Historical perspectives of the Riccati equations

Control Days
University of Padua

Based on a presentation at
IFAC WC Toulouse 2017
May 9th, 2019

Marc Jungers

UMR 7039

What is a Riccati equation?

Count Jacopo Francesco Riccati
(1676–1754)

$$a + bx + cx^2 = 0.$$

By extension Riccati equation(s)

Scalar equations:

$$a + bx(t) + cx^2(t) = \dot{x}(t).$$

$$a(t) + b(t)x(t) + c(t)x^2(t) = \dot{x}(t) + f(t).$$

Matrix equations:

$$Q + AX + XB + XSX = 0$$

$$Q + AX(t) + X(t)B + X(t)SX(t) = \dot{X}(t)$$

and any equations involving constant, linear and quadratic term.

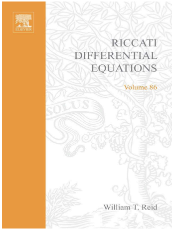
Links with other equations

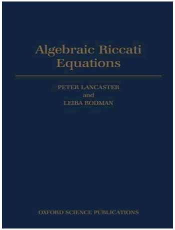
- Lyapunov equation: $Q + A'X + XA = \dot{X}$;
- Bernoulli equation: $bx + cx^2 = \dot{x}$;
- Coupled quadratic equations.

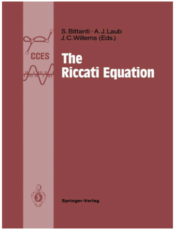
Keyword « Riccati » in google: 681 000 results and more than 6000 papers containing Riccati in their title (google scholar).

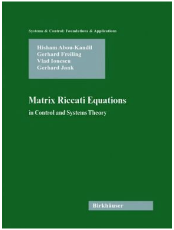
Some books in the literature about Riccati equations

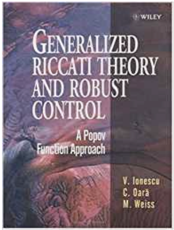
Topics for Riccati equations: algebraic ones, differential ones, coupled ones, numerical aspects and specific applications.

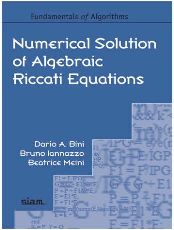


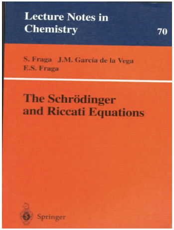


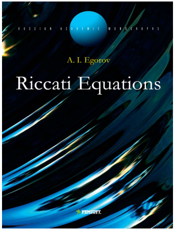












Motivation of this talk

The Riccati equations are particularly popular in control system theory and are mainly involved in LQ regulator, or optimal control, filtering, but also in game theory...

- How appears such an equation?
- Who are the scientists associated with this equation behind Riccati himself?
- What are the crucial dates of its study?
- What are the historical different approaches allowing to solve the Riccati equations?
- What are the historical anecdotes related to the Riccati equations?
- What are the links with control system theory?

This talk tries to give some highlights and answers to these questions.

Other motivation, more personal (i)

Count Jacopo Francesco Riccati (born 28 May 1676 at Venice, Venetian Republic, died 15 April 1754 in Treviso, Venetian Republic) came from the Colonna family from his mother's family side.

He was always attached to the region of Treviso: college in Brescia and he entered University of Padua in 1693 to start law degree (graduated in 1696).

University of Padua

Riccati was also Mayor from 1698 to 1729 of the city of Castelfranco Veneto.

Other motivation, more personal (ii)

At University of Padua, he was interested by Astronomy and was, in parallel of his law studies, the student of Stefano degli Angeli.

In 1695, Angeli offered a copy of Newton's *Philosophiae Naturalis Principia Mathematica*[1] dealing with physics and infinitesimal approach. That consists in the motivation of Riccati to study mathematics and physics.

Stefano degli Angeli
(1623–1697)

He rejected the position of Chair in mathematics at the University of Padua, due to his large ressources, but continued his research there. In particular Giuseppe Suzzi (1701–1764)[2] and Lodovico da Riva (1698–1746) were his private students and became professors in mathematics and astronomy at University of Padua.

[1] I. Newton. *Philosophiæ naturalis principia mathematica*. 1687.

[2] G. Suzzi. *Solutio generalis aequationum tertii gradus*. Patauii: Tipografia del Seminario Padova, 1747.

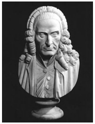
Other motivation, local influence (iii)

Working at University of Padua allowed Riccati to meet several distinguished scientists, among them:

- Giovanni Poleni, who was a professor at the University of Padua;
- Ramiro Rampinelli, a mathematician who was a professor at Rome and at Bologna;
- Bernardino Zendrini, a scientist working for the Republic of Venice.

He also influences others scientists as Maria Gaetana Agnesi[3] at Padua, specialist of algebra and analysis.

Giovanni Poleni (1683–1761)



Bernardino Zendrini (1679–1747)

Maria Gaetana Agnesi (1718–1799)

[3] M. G. Agnesi. *Instituzioni analitiche ad uso della gioventù italiana*. 1748.

Preliminary comments

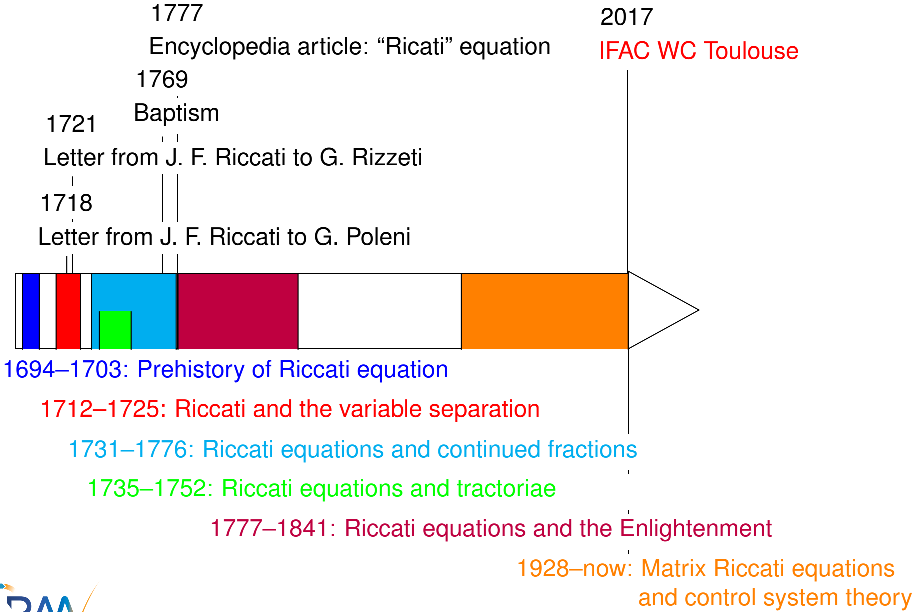
- Names in red are contributors to Riccati equations.
- Equations that are in green boxes are written in a modern way (and are consistent in the whole paper), the other equations are in the original form in the historical documents. *i.e.*

$$xx\mathrm{d}x + yy\mathrm{d}x = aa\mathrm{d}y; \qquad \boxed{a^2\dot{x}(t) = x^2(t) + t^2}$$

- This talk is based on a talk[4] in an historical session at IFAC World Congress 2017 at Toulouse, France, called 300 years developments for an essential tool in control system theory: the Riccati equations, co-organized by M. Jungers, S. Bittanti and P. Colaneri.

[4] M. Jungers. "Historical perspectives of the Riccati equations". In: *The 20th World Congress of the International Federation of Automatic Control*. Invited session "300 years developments for an essential tool in control system theory: the Riccati equations". 2017, pp. 9945–9956.

