the universal property of the Weierstrass family over $\mathbf{C} - \mathbf{R}$ to prove that for any pair (N, Γ) the moduli problem of (isomorphism classes of) Γ-structures on varying elliptic curves $E \to M$ (over varying M) admits a coarse moduli space Y_Γ of pure dimension 1 which is a fine moduli space whenever Γ-structures are rigid (which we proved occurs if and only if the congruence subgroup $K \cap \mathrm{SL}_2(\mathbf{Z})$ is torsion-free), and that the set $\pi_0(Y_\Gamma)$ is a torsor for the finite abelian group $(\mathbf{Z}/N\mathbf{Z})^\times/\det(\Gamma) = \widehat{\mathbf{Z}}^\times/\det(K)$.

The aim of this handout is to explain Deligne's important generalization of this circle of ideas to the case when K is an arbitrary compact open subgroup of $\mathrm{GL}_2(\mathbf{A}_f)$. His concept of "$K$-structure" will require passing to an "isogeny category" of elliptic curves, but for K contained in $\mathrm{GL}_2(\widehat{\mathbf{Z}})$ it will recover (in a suitable sense that we make precise) the concepts reviewed above. In the end the crucial constructions for the moduli problems of K-structures in Deligne's sense will reduce back to results in the more classical setting as above. The significance of Deligne's viewpoint is that it frees the theory from the restrictiveness of $\mathrm{GL}_2(\widehat{\mathbf{Z}})$ and the $\mathbf{Z}$-structure on GL_2 by opening the door to working with actions by $\mathrm{GL}_2(\mathbf{A}_f)$ and its subgroup $\mathrm{GL}_2(\mathbf{Q})$ (which makes no sense in terms of $E[N]$'s).

To emphasize the general group-theoretic aspects of the situation, we will write "G" to denote "GL_2".

Remark 1.1. Since we will be working with groups such as $G(\mathbf{A}_f)$ and $G(\mathbf{A})$ as topological groups, we refer the reader to §2 of my notes "Weil and Grothendieck approaches to adelic points" for a discussion of the topology on $\mathrm{GL}_r(\mathbf{A})$ and $\mathrm{GL}_r(\mathbf{A}_f)$ (generalizing the classical case $r = 1$ which corresponds to the idele group and its finite part respectively). This can be made explicit: two elements $T, T' \in \mathrm{GL}_r(\mathbf{A}_f)$ are close when they are close in terms of matrix entries *and* $\det(T)^{-1}, \det(T')^{-1} \in \mathbf{A}_f$ are close. (This is related to the fact that a presentation of the coordinate ring of the affine GL_r is $\mathbf{Z}[x_{ij}, y]/(y\det(x_{ij}) - 1)$, but it is unpleasant to take this explicit description as the *definition*, as it is too specific to the case of GL_r and so masks the underlying structures that matter.)

In Theorem 3.6 of those notes, the "restricted direct product" viewpoint is brought into the picture. This is useful too, but the definition given in §2 of those notes works directly with the topology of the adele ring $\mathbf{A}$ and so is more intrinsic (and treats it directly as a $\mathbf{Q}$-algebra, whereas the "restricted direct product" viewpoint involves the intervention of integral structure).

2. $\widehat{\mathbf{Z}}$-lattices and compact open subgroups

We begin by addressing the adelic counterpart of lattices in rational vector spaces. Nearly everything we do carries over verbatim to the ring of finite adeles of any global fields, but we stick to the case of $\mathbf{Q}$ to ease the notation (and it suffices for what follows). Whereas finite free $\mathbf{Z}$-modules L of rank r in $V = \mathbf{Q}^r$ always satisfy the condition that the natural map $\mathbf{Q} \otimes_{\mathbf{Z}} L \to V$ is an isomorphism, in the adelic setting the naive analogue fails even for $r = 1$:

Example 2.1. Let $\alpha \in \mathbf{A}_f$ be a finite idele whose local components are all nonzero but infinitely many of which are non-units (so $\alpha \notin \mathbf{A}_f^\times$), and let $L := \alpha \cdot \widehat{\mathbf{Z}} \subset \mathbf{A}_f$. This is a $\widehat{\mathbf{Z}}$-submodule that is finite free of rank 1, but the $\mathbf{A}_f$-linearization $\mathbf{A}_f \otimes_{\widehat{\mathbf{Z}}} L \to \mathbf{A}_f$ is *not* an isomorphism. Indeed, this map is identified with multiplication on $\mathbf{A}_f$ by the nonunit α. (Concretely, the difference with the classical case of $\mathbf{Z}$-lattices in $\mathbf{Q}$-vector spaces is that $\mathbf{A}_f$ is not the total ring of fractions of $\widehat{\mathbf{Z}}$; we only invert $\mathbf{Z} - \{0\}$ rather than all elements of $\widehat{\mathbf{Z}}$ that are not zero-divisors.)

The preceding example is avoided by imposing the topological condition of *openness*; observe that the ideal $\alpha\mathbf{A}_f$ in $\mathbf{A}_f$ is never open when $\alpha \in \mathbf{A}_f$ is a non-unit. (Check!) This leads to:

Lemma 2.2. *Let V be a finite free $\mathbf{A}_f$-module of rank $r \geq 0$, equipped with its natural topology. A compact open subgroup L in V is necessarily a $\widehat{\mathbf{Z}}$-submodule that is finite free of rank r as a $\widehat{\mathbf{Z}}$-module. Conversely, if L is a finite free $\widehat{\mathbf{Z}}$-submodule of V then it is a compact open subgroup of V if and only if the natural $\mathbf{A}_f$-linear map $\mathbf{A}_f \otimes_{\widehat{\mathbf{Z}}} L \to V$ is an isomorphism.*

Recall that if R is a topological ring, then any finite free R-module is equipped with a canonical topological R-module structure via any basis (the choice of which does not matter, since R is a topological ring). In this way, the natural topology on any finite free $\widehat{\mathbf{Z}}$-module makes it profinite. (The only aspect of this lemma which doesn't carry over to other global fields F is that in the lemma we don't have to assume L is a $\widehat{\mathbf{Z}}$-submodule. In general we should work with compact open submodules over the maximal compact subring $\prod_{v\nmid\infty} \mathscr{O}_{F,v}$ of the ring of finite adeles of F.)