Main timeline of Riccati equations: 300 years of history



Outline of the talk, following the timeline

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Bernoulli family

Jacob I (1654–1705)

Nicolas (1662–1716)

Johann I (1667–1748)

Nicolas I (1687–1759)

Nicolas II (1695–1726)

Daniel (1700–1782)

Johann II (1710–1790)

Johann III (1744–1807)

Daniel II (1751–1834)

Jacob II (1759–1789)

The story begins with the Bernoulli family before the work of Riccati

[Bernoulli, Joh. I., 1694][5] Exemplo res patebit: Esto proposita æquatio differentialis hæc

$$xx\mathrm{d}x + yy\mathrm{d}x = aa\mathrm{d}y; \qquad \boxed{a^2\dot{x}(t) = x^2(t) + t^2}$$

quæ an per separationem indeterminatarum construi possit nondum tentavi;

[Bernoulli, Jac. I., 1695][6] Problema: Æquationem

$$a\mathrm{d}y = yp\mathrm{d}x + by^n q\mathrm{d}x \qquad \boxed{a\dot{x}(t) = px(t) + bqx^n(t)}$$

(ubi a & b quantitates datas & constantes, n potestatem quamvis lit. y, p & q quantitates utcunque datas per x denotant) construere, saltem per quadraturas, hoc est, separare in illa literas indeterminatas x & y cumsuis differentialibus a se invicem.

[5] Bernoulli, Joh. I. “Modus Generalis construendi omnes æquationes differentiales primi gradus”. In: *Acta Eruditorum, Publicata Lipsiæ* (1694), pp. 435–437, page 436.

[6] Bernoulli, Jac. I. “Explicationes, Annotationes et Additiones ad ea, quæ in Actis sup. anni de Curva Elastica, Isochrona Paracentrica, & Velaria, hinc inde memorata, & partium controversa leguntur, ubi de Linea mediarum directionum, aliisque novis”. In: *Acta Eruditorum, Publicata Lipsiæ* (1695), pp. 537–553, page 553.

First result

First change of variable to simplify the problem provided by Johann I Bernoulli:

> *[Bernoulli, Joh. I., 1697]*[7] *Æquatio proposita est hæc:*
>
> $$a\mathrm{d}y = yp\mathrm{d}x + by^n q\mathrm{d}x \qquad \boxed{a\dot{x}(t) = px(t) + bqx^n(t)}$$
>
> *[...] Ut potestas n deprimatur, ponendum est* $y = v^{n:(1-n)}$*, unde proposita mutatur in hanc ulterius resolvendam* $\frac{1}{1-n}a\mathrm{d}v = vp\mathrm{d}x + bq\mathrm{d}x$.

In several letters, Johann I Bernoulli writes to Leibniz that he cannot solve these equations[8].

Gottfried Wilhelm Leibniz
(1646–1716)

[7] Bernoulli, Joh. I. “De conoidibus et sphæroidibus quædam. Solutio analytica Æquationis in Actis A. 1695, page 553 proposita”. In: *Acta Eruditorum Publicata Lipsiæ* (1697), pp. 113–118, page 115.

[8] Bernoulli, Jac. I. “Letter to G. W. Leibniz, 27 January 1697”. In: *See [46, Letter VIII pp. 48–52]* (), page 50.

First solution as a fraction of power series

[Bernoulli, Jac. I, 1703][9] *Reductio æquationis* $\mathrm{d}y = yy\mathrm{d}x + xx\mathrm{d}x$ $\boxed{\dot{x}(t) = x^2(t) + t^2}$ *ad aliam differentio-differentialem nihil habet mysterii; pono solummodo* $y = -\mathrm{d}z : z\mathrm{d}x$*; [...]*

$$-\mathrm{dd}z : z = xx\mathrm{d}x^2, \qquad \boxed{\frac{1}{z}\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = t^2}$$

[...]

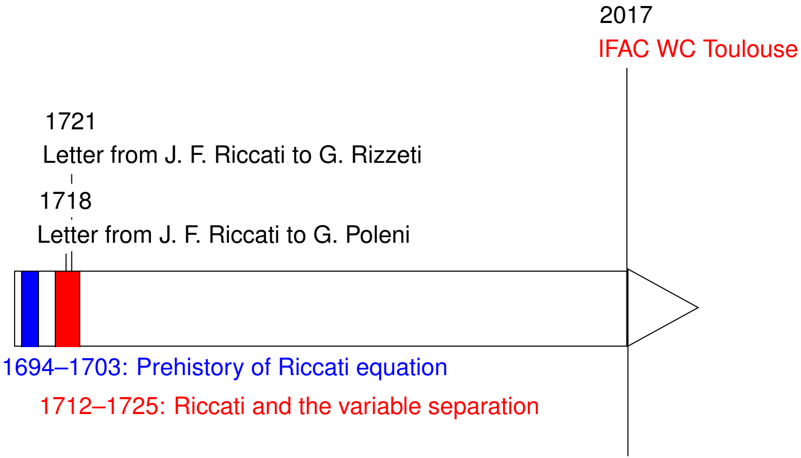
$$y = \frac{\frac{x^3}{3} - \frac{x^7}{3.4.7} + \frac{x^{11}}{3.4.7.8.11} - \frac{x^{15}}{3.4.7.8.11.12.15} + \frac{x^{19}}{3.4.7.8.11.12.15.16.19} - etc}{1 - \frac{x^4}{3.4} + \frac{x^8}{3.4.7.8} - \frac{x^{12}}{3.4.7.8.11.12} + \frac{x^{16}}{3.4.7.8.11.12.15.16} - etc}.$$

$$y = \frac{x^3}{3} + \frac{x^7}{3.3.7} + \frac{2x^{11}}{3.3.3.7.11} + \frac{13x^{15}}{3.3.3.3.5.7.7.11} + etc.$$

At that time, solving a differential equation consisted of giving a solution with a finite number of terms. This is perhaps why such an equation has not been called a *Bernoulli equation*.

[9] Bernoulli, Jac. I. “Letter to G. W. Leibniz, 3 October 1703”. In: *See [46, Letter XII pp. 72–79]* (), pages 74–75.

Main timeline of Riccati equations: 300 years of history



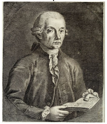
Outline of the talk, following the timeline

Riccati family

Jacopo Francesco
(1676–1754)

Elisabetta
Onigo–Riccati

Vincenzo
(1707–1775)



Giordano
(1709–1790)

OPERE
DEL CONTE
JACOPO RICCATI
NOBILE TREVIGIANO.
Tomo Primo.

IN LUCCA MDCCLXI.
APPRESSO JACOPO GIUSTI.
CON LICENZA DE' SUPERIORI.

They had 18 children (nine of whom died in childhood).

First interest of Jacopo Francesco Riccati (i)

Jacopo F. Riccati worked initially on problems related to plane curves determined by curvature properties. In 1712, he studied the problem of determining the curve having a radius of curvature that is assigned and that depends only on the coordinates. He showed that the curve is solution of a second order differential equation and that a suitable change of variable involving the slope of the curve leads to a first order differential equation[10].

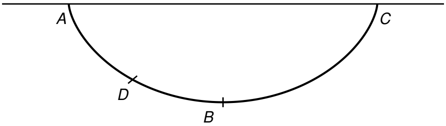
With a modern writing, let us consider a curve described by a function $y(x)$ in Cartesian coordinates. The radius of curvature r is given by $\frac{1}{r} = \frac{y''(x)}{(1+y'(x))^{3/2}}$.
$r = r(y)$ leads to a second order differential equation of the form $f(y, y', y'') = 0$ independent of x.
Introducing the slope $p(y) = y'(x)$, such that $y''(x) = p(y)p'(y)$, we obtain a first order differential equation of type $f(y, p(y), p'(y)) = 0$.
In several examples underlined by Jacopo F. Riccati, such an equation make appear a quadratic term.

[10] J. F. Riccati. “Soluzione generale del Problema inverso intorno ai raggi osculatori, cioè, data in qual si sia maniera per l’ordinata l’espressionne del raggio osculatore, determinar la curva, a cui convenga una tal’ espressionne”. In: *Giornale de’ Letterati D’Italia* Articolo VIII (1712), pp. 204–220.

First interest of Jacopo Francesco Riccati (ii)

Original problem: the motion on a cycloidal curve (or cycloidal pendulum) by taking into account a friction proportional to the velocity. The reasoning here is not clearly mentioned but is enlightened in a part of *Opere del Conte Jacopo Riccati* under the same problem. (*Opere del Conte Jacopo Riccati*[11,12,13,14]).

> *(vol. 3, p. 379) Determinar nella cicloide il moto d'un pendolo a cui si resista dal mezzo in ragione della velocità*



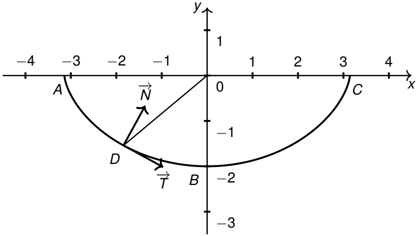
[11] J. F. Riccati. *Opere Del Conte Jacopo Riccati nobile treviagiano*. Vol. Tomo Primo. Lucca, appresso Jacopo Giusti, 1761.

[12] J. F. Riccati. *Opere Del Conte Jacopo Riccati nobile treviagiano*. Vol. Tomo Secondo. Lucca, appresso Jacopo Giusti, 1762.

[13] J. F. Riccati. *Opere Del Conte Jacopo Riccati nobile treviagiano*. Vol. Tomo Terzo. Lucca, appresso Jacopo Giusti, 1764.

[14] J. F. Riccati. *Opere Del Conte Jacopo Riccati nobile treviagiano*. Vol. Tomo Quarto. Lucca, appresso Jacopo Giusti, 1765.

First interest of Jacopo Francesco Riccati (iii)



Cycloid $x(\theta) = \theta + \sin(\theta)$, $y(\theta) = -1 - \cos(\theta)$ with $\theta \in [-\pi, \pi]$. $\vec{T} = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}$ is tangent and $\vec{N} = \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{pmatrix}$ is normal at $D(\theta)$.

The arc length is $s(\theta) = BD$. We have $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = 2|\cos(\theta/2)|\mathrm{d}\theta$ and by integrating $s(\theta) = 4\sin(\theta/2)$. In addition, $y(\theta) - y(0) = \dfrac{s^2(\theta)}{8}$.

With $u = \dfrac{\mathrm{d}s(\theta)}{\mathrm{d}t}$, the theorem of mechanical energy leads to

$$\frac{\mathrm{d}E_m}{\mathrm{d}t} = \frac{\mathrm{d}(mu^2/2 + mg(y(\theta) - y(0)))}{\mathrm{d}t} = -fu.u, \text{ which implies}$$

$$mu\mathrm{d}u + \frac{mg}{4}s\mathrm{d}s = -fu\mathrm{d}s.$$

The slope of the tangent vector $\vec{T}$: $t(\theta) = \dfrac{\mathrm{d}y}{\mathrm{d}x} = \tan(\theta/2)$, which is solution of the Riccati differential equation $t'(\theta) = \dfrac{\mathrm{d}t(\theta)}{\mathrm{d}\theta} = \dfrac{1}{2}(1 + t^2(\theta))$.

First interest of Jacopo Francesco Riccati (iv)

In a letter addressed to Marquess Giovanni Poleni (1683–1761), Jacopo F. Riccati provides a formalization of such an equation, based on mechanical problems.

> *[Riccati J.F. 1718]*[15,16]
>
> *Problema*
>
> *Determinar nella cicloide il moto d'un pendolo a cui si resista dal mezzo in ragione della velocità. [...]*
>
> $$-s\mathrm{d}s + 2gu\mathrm{d}s = u\mathrm{d}u.$$
>
> *[...]*
> *Sia ingionto di separar le variabili in tutte le equazioni differenziali del secondo grado*
>
> $$as^m\mathrm{d}s + bu^p s^q\mathrm{d}s = \mathrm{d}u. \qquad \dot{x}(t) = ax^m(t) + bt^q x^p(t)$$

[15] J. F. Riccati. "Letter to G. Poleni, 16 april 1718". In: *See [63, Letter 16, pp. 108–112]* (1718), pages 110.

[16] M. L. Soppelsa. *Jacopo Riccati – Giovanni Poleni, Carteggio (1715–1742)*. Archivio della corrispondenza degli scienziati italiani, 13. Firenze: Leo S. Olschki, 1997.

First interest of Jacopo Francesco Riccati (v)

In the same letter, he emphasizes four cases for which he obtained a solution, but he asked the question of existence of other solutions. Then he asked to Poleni to speak to Nicolas I Bernoulli.

> *[Riccati J.F. 1718]*[17,18]
> ***Canone primo***. $q = 1$.
> ***Canone secondo***. $m = \dfrac{p+q}{p-q}$. $q = \dfrac{1}{2}$, $p = 0$, $m = 1$.
> ***Canone terzo***. $m = -1$, $q = \dfrac{1}{2}$.
> ***Canone quarto***. $q = 2$; $m = -3p - 4$.

[17] J. F. Riccati. "Letter to G. Poleni, 16 april 1718". In: *See [63, Letter 16, pp. 108–112]* (1718), pages 110.

[18] M. L. Soppelsa. *Jacopo Riccati – Giovanni Poleni, Carteggio (1715–1742)*. Archivio della corrispondenza degli scienziati italiani, 13. Firenze: Leo S. Olschki, 1997.

Exchange of letters about this question

Nicolas I Bernoulli cited this letter and the work of Riccati to Pierre Rémond de Montmort in a letter dated May 18th 1718. He also recalled the early work of his uncles[19]! de Montmort asked in a letter to Nicolas I Bernoulli, dated December 31st 1718[20] if he can send him the solution of the mentioned cases.

Pierre Rémond de Montmort (1678–1719)

Nicolas I Bernoulli answered him in April 1st 1719 by explaining how to obtain the mentioned solution and noticed also that with his uncle, he found additional solutions.

> *[Bernoulli Nic. I, 1719]*[21]
>
> *Nous avons trouvé mon oncle et moi encore deux autres cas de la dite équation, dans lesquels on peut séparer les indéterminées savoir quand* $p = -1$ *et* $q = -1$ *et quand* $q = 2$ *et* $p = -3m - 4$*; ces deux cas se réduisent facilement celui là au* 3^e *et celuici au* 4^e *cas de Mr. Riccati.*

[19] Bernoulli Nic. I. "Letter to Pierre Rémond de Montmort, 18e Mai 1718". In: *Universitätsbibliothek Basel, Signatur L Ia 21:2Bl.251r–253r* (1718), page 252 verso.

[20] P. Rémond de Montmort. "Letter to Bernoulli Nic. I, 31 Décembre 1718". In: *Universitätsbibliothek Basel, Signatur L Ia 22:2 Nr.204* (1718).

[21] Bernoulli Nic. I. "Letter to Pierre Rémond de Montmort, 1 Avril 1719". In: *Universitätsbibliothek Basel, Signatur L Ia 21:2:Bl.264r-268v* (1719), p. 265.

Fundamental letter of Jacopo Riccati to Giovanni Rizzeti

In a letter dated January 1st 1721[22] and addressed to Giovanni Rizzeti (1675–1751), Jacopo F. Riccati came back to these generic differential equations.