Proof. Fix an $\mathbf{A}_f$-basis of V to identity V with $\mathbf{A}_f^r$; we may and do assume $r > 0$. Fix a compact open subgroup L. The $\widehat{\mathbf{Z}}$-submodule $L_0 = \widehat{\mathbf{Z}}^r$ is visibly compact and open, so the intersection $L \cap L_0$ is open in L_0. Hence, there is a finite set S of non-archimedean places of $\mathbf{Q}$ and integers $e_v \geq 0$ for $v \in S$ such that

$$\prod_{v\in S} \mathfrak{m}_v^{e_v} \times \prod_{v\notin S} \mathbf{Z}_v \subseteq L \cap L_0 \subseteq \prod_v \mathbf{Z}_v.$$

Thus, to prove that L is finite free of rank r as a $\widehat{\mathbf{Z}}$-module we can drop the factors away from S and thereby reduce our problem to the consideration of compact open subgroups L of $\mathbf{Q}_S^r$ with $\mathbf{Q}_S = \prod_{v\in S} \mathbf{Q}_v$; we seek to prove that such subgroups are necessarily submodules over $\mathbf{Z}_S = \prod_{v\in S} \mathbf{Z}_v$ and are finite free of rank r over $\mathbf{Z}_S$. A basis of open neighborhoods of 0 is given by $M_e := \prod_{v\in S} \mathfrak{m}_v^e$ for $e \to \infty$, so L must contain some M_e with finite index. Thus, L/M_e is finite, yet $\mathbf{Q}_S^r/M_e$ is exhausted by $M_{e'}/M_e$ for $e' \to -\infty$. Hence, L sits between M_e and $M_{e'}$ for some $e' \leq e$ (with e' possibly negative).

It follows that L is stable under multiplication by the open ideal $M_{e-e'}$ and hence is a $\mathbf{Z}_S$-submodule since $\mathbf{Z} \to \mathbf{Z}_S/M_{e-e'}$ is surjective (Chinese Remainder Theorem). Being a $\mathbf{Z}_S$-submodule, the primitive idempotents of $\mathbf{Z}_S$ (corresponding to the local factor rings) decompose L as a product $L = \prod_{v\in S} L_v$ for a $\mathbf{Z}_v$-submodule $L_v \subseteq \mathbf{Q}_v^r$. Since $\mathbf{Q}_S = \prod_{v\in S} \mathbf{Q}_v$ has the product topology, the condition that L is compact open in $\mathbf{Q}_S^r$ is equivalent to L_v being compact open in $\mathbf{Q}_v^r$ for each $v \in S$. Thus, by the classical case of local fields, we see that L is compact open if and only if each L_v is a finite free $\mathbf{Z}_v$-submodule of $\mathbf{Q}_v$ of rank r.

It remains to prove that if $L \subseteq V$ is a $\widehat{\mathbf{Z}}$-submodule that is finite free of rank $r > 0$ then it is compact open if and only if the $\mathbf{A}_f$-linear map $\mathbf{A}_f \otimes_{\widehat{\mathbf{Z}}} L \to V$ is an isomorphism. Fix an $\mathbf{A}_f$-basis of V to identify it with $\mathbf{A}_f^r$. Assume L is compact open. The preceding argument proves that for a suitable finite set S of non-archimedean places of $\mathbf{Q}$ we have

$$L = \prod_{v\in S} L_v \times \prod_{v\notin S} \mathbf{Z}_v^r$$

inside of $\mathbf{A}_f^r$, with L_v a $\mathbf{Z}_v$-lattice in $\mathbf{Q}_v^r$. Hence, for each $v \in$ we have $L_v = T_v(\mathbf{Z}_v^r)$ for some $T_v \in \mathrm{GL}_r(\mathbf{Q}_v)$, and defining $T_v = 1$ for $v \notin S$ yields an element $T = (T_v) \in \mathrm{GL}_r(\mathbf{A}_f)$ such that $L = T(\widehat{\mathbf{Z}}^r)$. Thus, to prove that the natural map $\mathbf{A}_f \otimes_{\widehat{\mathbf{Z}}} L \to V$ is an isomorphism we may first apply T^{-1} to $V = \mathbf{A}_f^r$ to reduce to the case when L is the standard $\widehat{\mathbf{Z}}$-submodule $\widehat{\mathbf{Z}}^r$ inside of $\mathbf{A}_f^r = V$. This case is clear.

Conversely, assume that $\mathbf{A}_f \otimes_{\widehat{\mathbf{Z}}} L \to V$ is an isomorphism. We seek to prove that L is a compact open subgroup of V. Pick a $\widehat{\mathbf{Z}}$-basis of L to identify it with $\widehat{\mathbf{Z}}^r$, so the $\mathbf{A}_f$-linear isomorphism hypothesis implies that we may use the same r-tuple as an $\mathbf{A}_f$-basis of V. This identifies the inclusion of L into V with the standard inclusion of $\widehat{\mathbf{Z}}^r$ into $\mathbf{A}_f^r$, which is visibly a compact open subgroup. ■

It is immediate from the final assertion in the lemma (and even shown in the proof of the lemma) that the action of $\mathrm{Aut}_{\mathbf{A}_f}(V) = \mathrm{GL}_r(\mathbf{A}_f)$ on $V = \mathbf{A}_f^r$ is transitive on the set of compact open subgroups. This is analogous to the familiar fact that $\mathrm{GL}_r(\mathbf{Q})$ acts transitively on the set of $\mathbf{Z}$-lattices in $\mathbf{Q}^r$, and so motivates the reasonableness of the following concept:

Definition 2.3. A $\widehat{\mathbf{Z}}$-*lattice* in a finite free $\mathbf{A}_f$-module V is a compact open $\widehat{\mathbf{Z}}$-submodule.

Observe that an intersection of two $\widehat{\mathbf{Z}}$-lattices in V is again a $\widehat{\mathbf{Z}}$-lattice. The relationship between $\widehat{\mathbf{Z}}$-lattices in finite free $\mathbf{A}_f$-modules and $\mathbf{Z}$-lattices in finite-dimensional $\mathbf{Q}$-vector spaces is very concrete when the $\mathbf{A}_f$-module is equipped with a $\mathbf{Q}$-structure:

Example 2.4. Let $V_{\mathbf{Q}}$ be a finite-dimensional $\mathbf{Q}$-vector space, and $V = \mathbf{A}_f \otimes_{\mathbf{Q}} V_{\mathbf{Q}}$ the associated finite free $\mathbf{A}_f$-module. For any $\mathbf{Z}$-lattice $L_{\mathbf{Z}}$ in $V_{\mathbf{Q}}$ we may view a $\mathbf{Z}$-basis of $L_{\mathbf{Z}}$ as a $\mathbf{Q}$-basis of $V_{\mathbf{Q}}$ and hence as an $\mathbf{A}_f$-basis of V. Thus, the natural map $\phi : \widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} L_{\mathbf{Z}} \to V$ is an isomorphism onto a $\widehat{\mathbf{Z}}$-lattice L in V. Moreover, since this map ϕ is identified with the inclusion $\widehat{\mathbf{Z}}^r \to \mathbf{A}_f^r$ upon using a $\mathbf{Z}$-basis of $L_{\mathbf{Z}}$ as a $\mathbf{Q}$-basis of $V_{\mathbf{Q}}$, it follows that $L \cap V_{\mathbf{Q}} = L_{\mathbf{Z}}$ (since $\widehat{\mathbf{Z}} \cap \mathbf{Q} = \mathbf{Z}$ inside of $\mathbf{A}_f$). Thus, $L_{\mathbf{Z}} \mapsto \widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} L_{\mathbf{Z}}$ is an injection from the set of $\mathbf{Z}$-lattices in $V_{\mathbf{Q}}$ to the set of $\widehat{\mathbf{Z}}$-lattices in V.