> *[J.F. Riccati]*[23]
>
> *1° formula*
>
> $$ax^m\mathrm{d}x + yy\mathrm{d}x = b\mathrm{d}y, \qquad \boxed{b\dot{x}(t) = x^2(t) + at^m} \tag{1}$$
>
> *2° formula*
>
> $$-axx\mathrm{d}x + byy\mathrm{d}x + cx\mathrm{d}x = g\mathrm{d}y. \qquad \boxed{g\dot{x}(t) = bx^2(t) - at^2 + ct}$$

He claimed also that he had already provided solutions to Equation (1) for an infinite number of particular values of *m* (without saying which ones) with variable separation techniques and asked to his friend to deal with Nicolas I Bernoulli, if he knows a generic solution.

[22] S. Bittanti. "The Riccati Equation". In: Springer Verlag, 1991. Chap. Count Riccati and the Early Days of the Riccati Equation, pp. 1–10.

[23] L. Grugnetti. "L'Equazione di Riccati". In: *Bollettino di Storia delle Matematische* VI.1 (1986), pp. 45–82, pages 57–58.

Come back of the Bernoulli family (i)

The Bernoulli family continued to play a crucial role in the history of Riccati equations. Nicolas III Bernoulli met Jacopo F. Riccati and also exchanged letters with him[24].

He encouraged him to submit his results[25], leading to the paper [J.F.Riccati, AEL, 1724][26] and mentioned that he found the solution for $m = -2$ and $m = -4$ (see also the publication [Bernoulli Nic. III., 1720][27] where the substitution $y = \frac{1}{z}$ is considered.).

[24] L. Grugnetti. "On the correspondance between Jacopo Riccati and Nicolaus Bernoulli". In: *The Riccatis and frontier culture in eighteenth-century Europe (Riccati e la cultura della Marca nel Settecento europeo) (Nuncius Library, Florence)*. Ed. by G. Piaia and M. L. Soppelsa. 1992.

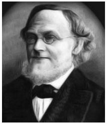
[25] In a letter dated August 26th 1721

[26] J. F. Riccati. "Animadversiones in aequationes differentiales secundi gradus". In: *Acta Eruditorum Lipsiae* VIII.II (1724), pp. 67–73.

[27] Bernoulli Nic. III. "Enodatio alicuius Problematis Geometricia Cel. Jac. Hermanno propositii; atque de Inveniendis Curvis Algebraicis ab eodem Viro propositis, quæ non sint indefinite rectificabiles, habeant tamen aliquos arcus rectificationem admittentes". In: *Acta Eruditorum Lipsiae* (1720), pp. 269–285.

Come back of the Bernoulli family (ii)

Nicolas III Bernoulli introduced in addition the issue of solvability of the differential Riccati equation to Christian Goldbach (1690–1764)[28]. They exchanged several letters, such[29].



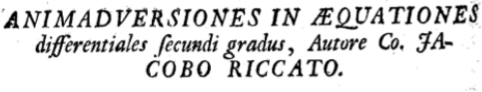
Christian Goldbach
(1690–1764)

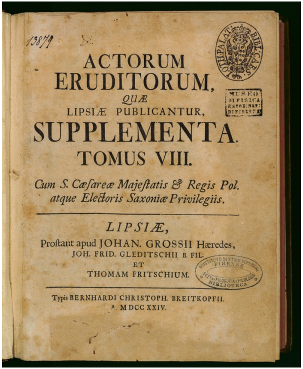
[28] C. Goldbach. "De casibus quibus integrari potest æquatio differentialis $ax^m \mathrm{d}x + byx^p \mathrm{d}x + cy^2 \mathrm{d}x = \mathrm{d}y$ observationes quædam". In: *Commentarii academiae scientiarum Petropolitanae* T. 1 (1726), pp. 185–197.

[29] C. Goldbach. "Letter to N. Bernoulli, 1721 "Sur l'équation du Comte de Riccati"". In: *See [27, Letter I pp. 97–98]* ().

Fundamental paper of Jacopo Riccati

The paper of Jacopo F. Riccati is published in 1724[30] and recalls the results already communicated in letters.


ANIMADVERSIONES IN ÆQUATIONES differentiales ſecundi gradus, Autore Co. JACOBO RICCATO.


ACTORUM ERUDITORUM, QUÆ LIPSIÆ PUBLICANTUR, SUPPLEMENTA. TOMUS VIII.

Cum S. Cæſareæ Majeſtatis & Regis Pol. atque Electoris Saxoniæ Privilegiis.

LIPSIÆ,
Proſtant apud JOHAN. GROSSII Hæredes, JOH. FRID. GLEDITSCHII B. FIL. ET THOMAM FRITSCHIUM.

Typis BERNHARDI CHRISTOPH. BREITKOPFII. A. MDCC XXIV.

[30] J. F. Riccati. “Animadversiones in aequationes differentiales secundi gradus”. In: *Acta Eruditorum Lipsiae* VIII.II (1724), pp. 67–73.

Come back of the Bernoulli family (iii)

Daniel I Bernoulli, with the help of Christian Goldbach and his brother Nicolas III Bernoulli published in answer to the paper [J.F. Riccati AEL, 1724][31] that appears in the same issue [D. Bernoulli, AEL, 1724]. In this paper he provided a solution hidden with a famous anagram, that is impenetrable until now.

DANIELIS BERNOULLI, JOH. FIL. MED.
Cand. Notata in præcedens ſchediaſma Ill. Co. Jacobi Riccati.

> *[D. Bernoulli, AEL, 1724][32] Solutio problematis ab Ill. Riccato propositi characteribus occultis involuta.*
>
> *24a, 6b, 6c, 8d, 33e, 5f, 2g, 4h, 33i, 6l, 21m, 26, 16o, 8p, 5q, 17r, 16s, 25t, 32u, 5x, 3y, +, -, –, ±, =, 4, 2, 1.*

- Daniel Bernoulli was in competition with his brother Nicolas III and they were used to hide their contributions with anagrams;
- The anagram is composed of 24 letters *a*, 6 letters *b*,... (huge number of combinations!) and fixes the date of the discover without being readable.

[31] J. F. Riccati. "Animadversiones in aequationes differentiales secundi gradus". In: *Acta Eruditorum Lipsiae* VIII.II (1724), pp. 67–73.

[32] D. Bernoulli. "Cand. Notata in præcedens schediasma Ill. Co. Jacobi Riccati". In: *Acta Eruditorum Lipsiae* VIII.II (1724), pp. 73–75, page 75.

Come back of the Bernoulli family (iv)

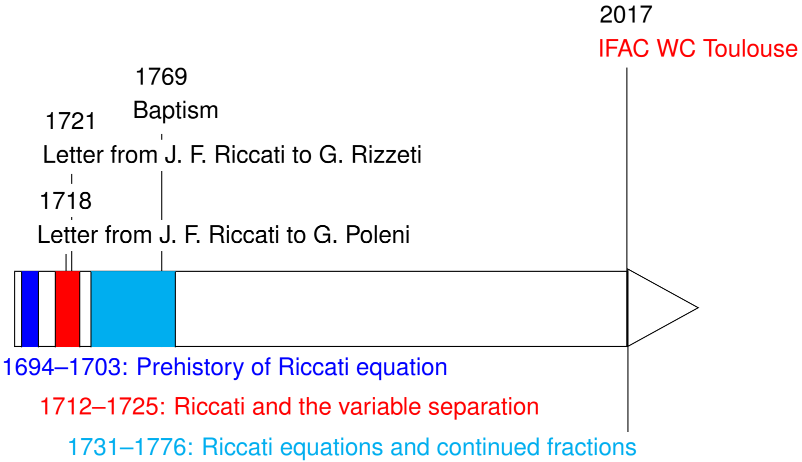
In 1725 (one year later), Daniel Bernoulli completed the possible exponents leading to a quadratic integrable Riccati equation.

> *[D. Bernoulli, AEL, 1725]*[33]
>
> *(A)* $ax^n du + uudx = bdu$ *[...]* $b\dot{x}(t) = x^2(t) + at^n$
>
> ***Lemma primum***. *Si formula (A) separationem indeterminatarum admittit in casu* ($n = m$), *admittet quoque in casu* ($n = \frac{-m}{m+1}$).
>
> ***Lemma primum***. *Si formula (A) separationem indeterminatarum admittit in casu* ($n = m$), *admittet quoque in casu* ($n = -m - 4$)*.[...]*
>
> *Sic alternando applicationem Lemmatum duorum in infinitum semper novi deteguntur valores pro exponente n, qui omnes continentur in formula catholica* $n = \frac{-4c}{2c \pm 1}$, *ubi c significare potest quemcunque numerum integrum sive affirmativum, sive negativum.*

We recognize the formula $n = \frac{-4c}{2c \pm 1}$ in the anagram. We have to wait the result of Liouville to prove the necessity of the sufficient conditions.

[33] D. Bernoulli. "Solutio problematis Riccatiani propositi". In: *Acta Eruditorum Lipsiae* VIII (1725), pp. 473–475, p. 473–474.

Main timeline of Riccati equations: 300 years of history



Outline of the talk, following the timeline

Continued fractions

A continued fraction is of the form

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots}}},$$

where $a_i, \forall i \in \mathbb{N}$ and $b_j, \forall j \in \mathbb{N}^*$ may be either rational numbers, real numbers or complex ones. This is a simple continued fraction when $b_j = 1, \forall j \in \mathbb{N}^*$. They are called finite or infinite depending on the number of levels in the fraction.

Leonhard Euler (1707–1783) is at the origin of a crucial study on continued fractions presented in 1737 and published in 1744[34], nevertheless he started to work on continued fractions to solve a Riccati equation.

Leonhard Euler
(1707–1783)

[34] L. Euler. "De Fractionibus Continuis Dissertatio". In: *In Commentarii academiae scientiarum Petropolitanae* 9 (1737). Presented to the St. Petersburg Academy on March 7, 1737 and published in 1744., pp. 98–137.

Continued fractions: Euler's contribution (i)

> *[L. Euler, 1731][35] Casus nuper formulae Riccatianae separabiles considerans, sequentem universalem detexi substitutionem, qua aequatio*
>
> $$a\,\mathrm{d}q = q^2\mathrm{d}p - \mathrm{d}p \tag{2}$$
>
> *ad hanc formam*
>
> $$a\,\mathrm{d}y = y^2\mathrm{d}x - x^{-\frac{4n}{2n+1}}\mathrm{d}x \tag{3}$$
>
> *reduci potest. Ponatur* $p = (2n+1)x^{\frac{1}{2n+1}}$ *atque*
>
> $$q = -\frac{a}{p} + \cfrac{1}{\cfrac{-3a}{p} + \cfrac{1}{\cfrac{-5a}{p} + \cfrac{1}{\cfrac{-7a}{p} + \cfrac{1}{\ddots\ \cfrac{-(2n-1)a}{p} + \cfrac{1}{x^{\frac{2n}{2n+1}}y}}}}} \tag{4}$$

[35] L. Euler. "Letter to C. Goldbach, 25 November 1731". In: *See [28, Letter XV pp. 56–60]* (), pp. 58–59.

Continued fractions: Euler's contribution (ii)

[L. Euler, 1731][36] *Reciproce etiam aequationem* $a\mathrm{d}y = y^2\mathrm{d}x - x^{\frac{-4n}{2n+1}}\mathrm{d}x$ *in hanc* $a\mathrm{d}q = q^2\mathrm{d}p - \mathrm{d}p$ *transformo hac substitutione* $x = \left(\frac{p}{2n+1}\right)^{2n+1}$ *et*

$$y(p:2n+1)^{2n} = \cfrac{1}{\cfrac{(2n-1)a}{p} + \cfrac{1}{\cfrac{(2n-3)a}{p} + \cfrac{1}{\cfrac{(2n-5)a}{p} + \cfrac{1}{\ddots\, \cfrac{}{\cfrac{3a}{p} + \cfrac{1}{\cfrac{a}{p} + q}}}}}} \quad (5)$$

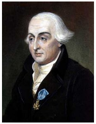
These solutions are related to the ones given by Riccati[37] and by D. Bernoulli[38].

[36] L. Euler. "Letter to C. Goldbach, 25 November 1731". In: *See [28, Letter XV pp. 56–60]* (), pp. 58–59.

[37] J. F. Riccati. "Letter to G. Poleni, 16 april 1718". In: *See [63, Letter 16, pp. 108–112]* (1718).

[38] D. Bernoulli. "Solutio problematis Riccatiani propositi". In: *Acta Eruditorum Lipsiae* VIII (1725), pp. 473–475.

Continued fractions: Lagrange's contribution (i)



Joseph Louis de Lagrange
(1736–1813)

Count Joseph Louis de Lagrange provided in 1776 a generic solution for a scalar Riccati equation with variable dependent weights[39].

If $y(x) = y_0(x)$ is the solution generic differential equation, then use the change of variable using iteratively two sequences $y_i(x)$ and $\xi_i(x)$ by the relations

$$y_i(x) = \frac{\xi_i(x)}{1 + y_{i+1}(x)}, \forall i \in \mathbb{N}.$$

If $\xi_i(x)$ is suitable chosen, then the equation verifies by $y_{i+1}(x)$ is easier to solve and it leads naturally to

$$y(x) = y_0(x) = \cfrac{\xi_0(x)}{1 + \cfrac{\xi_1(x)}{1 + \cfrac{\xi_2(x)}{1 + \cfrac{\xi_3(x)}{1 + \ddots}}}}.$$

He also provided a generic condition to have a finite continued fraction.

[39] J. L. de de Lagrange. "Sur l'usage des fractions continues dans le calcul intégral". In: *Nouveaux mémoires de l'Académie royale des sciences et belles-lettres de Berlin* (1776), pp. 301–332.

Continued fractions: Lagrange's contribution (ii)

Generic choice: $\xi_0(x) = a_0 x^{\alpha_0}$ with $\alpha_0 \in \mathbb{R}$ and $\xi_i(x) = a_i x^{\alpha_i}$ with $\alpha_i > 0$, $\forall i \in \mathbb{N}^*$. For small x, $y_i(x) \sim \xi_i(x) \sim a_i x^{\alpha_i}$, $\forall i \in \mathbb{N}$.
Assuming that $y(x) = a_0 x^{\alpha_0}$ and using this expression in the differential equation leads to a sum of terms of the form $A a_0^m x^{\alpha_0 m + n}$ with m and n integers.
Gathering the terms of the smallest exponent leads to a constraint on α_0 allowing to compensate the amplitudes.

Particular differential equation considered by Lagrange:

$$1 + 2mxy(x) - y^2(x) + nx^2 \frac{\mathrm{d}y(x)}{\mathrm{d}x} = 0.$$

$$\boxed{1 + 2mtx(t) - x^2(t) + nt^2 \dot{x}(t) = 0}$$

Replacing $y(x) = a_0 x^{\alpha_0}$ leads to terms with exponents 0, $1 + \alpha_0$ and $2\alpha_0$. In order to make equal the two smallest exponents, we impose $\alpha_0 = 0$ (the value $\alpha_0 = -1$ is avoided because two exponents are equal but not the smallest ones).
The amplitudes of these two terms of identical and smallest exponents verify $1 - a_0^2 = 0$. We thus choose $\xi_0(x) = 1$.
Iteratively, we obtain: $\xi_1(x) = -mx$, $\xi_2(x) = \dfrac{m-n}{2}x$, $\xi_3(x) = -\dfrac{m+n}{2}x$, $\xi_4(x) = \dfrac{m-2n}{2}x$, $\xi_5(x) = -\dfrac{m+2n}{2}x$, $\xi_6(x) = \dfrac{m-3n}{2}x$, ...
The continued fraction is finite when $m = \lambda n$ where λ is a relative integer.