We claim that this injective correspondence is actually surjective, and moreover that it respects inclusions and intersections in both directions (i.e., every L arises from some $L_{\mathbf{Z}}$, $L \subseteq L'$ if and only if $L_{\mathbf{Z}} \subseteq L'_{\mathbf{Z}}$, and $L \cap L' = \widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} (L_{\mathbf{Z}} \cap L'_{\mathbf{Z}})$). Once bijectivity and the compatibility with inclusions are established, the behavior for intersections follows formally (since the intersection is characterized in terms maximality with respect to inclusions).

Fix a choice of $\mathbf{Z}$-lattice $L_{\mathbf{Z}}$ in $V_{\mathbf{Q}}$, so we get a $\widehat{\mathbf{Z}}$-lattice L in V. The proof of Lemma 2.2 shows that any $\widehat{\mathbf{Z}}$-lattice L' in V satisfies $L' = T(L)$ for some $T \in \mathrm{GL}_r(\mathbf{A}_f)$. In particular, by considering "local denominators" in the matrix entries of T and T^{-1}, we get a nonzero $n \in \mathbf{Z}$ such that $nL \subseteq L' \subseteq (1/n)L$. In particular, $L'_{\mathbf{Z}} := L' \cap V_{\mathbf{Q}}$ is sandwiched between $nL \cap V_{\mathbf{Q}} = nL_{\mathbf{Z}}$ and $(1/n)L \cap V_{\mathbf{Q}} = (1/n)L_{\mathbf{Z}}$, so it is a $\mathbf{Z}$-lattice in $V_{\mathbf{Q}}$. The natural map $\widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} L'_{\mathbf{Z}} \to L'$ is injective (by comparing with $(1/n)L_{\mathbf{Z}}$ and $(1/n)L$), and we claim it is an isomorphism. Scaling through by n, we simply need to observe that the natural map

$$L_{\mathbf{Z}}/n^2 L_{\mathbf{Z}} \to L/n^2 L$$

is an isomorphism linear over the ring isomorphism $\mathbf{Z}/(n^2)\mathbf{Z} \simeq \widehat{\mathbf{Z}}/(n^2)\widehat{\mathbf{Z}}$, so scalar extension by $\mathbf{Z} \to \widehat{\mathbf{Z}}$ sets up a bijection between the set of $\mathbf{Z}$-submodules of $L_{\mathbf{Z}}$ containing $n^2 L_{\mathbf{Z}}$ and the set of $\widehat{\mathbf{Z}}$-submodules of L containing $n^2 L$. This establishes the desired bijectivity among sets of $\mathbf{Z}$-lattices and sets of $\widehat{\mathbf{Z}}$-lattices.

Our proof of the bijectivity gives more: it shows that $L \mapsto L \cap V_{\mathbf{Q}}$ is the inverse correspondence, so the behavior with respect to inclusions is now clear.

Note that in the case of the standard $\widehat{\mathbf{Z}}$-lattice $L_0 = \widehat{\mathbf{Z}}^r$ in $\mathbf{A}_f^r$, the stabilizer of L_0 in $\mathrm{GL}_r(\mathbf{A}_f)$ is $K_0 = \mathrm{GL}_r(\widehat{\mathbf{Z}})$. By Exercise 2(i) in HW5 this is a maximal compact subgroup of $\mathrm{GL}_r(\mathbf{A}_f)$, and every compact subgroup of $\mathrm{GL}_r(\mathbf{A}_f)$ lies in a conjugate of K_0 (so every compact subgroup lies in a maximal one, and the maximal ones are all open). For $g \in \mathrm{GL}_r(\mathbf{A}_f)$ the stabilizer of the $\widehat{\mathbf{Z}}$-lattice $g(L_0)$ is gK_0g^{-1}, so this proves:

Proposition 2.5. *Let V be a finite free $\mathbf{A}_f$-module of rank $r \geq 0$, and let* $\mathrm{GL}(V) = \mathrm{Aut}_{\mathbf{A}_f}(V) = \mathrm{GL}_r(\mathbf{A}_f)$ *equipped with its natural structure of topological group. The map $L \mapsto \mathrm{Stab}_{\mathrm{GL}(V)}(L)$ is a bijection from the set of $\widehat{\mathbf{Z}}$-lattices in V to the set of maximal compact open subgroups of* $\mathrm{GL}(V)$*.*

For a local system V of finite free $\mathbf{A}_f$-modules of rank $r \geq 0$ over a complex manifold M, we define a *relative* $\widehat{\mathbf{Z}}$*-lattice* in V to be a local system L of finite free $\widehat{\mathbf{Z}}$-modules over M equipped with an isomorphism

$$\underline{\mathbf{A}}_f \otimes_{\underline{\widehat{\mathbf{Z}}}} L \simeq V.$$

This isomorphism condition amounts to saying that the local system L is a subsheaf of V such that it is compact open on fibers over M. In view of the dictionary between local systems on M and representations of the fundamental group of each connected component of M, to specify L amounts to specifying a compact open subgroup of the fiber V_{m_i} of V over a single point m_i in each connected component M_i of M.

To give an example, let $E \to M$ be an elliptic curve over a complex manifold M, so $E[N]$ is a local system of finite free $\mathbf{Z}/N\mathbf{Z}$-modules of rank 2 over M for any $N \geq 1$. Since $E[N]$ is identified with the mod-N reduction of the local system $\underline{\mathrm{H}}_1(E/M)$ of finite free $\mathbf{Z}$-modules of rank 2, compatibly with change in N, the inverse limit sheaf $T_f(E) := \varprojlim E[N]$ over M is also a locally constant sheaf:

$$T_f(E) = \underline{\widehat{\mathbf{Z}}} \otimes_{\underline{\mathbf{Z}}} \underline{\mathrm{H}}_1(E'/M).$$

Hence, $V_f(E) := \mathbf{Q} \otimes_{\mathbf{Z}} T_f(E)$ is a locally constant sheaf of finite free $\mathbf{A}_f$-modules of rank 2 over M.

Observe that $T_f(E)$ is a relative $\widehat{\mathbf{Z}}$-lattice in $V_f(E)$. Our interest in relative $\widehat{\mathbf{Z}}$-lattices is due to the following fact which essentially reverses this process and was asserted in class (with proof not given there, so we give it here):

Proposition 2.6. *Let M be a complex manifold. The functor $E \mapsto (E_{\mathbf{Q}}, T_f(E))$ from the category of elliptic curves over M to the category of pairs (E', L) consisting of object E' in $\mathscr{E}ll^0_M$ and a relative $\widehat{\mathbf{Z}}$-lattice $L \subset V_f(E')$ is an equivalence of categories.*

More specifically, for any elliptic curve E' over M and relative $\widehat{\mathbf{Z}}$-lattice L in $V_f(E')$ there is a pair (E, h) consisting of an elliptic curve E over M and isomorphism $h : E_{\mathbf{Q}} \simeq E'_{\mathbf{Q}}$ in $\mathscr{E}ll^0_M$ such that $V_f(h)$ carries $T_f(E)$ isomorphically onto L, and for elliptic curves E_1 and E_2 over M a map $F : (E_1)_{\mathbf{Q}} \to (E_2)_{\mathbf{Q}}$ in $\mathscr{E}ll^0_M$ arises from a morphism $E_1 \to E_2$ of elliptic curves if and only if $V_f(F)$ carries $T_f(E_1)$ into $T_f(E_2)$. In particular, the pair (E, h) is uniquely determined by (E', L) unique up to unique isomorphism (viewing E as an elliptic curve over M, not just an object in the isogeny category over M).

Proof. Due to the asserted uniqueness of (E, h) up to unique isomorphism, it suffices to work locally on M. Thus, we may arrange that M is connected, so we can view local systems as representations of the fundamental group. Hence, by the relative uniformization of elliptic curves and Example 2.4, we just need to make two observations concerning fibers of a local system: (i) if $L_{\mathbf{Z}}$ is a finite free $\mathbf{Z}$-module equipped with an action by $\pi_1(M)$ then a $\widehat{\mathbf{Z}}$-lattice L' in $V := \mathbf{A}_f \otimes_{\mathbf{Z}} L_{\mathbf{Z}}$ is $\pi_1(M)$-stable if and only if the corresponding $\mathbf{Z}$-lattice $L'_{\mathbf{Z}}$ in $V_{\mathbf{Q}} := \mathbf{Q} \otimes_{\mathbf{Z}} L_{\mathbf{Z}}$ is $\pi_1(M)$-stable, and (ii) if $L_{\mathbf{Z}}$ and $L'_{\mathbf{Z}}$ are finite free $\mathbf{Z}$-modules equipped with $\pi_1(M)$-actions then a $\pi_1(M)$-equivariant $\mathbf{Q}$-linear map $V_{\mathbf{Q}} \to V'_{\mathbf{Q}}$ between the associated finite-dimensional $\mathbf{Q}$-vector spaces arises from a $\pi_1(M)$-equivariant $\mathbf{Z}$-linear map $L_{\mathbf{Z}} \to L'_{\mathbf{Z}}$ if and only if the associated $\mathbf{A}_f$-linear map $V \to V'$ carries the associated $\widehat{\mathbf{Z}}$-lattice $L = \widehat{\mathbf{Z}} \otimes_{\mathbf{Z}} L_{\mathbf{Z}}$ into the associated $\widehat{\mathbf{Z}}$-lattice L' (since $L' \cap V' = L'_{\mathbf{Z}}$). ■

Remark 2.7. In the preceding argument we could have shrank M to be a ball, and then all local systems on M are constant and the fundamental group would have dropped out of consideration. The reason to formulate the argument as we did (localizing M to make it connected but not localizing any further) is because in this form the argument can be carried over to the algebraic theory (using adelic étale sheaves in the role of the local systems and using étale fundamental groups in the role of topological fundamental groups).

3. Level structures attached to compact open subgroups of $\mathrm{GL}_2(\mathbf{A}_f)$.

In class the *isogeny category* $\mathscr{E}ll^0_M$ of elliptic curves over a complex manifold M was defined, and we saw that the functor $E \rightsquigarrow V_f(E)$ on elliptic curves over M (with values in the category of local systems of finite free $\mathbf{A}_f$-modules of rank 2) naturally extends to a functor on $\mathscr{E}ll^0_M$ that commutes with base change on M in the evident manner.

Lemma 3.1. *Let E' be an object in $\mathscr{E}ll^0_M$, and define the sheaf of sets $I_{E'/M}$ on M via*

$$I_{E'/M}(U) = \mathrm{Isom}_{\mathbf{A}_f\text{-lin}/U}(\mathbf{A}_f^2 \times U, V_f(E'_U))$$

(with the evident restriction maps). The left $G(\mathbf{A}_f)$-action via $g.\phi = \phi \circ g^t$ makes $I_{E'/M}$ into a left $G(\mathbf{A}_f)$-torsor over M; i.e., every $m \in M$ admits an open neighborhood U such that $I_{E'/M}(U) \neq \emptyset$, and the natural map of locally constant sheaves

$$\underline{\mathrm{GL}_2(\mathbf{A}_f)} \times I_{E'/M} \to I_{E'/M} \times I_{E'/M}$$

over M defined by $(g, \phi) \mapsto (\phi, g.\phi)$ is an isomorphism.

Proof. We may work locally on M so that the elliptic curve E' over M admits an $\underline{\mathrm{H}}_1$-trivialization. Since

$$V_f(E') = \underline{\mathbf{A}}_f \otimes_{\underline{\mathbf{Z}}} \underline{\mathrm{H}}_1(E'/M)$$

as local systems of finite free $\mathbf{A}_f$-modules of rank 2 over M, it follows that $V_f(E')$ is a constant sheaf. This makes the assertion clear. ■

In view of this lemma, it makes sense to form the quotient sheaf $H\backslash I_{E'/M}$ for any subgroup $H \subseteq G(\mathbf{A}_f)$. (Recall that G denotes GL_2.)

Definition 3.2 (Deligne)**.** For a compact open subgroup K in $G(\mathbf{A}_f)$ and an object E' in $\mathscr{E}ll^0_M$ a *K-structure* on E' is a global section $\alpha \in \mathrm{H}^0(M, K\backslash I_{E'/M})$.

The notions of isomorphism (relative to $\mathscr{E}ll^0_M$!) and base change (with respect to holomorphic maps $M' \to M$) for K-structures are defined in the evident manner. The following example illustrates the subtlety of this concept in the "familiar" case that $K \subseteq G(\widehat{\mathbf{Z}})$:

Example 3.3. Suppose $K \subseteq G(\widehat{\mathbf{Z}})$, so K contains $K(N)$ for some $N \geq 1$ and $K/K(N)$ is identified with a subgroup $\Gamma \subseteq G(\mathbf{Z}/N\mathbf{Z})$. Then a Γ-structure on E (with $N \geq 1$ understood from the context) is a global section of $\Gamma\backslash I_{E/M,N}$. We can express this in terms of K by using the sheaf

$$I_{E/M,\infty} := \mathscr{I}som(\widehat{\mathbf{Z}}^2 \times M, T_f(E))$$

of $\widehat{\mathbf{Z}}$-linear trivializations of the total Tate module local system $T_f(E)$. Namely, there is a natural quotient map $I_{E/M,\infty} \twoheadrightarrow I_{E/M,N}$ equivariant with respect to $G(\widehat{\mathbf{Z}}) \twoheadrightarrow G(\mathbf{Z}/N\mathbf{Z})$, and this induces an isomorphism of quotient sheaves

$$K\backslash I_{E/M,\infty} \simeq \Gamma\backslash I_{E/M,N}.$$

Note that $I_{E/M,\infty}$ is a $G(\widehat{\mathbf{Z}})$-torsor inside of the $G(\mathbf{A}_f)$-torsor

$$I_{E_{\mathbf{Q}}/M} = \mathscr{I}som(\mathbf{A}_f^2 \times M, V_f(E)),$$

so $I_{E/M,\infty}$ is much smaller than $I_{E_{\mathbf{Q}}/M}$ in general. A Γ-structure on E over M is clearly identified with an element

$$\alpha \in \mathrm{H}^0(M, K\backslash I_{E/M,\infty});$$

these will be called *classical K-structures*. They are special kinds of K-structures in the sense that they locally arise from $\mathbf{A}_f$-linear isomorphism $\mathbf{A}_f^2 \times U \simeq V_f(E_U)$ that respects the "integral structure" on both sides; i.e., carries $\widehat{\mathbf{Z}}^2 \times U$ isomorphically over to $T_f(E_U)$.