Continued fractions: Lagrange's contribution (iii)

Link to Riccati equation[40]: From differential equation

$$1 + 2mxy(x) - y^2(x) + nx^2 \frac{\mathrm{d}y(x)}{\mathrm{d}x} = 0,$$

and using the change of variables $x = at^\alpha$ and $y = bt^\beta u$ leads to the generic Riccati equation:

$$\frac{\mathrm{d}u}{\mathrm{d}t} - At^p u^2 + Bt^q = 0, \qquad \boxed{\dot{x}(t) = At^p x^2(t) - Bt^q}$$

with $\alpha = -\dfrac{p+q+2}{2}$, $\beta = \dfrac{p-q}{2}$, $\dfrac{m}{n} = \dfrac{p-q}{2(p+q+2)}$, $a = \dfrac{\alpha}{n\sqrt{AB}}$, $b = \sqrt{\dfrac{A}{B}}$.
The finiteness of the continued fraction is ensured when

$$q = \frac{(1-2\lambda)p - 4\lambda}{1+2\lambda},$$

with λ a positive or negative integer. We recognize the result of Riccati and Bernoulli for $p = 0$: $q = \dfrac{-4\lambda}{1+2\lambda}$.

[40] J. L. de de Lagrange. "Sur l'usage des fractions continues dans le calcul intégral". In: *Nouveaux mémoires de l'Académie royale des sciences et belles-lettres de Berlin* (1776), pp. 301–332, pp. 327.

Baptism of Riccati equations

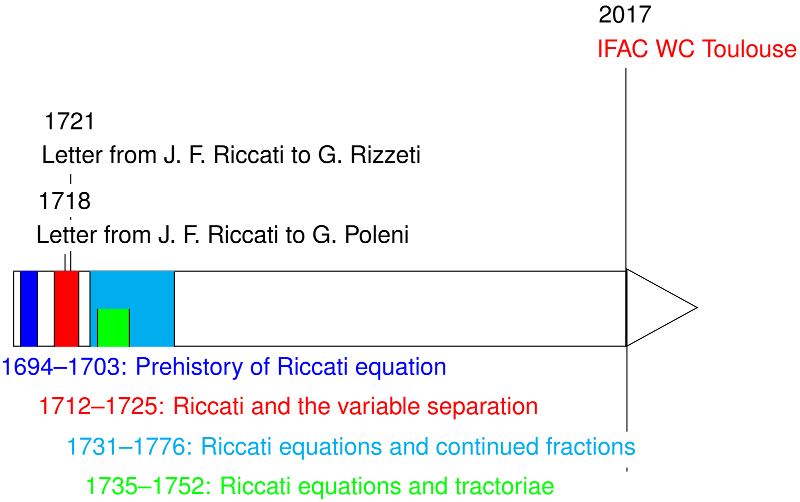
Jean le Rond d'Alembert
(1717–1783)

Jean le Rond d'Alembert was the first to baptize with the name of Riccati this kind of quadratic differential equations in a letter[41] to Count Joseph-Louis Lagrange. Before the Riccati equation was referred to *the equation studied or introduced by Count Riccati*. The topic of this letter deals with the vibrating string.

> *L'équation générale est* $\frac{\mathrm{dd}y}{X\mathrm{d}x^2} = \frac{\mathrm{dd}y}{\mathrm{d}t^2}$*; [...] Pour intégrer cette équation, [...] la question se réduira à intégrer l'équation* $\frac{\mathrm{dd}\zeta}{\mathrm{d}x^2} = \frac{-\lambda^2 X \pi^2 \zeta}{2aLe}$*. En faisant* $\zeta = e^{\int p\mathrm{d}x}$*, cette équation tombe dans le cas si connu de Riccati [...].*

[41] D'Alembert. "Letter to De La Grange". In: *See [2, pp. 235–254]* (1769).

Main timeline of Riccati equations: 300 years of history



Outline of the talk, following the timeline

Notion of tractoria (i)

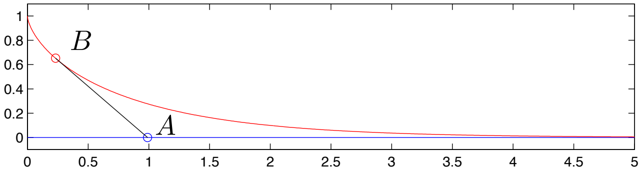
The continued fraction is not the only approach used by Euler to solve a differential Riccati equation.

Actually, he was inspired by the tractional motion introduced by Leibniz and by the work of Poleni, who invented the first instruments that work for plotting the integral of a graphically defined function and called integraphs.

Giovanni Poleni
(1683–1761)

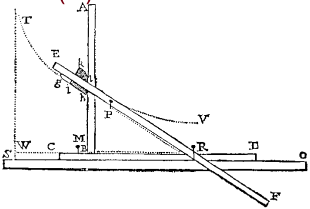
Gottfried Wilhem Leibniz
(1646–1716)

Notion of tractoria (ii)



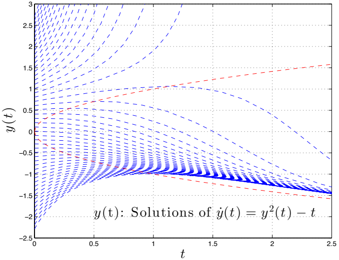
A [*Tractoria*] *or equitangential curve, or curve of pursuit is defined as follows. Consider a curve called basis (the line is called constant basis and the others curves, variable bases) and a point A on this curve. Select one initial point B in the space. The idea is that the segment [AB] is rigid (with constant or variable length) and to move the point A on the basis. The tractoria is the locus of the points B. That implies that the line (AB) is always the tangent of the tractoria at point B.*

Notion of tractoria (iii)



In a letter to G. Poleni[42], Euler imagines an instrument allowing easily to draw a tractoria based on an arbitrary curve and thus that could be used to solve some Riccati equations, with a constant basis. A presentation to Saint-Petersburg Academy is given in 1736 and is only published in 1741[43].

Vincenzo Riccati provided a geometrical method to solve the equation investigated by his own father thanks to tractoria induced by a variable basis, with a variable length of the segment[44].

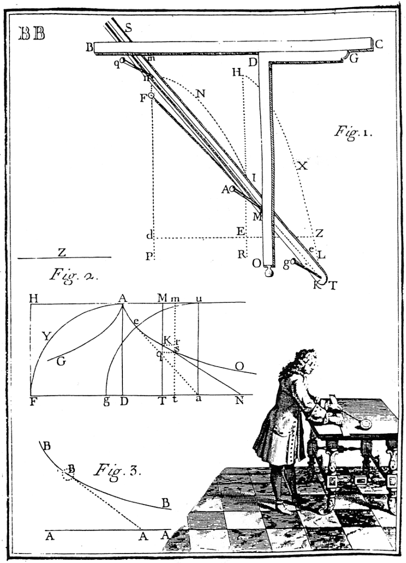


[42] L. Euler. "Letter to G. Poleni, 18 october 1735". In: (1735).

[43] L. Euler. "De constructione æquationum ope motus tractorii aliisque ad methodum tangentium inversam pertinentibus". In: *Commentarii academiae scientiarum Petropolitanae* 8 (1741), pp. 66–85.