For K contained in $G(\widehat{\mathbf{Z}})$, a K-structure on E (viewed as an object in $\mathscr{E}ll^0_M$) need *not* arise from a classical K-structure in the sense of the preceding example; for instance, $G(\mathbf{Q})$ acts on K-structures without changing the underlying elliptic curve. For this reason, the notion of K-structure may seem too abstract. But in fact there is a *functorial* way to pass to an isogenous elliptic curve over M making the K-structure become classical. This is made precise by the following result, which essentially inverts the procedure in Example 3.3 and is the key to linking up Deligne's notion of K-structure with the more familiar finite-level Γ-structures that we understand:

Proposition 3.4. *Let $E \to M$ be an elliptic curve, and let K be a compact open subgroup of $G(\widehat{\mathbf{Z}})$. Let α be a K-structure on $E_{\mathbf{Q}}$. There is a canonically associated pair (E_α, h_α) consisting of an elliptic curve E_α over M and an isomorphism $h_\alpha : E_{\mathbf{Q}} \simeq (E_\alpha)_{\mathbf{Q}}$ in $\mathscr{E}ll^0_M$ such that the resulting isomorphism $I_{E_{\mathbf{Q}}/M} \simeq I_{(E_\alpha)_{\mathbf{Q}}/M}$ carries α to a classical K-structure on E_α. The pair (E_α, h_α) is functorial in isomorphisms between pairs $(E_{\mathbf{Q}}, \alpha)$ and is naturally compatible with base change over M.*

Note that h_α is an isomorphism in $\mathscr{E}ll^0_M$, and it need not arise from an isogeny in either direction between E and E_α; in terms of homology lattices, this corresponds to the fact that any two lattices in a finite-dimensional $\mathbf{Q}$-vector space are commensurable with respect to each other but neither may contain the other.

Proof. Since the construction will be canonical and compatible with base change, it suffices to carry out the construction locally on M (provided that we verify the canonicity!). Thus, we may assume that α arises from an $\mathbf{A}_f$-linear isomorphism of local systems $\phi : \mathbf{A}_f^2 \times M \simeq V_f(E)$ over M that is well-defined up to local precomposition with an element of K. But the K-action on $\mathbf{A}_f^2$ carries $\widehat{\mathbf{Z}}^2$ isomorphically onto itself (since we assume $K \subseteq G(\widehat{\mathbf{Z}})$), so the image $L = \phi(\widehat{\mathbf{Z}}^2 \times M) \subset V_f(E)$ is well-defined (i.e., depends on α but not ϕ). Applying Lemma 2.2 on fibers over each connected component of the manifold M (and using local constancy), it follows that L is a relative $\widehat{\mathbf{Z}}$-lattice in $V_f(E)$. Thus, by Proposition 2.6 there is a unique pair (E_α, h_α) consisting of an elliptic curve E_α over M and an isomorphism $h_\alpha : E_{\mathbf{Q}} \simeq (E_\alpha)_{\mathbf{Q}}$ such that $V_f(h_\alpha)$ carries L isomorphically onto $T_f(E_\alpha) \subset V_f((E_\alpha)_{\mathbf{Q}})$.

By construction of L via ϕ, we obtain a $\widehat{\mathbf{Z}}$-linear isomorphism $\widehat{\mathbf{Z}}^2 \times M \simeq T_f(E_\alpha)$ over M which is well-defined up to local precomposition with an element of K. In other words, we have a global section of $I_{E_\alpha/M,\infty}$ well-defined up to the action by K locally over M, which is to say that we have a classical K-structure on E_α. Going back through the construction, we see that in fact that isomorphism

$$K\backslash I_{E_{\mathbf{Q}}/M} \simeq K\backslash I_{(E_\alpha)_{\mathbf{Q}}/M}$$

carries α to a global section of $K\backslash I_{E_\alpha/M,\infty}$. Thus, the pair (E_α, h_α) is readily seen to satisfy all of the asserted properties (including that it globalizes beyond our initial localization on M to lift α to some ϕ). ■

Corollary 3.5. *Let K be a compact open subgroup of $G(\mathbf{A}_f)$. For any complex manifold M, let $F_K(M)$ be the set of isomorphism classes of pairs (E', α) consisting of E' in $\mathscr{E}ll^0_M$ and a K-structure α on E'. Make this into a contravariant functor via base change.*

All pairs (E', α) admit no nontrivial automorphisms if and only if $K \cap G'(\mathbf{Q})$ is torsion-free (where $G' = \mathrm{SL}_2$), and F_K admits a coarse moduli space Y_K of pure dimension 1. The set $\pi_0(Y_K)$ is a torsor for the finite abelian group

$$\mathbf{Q}^\times\backslash\mathbf{A}_f^\times/\det(K) = \widehat{\mathbf{Z}}^\times/\det(K),$$

and F_K admits a fine moduli space whenever all pairs (E', α) are rigid.

Proof. First assume $K \subseteq G(\widehat{\mathbf{Z}})$, and pick $N \geq 1$ such that K contains $K(N)$. Let $\Gamma = K/K(N) \subseteq G(\mathbf{Z}/N\mathbf{Z})$.

The construction in Proposition 3.4 defines a bijection between the set of isomorphism classes of pairs consisting of objects in $\mathscr{E}ll^0_M$ equipped with a K-structure and the set of isomorphism classes of pairs consisting of objects in $\mathscr{E}ll_M$ equipped with a *classical* K-structure, and this respects base change in M. Hence, F_K is identified with the functor F_Γ of Γ-structures on varying elliptic curves over complex manifolds (no isogeny category!). This solves our problems for such K, since $K \cap G'(\mathbf{Q}) = K \cap G'(\mathbf{Z})$ is the preimage in $G'(\mathbf{Z})$ of $\Gamma \cap G'(\mathbf{Z}/N\mathbf{Z})$ (because K is the full preimage of Γ in $G(\mathbf{Z})$).

Now consider general K. Pick $g \in G(\mathbf{A}_f)$ such that $K' := gKg^{-1} \subseteq G(\widehat{\mathbf{Z}})$. Note that $\det(K) = \det(K')$ in the *commutative* quotient $\mathbf{A}_f^\times$ of $G(\mathbf{A}_f)$. Thus, it suffices to naturally identified the functors F_K and $F_{K'}$. This is achieved by noting that the left action by g on the sheaf $I_{E'/M}$ (for any E' in $\mathscr{E}ll_M^0$) carries ks to $(gkg^{-1})(gs)$ on local sections and hence induces an isomorphism $K\backslash I_{E'/M} \simeq K'\backslash I_{E'/M}$ of quotient sheaves over M. Because this procedure commutes with base change on M, is defines a bijection of sets $F_K(M) \to F_{K'}(M)$ that is functorial in M, as desired. ■

If you think about it, you'll see that the passage from E to E_α in Proposition 3.4 is quite cumbersome to make explicit repeatedly in arguments, and this is why the concept of K-structure on an object in $\mathscr{E}ll_M^0$, especially its *functoriality* relative to isomorphisms in $\mathscr{E}ll_M^0$, is genuinely more convenient than the concept of a classical K-structure on an elliptic curve over M when we wish to bring in $G(\mathbf{A}_f)$-actions (including $G(\mathbf{Q})$-actions as a special case).

4. Adelic moduli spaces

Let K be an arbitrary compact open subgroup of $G(\mathbf{A}_f)$, not assumed to lie in $G(\widehat{\mathbf{Z}})$. We have associated to this K the (generally disconnected) complex manifold

$$M_K = G(\mathbf{Q})\backslash G(\mathbf{A})/(Z(\mathbf{R})K_\infty^0 \times K)$$

where Z denotes the center of G and K_∞^0 is the identity component of a chosen maximal compact subgroup of $G(\mathbf{R})$ (equivalently, K_∞^0 is a maximal compact subgroup of $G'(\mathbf{R})$). We let $\tau_0 \in \mathbf{C} - \mathbf{R}$ be the point corresponding to the choice of K_∞ and a choice of $i = \sqrt{-1} \in \mathbf{C}$ (i.e., τ_0 is the unique point in the i-component of $\mathbf{C} - \mathbf{R}$ with stabilizer K_∞^0 in $G'(\mathbf{R})$). In most expositions of the relationship between the classical and adelic theories of modular curves, τ_0 is chosen to be i, but this is entirely unnecessary and somewhat misleading; its only purpose is to correspondingly choose K_∞^0 to be the standard maximal compact subgroup $\mathrm{SO}_2(\mathbf{R})$ ("the" circle) in $\mathrm{SL}_2(\mathbf{R})$ rather than one of its conjugates. Throughout this section, we fix our choices of i and τ_0.

The complex manifold M_K is a disjoint union of finitely many quotients of the half-plane $G'(\mathbf{R})/K_\infty^0$ by the action of some congruence subgroups of $G'(\mathbf{Q})$, as indicated in class. A canonical identification of M_K as the coarse moduli space Y_K for F_K is provided by:

Theorem 4.1. *There is a unique identification of M_K as the coarse moduli space for F_K by assigning to the double coset class of any $g = (g_\infty, g_f) \in G(\mathbf{R}) \times G(\mathbf{A}_f) = G(\mathbf{A})$ the pair $(\mathbf{C}/\Lambda_\tau, \alpha_g)$ where $\tau = g_\infty(\tau_0)$ and α_g is the element in $K\backslash \mathrm{Isom}(\mathbf{A}_f^2, \mathbf{A}_f \otimes_{\mathbf{Z}} \Lambda_\tau) = K\backslash \mathrm{GL}_2(\mathbf{A}_f)$ (via the ordered $\mathbf{Z}$-basis $\{\tau, 1\}$ of Λ_τ and inner composition against the transpose matrix) corresponding to Kg_f^{-1}.*

Before we prove Theorem 4.1, we make some remarks to demonstrate that the situation is clearer if we don't try to recast everything in terms of classical level-structures. The second remark below, a well-posedness property of the description in Theorem 4.1, will actually be used in the proof of Theorem 4.1 to reduce to the consideration of special g.

Remark 4.2. Suppose $K \subseteq G(\widehat{\mathbf{Z}})$, so by Proposition 2.6 there is (up to unique isomorphism) a pair (E_g, ϕ_g) consisting of an elliptic curve E_g and an isomorphism $\phi_g : (E_g)_{\mathbf{Q}} \simeq (\mathbf{C}/\Lambda_{[g_\infty](\tau_0)})_{\mathbf{Q}}$ in $\mathscr{E}ll^0$ carrying $T_f(E_g)$ onto the $\widehat{\mathbf{Z}}$-lattice $(g_f^{-1})^t(\widehat{\mathbf{Z}}^2) \subset \mathbf{A}_f^2 = \mathbf{A}_f \otimes_{\mathbf{Z}} \Lambda_{[g_\infty](\tau_0)}$. Via Proposition 3.4, the preceding theorem says that the double coset of g (for such K) corresponds to the classical K-structure on E_g given by the K-orbit of the tautological isomorphism $\widehat{\mathbf{Z}}^2 \simeq (g_f^{-1})^t(\widehat{\mathbf{Z}}^2) = T_f(E_g)$.

Remark 4.3. Let's now verify for general K that the statement of Theorem 4.1 "makes sense" by directly proving that the pair $(\mathbf{C}/\Lambda_\tau, \alpha_g)$ defined according to the theorem is unaffected up to isomorphism (working in the isogeny category!) when modifying g by left multiplication by $G(\mathbf{Q})$ or right multiplication by $Z(\mathbf{R})K_\infty^0 \times K$. The archimedean component g_∞ intervenes through $g_\infty(\tau_0)$, which is unaffected by right multiplication by the $G(\mathbf{R})$-stabilizer $Z(\mathbf{R})K_\infty^0$ of τ_0, and clearly Kg_f^{-1} is unaffected by replacing g_f with $g_f k$. Hence, the right multiplication aspects are fine.