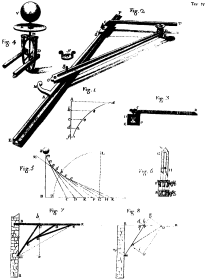
[44] V. Riccati. *De usu motus tractorii in constructione Aequationum Differentialium Commentarius*. Bononiae Typ. Laelii a Vulpe, 1752.

Notion of tractoria (iv)

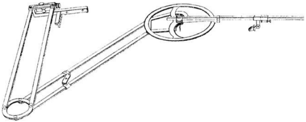


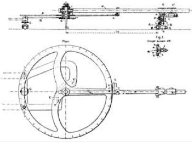
Tractoria: more modern tools

Giambattista Suardi (1711–1767)[45]



Louis Ferdinand Gustave Jacob (1907;1911):[46,47]





INTÉGROMÈTRE

A

LAME COUPANTE

PAR

LE COLONEL JACOB

DE L'ARTILLERIE COLONIALE

Extrait du Mémorial de l'artillerie navale. — 2e livr. de 1909.

UNIVERSITÉ DE PARIS
FACULTÉ DES SCIENCES

CABINET DU DÉPARTEMENT
DES SCIENCES MATHÉMATIQUES

PARIS

IMPRIMERIE NATIONALE

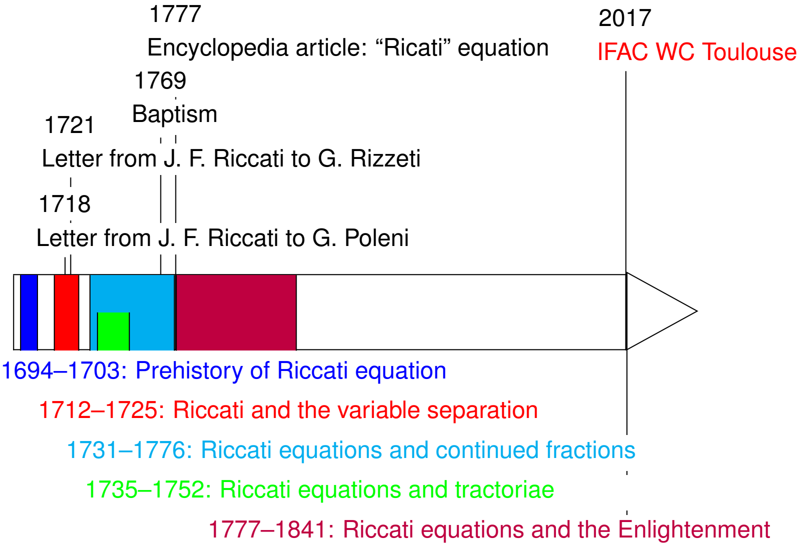
1909

[45] G. Suardi. *Nuovi istromenti per la descrizione di diverse curve antiche e moderne*. Rizzardi, Brescia, 1752.

[46] L. F. G. Jacob. “Intégromètre à lame coupante”. In: *Comptes rendus de l'Académie des Sciences (CRAC)* (1907), 898–900 and 1254.

[47] L. F. G. Jacob. *Le calcul mécanique: appareils arithmétiques et algébriques, intégrateurs*. O. Doin et fils, Paris, 1911.

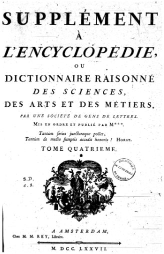
Main timeline of Riccati equations: 300 years of history



Outline of the talk, following the timeline

Article of Condorcet

Nicolas de Condorcet
(1743–1794)



SUPPLÉMENT
À
L'ENCYCLOPÉDIE,
OU
DICTIONNAIRE RAISONNÉ
DES SCIENCES,
DES ARTS ET DES MÉTIERS,
PAR UNE SOCIÉTÉ DE GENS DE LETTRES.
Mis en ordre et publié par M***.
Tantùm series juncturaque pollet,
Tantùm de medio sumptis accedit honoris! Horat.
TOME QUATRIEME.
A AMSTERDAM,
Chez M. M. REY, Libraire.
M. DCC. LXXVII.

The period of Enlightenment, is the occasion to make the point of the current knowledge concerning the Riccati equations. In 1777, the article[48] published in the supplementary material of the French Encyclopedia[49] and authored by Marquess Nicolas de Condorcet (1743–1794) with his signature “ (o) ” relates the knowledge at this time about the Riccati equations. It should be noted that this is the first occurrence of spelling mistake for the name Riccati in the literature with only one letter *c*.

[48] Condorcet. “Supplément à l'Encyclopédie”. In: vol. Tome 4. 1777. Chap. Ricati, p. 648.
[49] D. Diderot. “Supplément à l'Encyclopédie”. In: *Dictionnaire raisonné des sciences, des arts et métiers* Tome 4 (1777).

Necessity of the condition: the crucial contribution of Liouville

Joseph Liouville
(1809–1882)

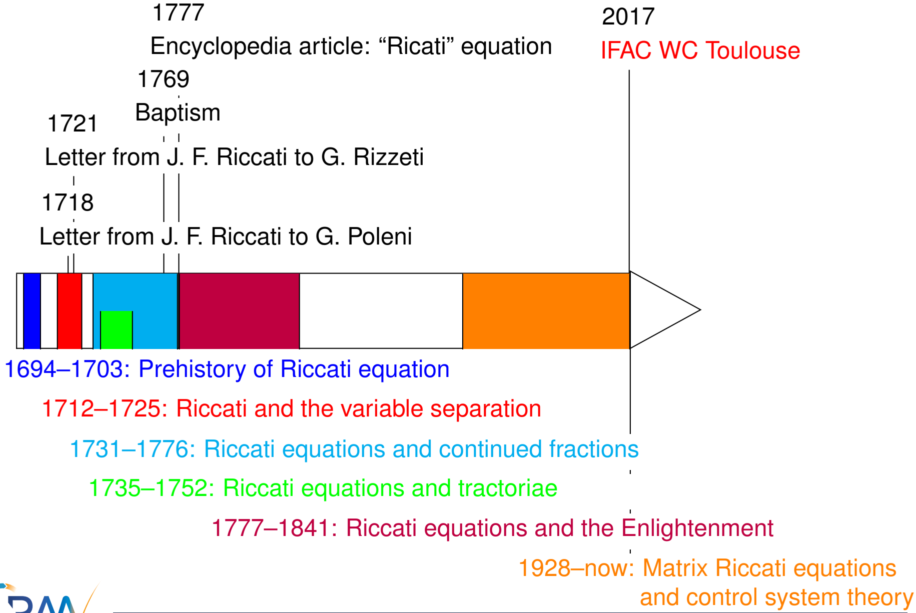
Joseph Liouville obtained solvability conditions for the second order equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x}(x) = P(x)y(x), \qquad \boxed{\ddot{x}(t) = P(t)x(t)}$$

where $P(x)$ is a polynomial in x. He then shows that the Riccati equation can be transformed into the differential equation of the latter form with the polynomial $P(x)$ which is a monomial x^m. The only possible values for the exponent m, called *module*, are of the form $m = \frac{-4h}{2h \pm 1}$, where h is a positive integer. This condition has been already mentioned as a sufficient condition, nevertheless Liouville proved that it is also necessary[50].

[50] J. Liouville. "Remarques nouvelles sur l'équation de Riccati". In: *Journal de Mathématiques Pures et Appliquées* VI (1841), pp. 1–13, pp. 1–2.

Main timeline of Riccati equations: 300 years of history



Outline of the talk, following the timeline

Contribution of Johann Radon

Johann Karl August Radon
(1887–1956)

Zum Problem von Lagrange.

Vier Vorträge von JOHANN RADON in Erlangen,
gehalten im Mathematischen Seminar der Hamburgischen Universität
(7.—24. Juli 1928).

Johann Radon (1887–1956) in the papers[51,52] studied the problem of Lagrange. The last paper is of particular interest for Riccati equations for several reasons:

- this is the first occurrence of a matrix differential Riccati equation,
- a linearization into an extended dimension linear system is provided and finally the related Hamiltonian matrix is emphasized.

---

[51] J. Radon. "Über die Oszillationstheoreme der kunjugierten Punkte beim Probleme von Lagrange. (German)". In: *Münchener Sitzungsberichte* 57 (1927), pp. 243–257.

[52] J. Radon. "Zum Problem von Lagrange. (German)". In: *Hamburger Math. Einzelschr.* 6 (1928), pp. 273–299.

Other matrix Riccati equations

William Marvin Whyburn (1920–1977) characterizes the solution[53] of the matrix equation in the square variable $Y(x)$

$$\frac{\mathrm{d}Y}{\mathrm{d}x}(x) + Y(x)Y(x) = R(x). \qquad \boxed{\dot{X}(t) = -X^2(t) + R(t)}$$

William Thomas Reid (1907–1977) extends the result of Whyburn[54] but with weakened assumptions and finally[55] copes with

$$W' + WA(x) + D(x)W + WB(x)W = C(x),$$

$$\boxed{\dot{X}(t) + X(t)A(t) + D(t)X(t) + X(t)B(t)X(t) - C(t) = 0}$$

where $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are square matrices and continuous on the integration interval.

---

[53] W. M. Whyburn. "Matrix Differential Equations". In: *American Journal of Mathematics* 56.1/4 (1934), pp. 587–592.

[54] W. T. Reid. "Some remarks on linear differential systems". In: *Bull. Amer. Math. Soc.* 45 (1939), pp. 414–419.

[55] W. T. Reid. "A matrix differential equation of Riccati type". In: *American Journal of Mathematics* 68.2 (1946), pp. 237–246.

Non-symmetric Riccati equation

The first nonsymmetric Riccati equations, that is with a rectangular matrix variable, occur with a vector variable[56,57]. The generic case of a rectangular variable has been introduced and studied by Levin[58].

Other manipulations are proposed by William E. Roth to transform the rectangular Riccati equation into a monolateral one.

---

[56] W. E. Roth. "On the matrix equation $X^2 + AX + XB + C = 0$". In: *Proc. Amer. Math. Soc.* 1.5 (1950), pp. 586–589.

[57] W. J. Coles. "Linear and Riccati systems". In: *Duke Math. J.* 22 (1955), pp. 333–338.

[58] J. J. Levin. "On the matrix Riccati equation". In: *Proc. Amer. Math. Soc.* 10 (1959), pp. 519–524.

Numerical techniques to solve algebraic Riccati equations

The numerical aspects are more concerned with the algebraic Riccati equations. Several techniques have been provided, that are straightforward and that avoid the requirement of particular solutions. One can cite:

- the essential work of Alan J. Laub[59] based on graph invariant subspaces of a characteristic matrix and its Schur decomposition;
- Iterative method, the main being provided by David L. Kleinman[60] that solved a Lyapunov equation at each step by fixing the quadratic term;
- The technique developed by considering a Newton's scheme[61];
- The approach considering the matrix sign function[62] by J. Roberts.

[59] A. J. Laub. "A Schur method for solving the algebraic Riccati equation". In: *IEEE Transactions on Automatic Control* 30 (1979), pp. 97–108.

[60] D. L. Kleinman. "On an Iterative Technique for Riccati Equation Computations". In: *IEEE Transactions on Automatic Control* 13.1 (1968), pp. 114–115.

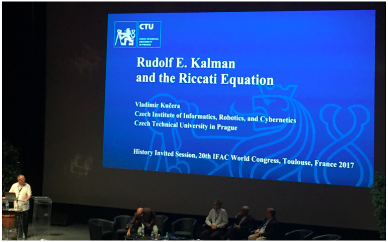
[61] N. R. Sandell. "On Newton's Method for Riccati Equation Solution". In: *IEEE Transactions on Automatic Control* 19.3 (1974), pp. 254–255.

[62] J. Roberts. "Linear model reduction and solution of the algebraic Riccati equation by use of the sign function". In: *International Journal of Control* 32 (1980), pp. 677–687.

Matrix Riccati equations in control system theory (i)

Linear Quadratic Regulator:

- Rudolph E. Kalman (1930–2016)[63] provides a fundamental result related to Riccati equation[64,65].



- Necessary conditions of optimality (the Pontryagin's Minimum Principle (PMP);
- Sufficient ones (Dynamic Programming (DP)).

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[63] See the historical invited session dedicated to the work of R. Kalman at IFAC WC 2017, Toulouse, France.

[64] E.R. Kalman. "Contributions to the theory of optimal control". In: *Bol. Soc. Mat. Mexicana* 2.5 (1960), pp. 102–119.

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Matrix Riccati equations in control system theory (ii)

but also (not exhaustively):

- Filtering with the Kalman–Bucy filter[66];
- $\mathcal{H}_2$-control[67];
- $\mathcal{H}_\infty$-control[68];
- Kalman-Yakubovich-Popov (KYP) lemma;
- Non symmetric or coupled Riccati equations are a key tool in game theory: $\mathcal{H}_\infty$-control or zero-sum games[69], Nash and Stackelberg strategies[70] or finally the mixed $\mathcal{H}_2/\mathcal{H}_\infty$-control[71].

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[66] R. E. Kalman. "A New Approach to Linear Filteting and Prediction Problems". In: *Transactions of the ASME–Journal of Basic Engineering* 82.Series D (1960), pp. 35–45.

[67] J. C. Willems. "Least squares stationary optimal control and the algebraic Riccati equation". In: *IEEE Transactions on Automatic Control* 16.6 (1971), pp. 621–634.

[68] B. A. Francis. *A Course in $\mathcal{H}_\infty$ Control Theory*. Vol. 88. Lectures Notes in Control and Information Sciences. New York: Springer-Verlag, 1987.

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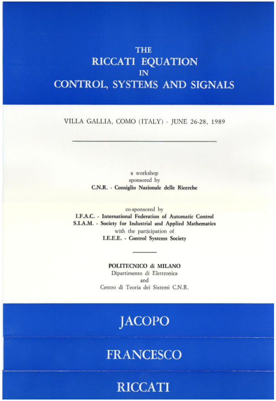
[70] H. Abou-Kandil et al. *Matrix Riccati Equations in Control and Systems Theory*. Birkhäuser, 2003.

[71] D.J.N. Limebeer, B.D.O. Anderson, and H. Hendel. "A Nash Game Approach to Mixed $\mathcal{H}_2/\mathcal{H}_\infty$ Control". In: *IEEE Transactions on Automatic Control* 39.1 (1994), pp. 69–82.

Related workshop

Proceedings of the IFAC-IEEE-SIAM workshop, held in Como, Italy, in 1989[72].



THE RICCATI EQUATION

1989

[72] S. Bittanti, ed. *The Riccati Equation in Control, Systems, and Signals*.. 183. Proceedings of the IFAC/IEEE/SIMA Workshop on the Riccati Equation in Control, Signals and Systems held in Como, June 26-29, 1989. Pitagora Editrice, Bologna, 1989.

Conclusion

Conclusion:

- The origin of the Riccati equations has been humbly emphasized;
- During 300 years, famous mathematicians have contributed on Riccati equations: the Bernoullli family, C. Goldbach, L. Euler, J. L. de Lagrange, N. de Condorcet, J. Le Rond d'Alembert, J. Liouville, J. K. Radon, G. Poleni, G. W. Leibniz...

Extracts of original documents are provided in the paper and copy of original documents are available on request `marc.jungers@univ-lorraine.fr`.

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Thank you for your attention

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