Consider $\gamma g = (\gamma g_\infty, \gamma g_f)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathbf{Q})$. Note that $[\gamma g_\infty](\tau_0) = [\gamma](\tau)$. The matrix entries for γ typically do not lie in $\mathbf{Z}$ (and even if they did, typically γ is *not* in $G(\mathbf{Z})$), so $\mathbf{C}/\Lambda_{[\gamma](\tau)}$ and $\mathbf{C}/\Lambda_\tau$ are generally not isomorphic as elliptic curves. Nonetheless, multiplication by $1/(c\tau+d)$ on $\mathbf{C}$ does carry $(\Lambda_\tau)_\mathbf{Q}$ isomorphically onto $(\Lambda_{[\gamma](\tau)})_\mathbf{Q}$ since

$$[\gamma](\tau) = a \cdot \frac{\tau}{c\tau+d} + b \cdot \frac{1}{c\tau+d}, \;\; 1 = c \cdot \frac{\tau}{c\tau+d} + d \cdot \frac{1}{c\tau+d}. \tag{1}$$

In other words, the matrix expressing the $\mathbf{Q}$-linear isomorphism $z \mapsto z/(c\tau+d)$ from $(\Lambda_\tau)_\mathbf{Q}$ to $(\Lambda_{[\gamma](\tau)})_\mathbf{Q}$ is $(\gamma^t)^{-1} = (\gamma^{-1})^t$ when using the respective ordered bases $\{\tau, 1\}$ and $\{[\gamma](\tau), 1\}$. (Beware that this isomorphism between 2-dimensional $\mathbf{Q}$-vector spaces inside of $\mathbf{C}$ typically does not carry either of the $\mathbf{Z}$-lattices Λ_τ or $\Lambda_{[\gamma](\tau)}$ into the other!) Hence, the resulting isomorphism $(\mathbf{C}/\Lambda_\tau)_\mathbf{Q} \simeq (\mathbf{C}/\Lambda_{[\gamma](\tau)})_\mathbf{Q}$ in $\mathscr{E}ll^0$ (which usually does not arise from an actual homomorphism of elliptic curves in either direction!) induces the identification of $\mathrm{Isom}(\mathbf{A}_f^2, \mathbf{A}_f \otimes_\mathbf{Q} (\Lambda_\tau)_\mathbf{Q}) = G(\mathbf{A}_f)$ with $\mathrm{Isom}(\mathbf{A}_f^2, \mathbf{A}_f \otimes_\mathbf{Q} (\Lambda_{[\gamma](\tau)})_\mathbf{Q}) = G(\mathbf{A}_f)$ via right multiplication by γ^{-1} (because these equalities with $G(\mathbf{A}_f)$ involve inner composition against the matrix-transpose, canceling out the appearance of transpose for γ^t as in (1) and swapping the order of multiplication). Since $Kg_f^{-1}\gamma^{-1} = K(\gamma g_f)^{-1}$, the isomorphism $(\mathbf{C}/\Lambda_\tau)_\mathbf{Q} \simeq (\mathbf{C}/\Lambda_{[\gamma](\tau)})_\mathbf{Q}$ carries α_g to $\alpha_{\gamma g}$, as desired.

Now we take up the proof of Theorem 4.1:

Proof. Letting $*$ denote the 1-point complex manifold, the underlying set $|M_K|$ of M_K is identified with $\mathrm{Hom}(*, M_K)$ and the set $F_K(*)$ is identified with the set of isomorphism classes of classical objects (E, α). Thus, the uniqueness is immediate from the very definition of a coarse moduli space (viewing the selection of a point in a complex manifold as a holomorphic map from a 1-point space into the manifold). The real content is to construct an element $\xi \in F_K(M_K)$ such that the induced map of sets $M_K \to F_K(*)$ is bijective (as then the identification $F_K(M_K) = \mathrm{Hom}(M_K, Y_K)$ carries ξ to a holomorphic map $M_K \to Y_K$ between pure 1-dimensional complex manifolds which is bijective on underlying sets and hence is a holomorphic isomorphism due to basic facts in one complex variable).

Observe that $K \cap G(\widehat{\mathbf{Z}})$ is a compact open subgroup of $G(\mathbf{A}_f)$ that is contained in $G(\widehat{\mathbf{Z}})$ as well as in K. Thus, it contains a compact open subgroup K' that is normal in K. Consider the natural map of functors $F_{K'} \to F_K$ defined by $(E, \alpha') \mapsto (E, \alpha)$, where E is an object in $\mathscr{E}ll_M^0$ and α is the image of α' under the map on global sections induced by the quotient map of sheaves $K'\backslash I_{E/M} \twoheadrightarrow K\backslash I_{E/M}$ over M. The corresponding map on coarse moduli spaces $Y_{K'} \to Y_K$ is given as expected on underlying sets (carrying a K'-orbit of $\mathbf{A}_f$-linear isomorphisms to the corresponding orbit under the larger group K).

Suppose that the proposed identification of $M_{K'}$ as the coarse moduli space for $F_{K'}$ is correct (as would be the case if we settle the case of compact open subgroups of $G(\widehat{\mathbf{Z}})$). Consider the resulting holomorphic map $M_{K'} = Y_{K'} \to Y_K$ of coarse moduli spaces arising from the natural transformation $F_{K'} \to F_K$. This carries the double coset of $g \in G(\mathbf{A})$ to the pair $(\mathbf{C}/\Lambda_{[g_\infty](\tau_0)}, Kg_f^{-1})$, so it is invariant under the holomorphic natural *left* action by the finite group K/K' on $M_{K'}$ via right multiplication through inversion. This is the same as invariance under the natural *right* action by K/K' on $M_{K'}$, so we thereby get a holomorphic map $M_{K'}/(K/K') \to Y_K$ between pure 1-dimensional complex manifolds. If this is bijective then it is a holomorphic isomorphism, and so the evident holomorphic isomorphism $M_{K'}/(K/K') \simeq M_K$ would thereby identify M_K analytically with Y_K in the desired manner on underlying sets. To verify the required bijectivity of $M_{K'}/(K/K') \to Y_K$ on underlying sets, we use the identification of $|Y_K|$ with $F_K(*)$ (and similarly for K') to reduce to the assertion that the K-structures on a classical object in $\mathscr{E}ll^0$ are precisely the K/K'-orbits of K'-structures. This in turn is just the obvious fact that the quotient of $K'\backslash G(\mathbf{A}_f)$ by the natural left action of the finite group $K/K' = K'\backslash K$ is $K\backslash G(\mathbf{A}_f)$.

Now we may and do assume $K \subseteq G(\widehat{\mathbf{Z}})$. As in the preceding argument, we may replace K with a compact open normal subgroup, such as $K(N)$ for some $N \geq 1$. Hence, we may and do assume $K = K(N)$ for some $N \geq 1$, so

$$M_{K(N)} \simeq G(\mathbf{Z})\backslash(G(\mathbf{R}) \times G(\widehat{\mathbf{Z}}))/K(N) = G(\mathbf{Z})\backslash(G(\mathbf{R}) \times G(\mathbf{Z}/N\mathbf{Z})).$$

The proof of Proposition 3.4 identifies $F_{K(N)}$ with the moduli functor for isomorphism classes of full level-N structures (the "classical" $K(N)$-structures!) on elliptic curves, for which we know *how* $M_{K(N)}$ is a coarse moduli space. The mechanism by which this is done carries the left action by $\gamma \in G(\mathbf{Z}/N\mathbf{Z})$ on full level-N structures (through inner composition with γ^t) over to right translation by γ^{-1}. In Remark 4.3 we showed that the assertion of Theorem 4.1 only depends on $g \in G(\mathbf{A})$ through its double coset class. Hence, left multiplication by $G(\mathbf{Q})$ is harmless and thereby reduces our task to considering $g \in G(\mathbf{R}) \times G(\widehat{\mathbf{Z}})$, consistent with the above description of M_K. Any $g \in G(\mathbf{R}) \times G(\widehat{\mathbf{Z}})$ is obtained from $(g_\infty, 1)$ via right translation by $g_f = ((g_f)^{-1})^{-1}$, so the associated elliptic curve with full level-N structure is $\mathbf{C}/\Lambda_\tau$ for $\tau = [g_\infty](\tau_0)$ and the full level-N structure

$$(g_f^{-1})^t \bmod N : (\mathbf{Z}/N\mathbf{Z})^2 \simeq (\mathbf{Z}/N\mathbf{Z})^2 = \Lambda_\tau / N\Lambda_\tau \simeq (\mathbf{C}/\Lambda_\tau)[N]$$

(using the ordered N-torsion basis $\{\tau/N, 1/N\}$).

The proof of Proposition 3.4 identifies F_K with the moduli functor for isomorphism classes of full level-N structures without changing the underlying elliptic curve when the K-structure is *classical*. Thus, we have obtained an identification of M_K as a coarse moduli space for F_K in a manner that has the desired effect on the double coset representatives. ■

Now we finally reap the fruit of our labors by bringing in the $G(\mathbf{A}_f)$-action to move between moduli spaces. Consider compact open subgroups K and K' in $G(\mathbf{A}_f)$ and any $g \in G(\mathbf{A}_f)$ such that $gK'g^{-1} \subseteq K$. There is a natural transformation

$$J_{K',K}(g) : F_{K'} \to F_K$$

via $(E, \alpha) \mapsto (E, \alpha \circ g^t)$. This makes sense because α is represented locally over M by a K'-orbit of $\mathbf{A}_f$-linear V_f-trivializations ϕ under the action $k'.\phi = \phi \circ {k'}^t$ ($k' \in K'$) and

$$\phi \circ {k'}^t \circ g^t = \phi \circ (gk')^t = \phi \circ (kg)^t = \phi \circ g^t \circ k^t$$

where $k = gk'g^{-1} \in K$ (so the $\phi \circ {k'}^t \circ g^t$ for fixed $g \in G(\mathbf{A}_f)$ and varying $k' \in K'$ all lie in a common K-orbit). Note also that the natural transformation $J_{K',K}(g)$ only depends on g through the double coset KgK' (and the condition "$gK'g^{-1} \subseteq K$" is unaffected by replacing g with any kgk').

Example 4.4. The case $g = 1 \in G(\mathbf{A}_f)$ amounts to requiring $K' \subseteq K$ in $G(\mathbf{A}_f)$, in which case the map $J_{K',K}(1) : F_{K'} \to F_K$ is the "forgetful" map that demotes K'-structures to K-structures (without affecting the elliptic curve). Another interesting case which arises for *arbitrary* $g \in G(\mathbf{A}_f)$ is to take $K' = K \cap g^{-1}Kg$ (which certainly lies in $G(\widehat{\mathbf{Z}})$ when K does, but is not conveniently described in purely $G(\widehat{\mathbf{Z}})$-terms when $g \notin G(\widehat{\mathbf{Z}})$, as is of much interest for Hecke operators). Then visibly $gK'g^{-1} = (gKg^{-1}) \cap K \subseteq K$ and also $K' \subseteq K$, so we get *two* natural transformations

$$J_{K',K}(1), J_{K',K}(g) : F_{K'} \rightrightarrows F_K.$$

These underlie the adelic moduli-theoretic description of Hecke operators.

An important application of our pointwise description of *how* $M_{K'}$ and M_K are respectively identified as coarse moduli spaces for $F_{K'}$ and F_K in Theorem 4.1 is that we get an adelic description of $J_{K',K}(g)$ that is analogous to what we saw long ago for the $G(\mathbf{Z}/N\mathbf{Z})$-action on the moduli space of full level-N structures:

Corollary 4.5. *Consider compact open subgroups $K, K' \subset G(\mathbf{A}_f)$ and $g \in G(\mathbf{A}_f)$ such that $gK'g^{-1} \subseteq K$. Under the respective identifications of $M_{K'}$ and M_K as coarse moduli spaces for $F_{K'}$ and F_K as in Theorem* 4.1*, the holomorphic map*

$$M_{K'} = G(\mathbf{Q})\backslash G(\mathbf{A})/(Z(\mathbf{R})K_\infty^0 \times K') \to G(\mathbf{Q})\backslash G(\mathbf{A})/(Z(\mathbf{R})K_\infty^0 \times K) = M_K$$

corresponding to the natural transformation $J_{K',K}(g) : F_{K'} \to F_K$ is induced by $x \mapsto x(1, g^{-1})$ on $G(\mathbf{A})$. This is a branched finite analytic covering.

By a branched *finite* analytic covering, we mean a holomorphic map that is proper and surjective with finite fibers (so locally on the base, in the 1-dimensional case it looks like a finite disjoint union of copies of standard power maps between unit disks; in terms of the higher-dimensional theory of complex manifolds these are the analogues of "finite flat morphisms" in algebraic geometry). Note that the proposed explicit map on adelic double coset spaces is well-defined, precisely because $x_f k' g^{-1} = x_f g^{-1} k$ for $k := gk'g^{-1} \in K$ and $x_f \in G(\mathbf{A}_f)$.

Proof. Pick $g' = (g'_\infty, g'_f) in G(\mathbf{A})$ and let $\tau' = [g'_\infty](\tau_0)$. By Theorem 4.1 the double coset class of g' in $M_{K'}$ corresponds to the pair $(\mathbf{C}/\Lambda_{\tau'}, \alpha_{g'})$ where $\alpha_{g'}$ goes over to the K'-coset class of ${g'_f}^{-1}$ in $K \backslash G(\mathbf{A}_f)$. By the moduli-theoretic definition of $J_{K',K}(g)$, this pair is carried to the point in M_K corresponding to the K-structure represented by $(\mathbf{C}/\Lambda_{\tau'}, \alpha_{g'} \circ g^t)$, which in turn is the K-coset of $g{g'_f}^{-1} = (g'_f g^{-1})^{-1}$. In other words, the point we get in M_K is the double coset class of $g'(1, g^{-1})$.

It remains to prove that the map $M_{K'} \to M_K$ is a branched finite analytic covering. It is a composition of an isomorphism $M_{K'} \simeq M_{gK'g^{-1}}$ and a "forgetful" map $M_{gK'g^{-1}} \to M_K$, so we are reduced to the special case $g = 1$ (upon renaming $gK'g^{-1}$ as K'). Picking an element of $G(\mathbf{A}_f)$ that conjugates K into $G(\widehat{\mathbf{Z}})$, we may reduce to the case $K \subseteq G(\widehat{\mathbf{Z}})$. It is harmless to shrink K' to some subgroup of the form $K(N)$. Then by inspection, M_K is the quotient of $M_{K(N)}$ by the natural action of the finite group $K/K(N)$. This is a branched finite analytic covering, by the theory of quotients of 1-dimensional complex manifolds equipped with an action of a finite group (as discussed in an earlier handout on such quotients). ■