On Proximinality of Convex sets in Super Spaces

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ABSTRACT

In this paper, we show that a closed convex set C of a Banach space is strongly proximinal (proximinal, resp.) in every Banach space isometrically containing it if and only if C is locally (weakly, resp.) compact. As a consequence, it is proved that local compactness of C is also equivalent to that for every Banach space Y isometrically containing it, the metric projection from Y to C is nonempty set-valued and upper semi-continuous.

Key words and phrases: proximinality, convex set, local compactness, Banach space

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1. Introduction

A closed subspace X of a real Banach space Y is said to be *proximinal* in Y if the metric projection $P_X : Y \to X$ defined by

$$P_X(y) = \{x \in X : \|y-x\| = \inf_{u\in X} \|y-u\| \equiv d(y, X)\}, \quad y \in Y \tag{1.1}$$

is nonempty valued at each point $y \in Y$. Clearly, if X is reflexive, then it is *proximinal*. But the converse is not true. A Banach space X is said to be *super proximinal* if it is *proximinal* in every Banach space Y isometrically containing X, i.e. TX is *proximinal* in Y if T is a linear isometry from X into Y. It is easy to observe that every reflexive space is *super proximinal* since a reflexive space is again a reflexive subspace in its super spaces. Pollul [18] first showed that the converse version is valid (see, also [4]), and Singer [20] gave Pollul's result a different proof.

A stronger notion under the name *strongly proximinal* first appeared in Godefroy and Indumathi [14] : A *proximinal* subspace X of a Banach space Y is said to be *strongly proximinal* in Y provided the metric projection $P_X : Y \to X$ satisfies that for every $\varepsilon > 0$ and $y \in Y$ there is $\delta > 0$ such that

$$P_X(y;\delta) \subset P_X(y)+\varepsilon B_X, \tag{1.2}$$

where

$$P_X(y;\delta) = \{u \in X : \|y-u\| < d(y;X) + \delta\}.$$

Since then, properties of strong proximinality have been extensively studied. We refer the readers to Godefroy and Indumathi [14], Godefroy, Indumathi and Lust-Piquard [15], Indumathi [16], Dutta and Narayana [9], [10] for strong proximinality of finite-codimensional subspaces; to Dutta and Shunmugaraj [11], [12] for characterizations of a closed convex set to be strongly proximinal and a quantitative study of strong proximinality; to Fonf, Lindenstrauss and Veselý [13] for properties of strong proximinality in polyhedral spaces; to Rao [19] for proximinality questions for higher-ordered dual spaces; to Bandyopadhyay, Li, Lin, and Narayana [1] for generalizations and applications of strong proximinality.

Strong proximinality of Banach spaces was further studied in Narayana [17]. Among other things, the author proved that a finite-dimensional subspace X of a Banach space Y is always strongly proximinal in Y, but every infinite-dimensional Banach space X can be isometrically embedded as a non-strongly proximinal hyperplane in some space Y; or equivalently, a Banach space X is strong proximinal in every super space if and only if it is finite dimensional.

The aim of this paper is to give Pollul's theorem [18] and Narayana's theorem [17] mentioned above a localized version, i.e. we show that a closed convex set C

of a Banach space is strongly proximinal (proximinal, resp.) in every Banach space isometrically containing it if and only if C is locally (weakly, resp.) compact. Local compactness of a closed convex set C is also shown to be equivalent to that for every superspace Y of C, i.e. the space is isometrically containing C, the metric projection P_C from Y to C is nonempty set-valued and *upper semi-continuous*, where $P_C : Y \to C$ is defined by

$$P_C(y) = \{x \in C : \|y-x\| = \inf_{u\in C} \|y-u\| \equiv d(y, C)\}, \quad y \in Y \tag{1.3}$$

Recall that the metric projection P_C from Y to C is said to be *upper semi-continuous*(in short u.s.c.) on Y, if for any $y \in Y$ and open set U of Y satisfying $P_C(y) \subset U$, there exists some $\delta > 0$ such that $P_C(z) \subset U$, for any $z \in Y$ with $\|z - y\| < \delta$. P_C is said to be *upper $Hausdorff$ semi-continuous*(in short u.H.s.c.) on Y, if for any $y \in Y$ and $\varepsilon > 0$, there exists some $\delta > 0$ such that $P_C(z) \subset P_C(y) + \varepsilon B_Y$, for any $z \in Y$ with $\|z - y\| < \delta$.

2. Super proximinality of closed convex sets

In this section, we shall discuss super proximinality of closed convex sets of Banach spaces, i.e. proximinality of closed convex sets in their superspaces. A set A in a Banach space X is said to be locally (weakly, resp.) compact if for every point $a \in A$ there is $\delta > 0$ such that $A \cap \{x \in X : \|x - a\| \le \delta\}$ is (weakly, resp.) compact. We denote by $B(a, \delta) = \{x \in X : \|x - a\| \le \delta\}$, the closed ball centered at a with radius δ. For a Banach space X, X^* denotes its dual, and B_X (S_X, resp.) stands for its closed unit ball(unit sphere, resp.). Let $\text{co}(A)$ ($\overline{\text{co}}(A)$, resp.) be the convex (closed convex, resp.) hull of a set $A \subset X$.

Proposition 2.1 Let C be a nonempty closed convex set of a Banach space X. Then the following statements are equivalent.

i) C is locally weakly compact;
ii) $C \cap rB_X$ is weakly compact for all $r > 0$;
iii) $C \cap B(x_0, r_0)$ is weakly compact for some $x_0 \in C$ and $r_0 > 0$;
iv) there is $r > \inf_{x\in C} \|x\|$ such that $C \cap rB_X$ is weakly compact.

Proof. i)$\Longrightarrow$ *ii*). Given $a \in C$, we denote by $C_a = C - a$. Locally weak compactness of C implies that $C_a \cap r_0 B_X$ is weakly compact for some $r_0 > 0$. Note that for every $r > 0$ by letting $s = r + \|a\|$ we have $C \cap rB_X \subset C_a \cap sB_X + a$. It suffices to show $C_a \cap sB_X$ is weakly compact for all $s \ge 0$. Nondecreasing monotonicity of sets $A(s) \equiv C_a \cap sB_X$ in $s \in \mathbb{R}^+$ allows us to assume $s \ge r_0$. Given $s \ge r_0$, convexity of C_a together with $0 \in C_a$ entails $\lambda[C_a \cap sB_X] \subset \{x \in C_a : \|x\| \le \lambda s\}$ for all $\lambda \in [0, 1]$. Let $\lambda = r_0/s$. Then $\lambda[C_a \cap sB_X] \subset \{x \in C_a : \|x\| \le r_0\}$. $C_a \cap sB_X$ is necessarily weakly compact since $\{x \in C_a : \|x\| \le r_0\}$ is weakly compact.

ii)$\Longrightarrow$ iii). It is trivial.

iii) $\Longrightarrow$ ii). Since $C \cap B(x_0, r_0)$ is weakly compact for some $x_0 \in C$ and $r_0 > 0$, $C \cap B(x_0, r_0) - x_0 = C_{x_0} \cap r_0 B_X$ is again weakly compact, where $C_{x_0} = C - x_0$. By a simple argument much like the proof of "i)$\Longrightarrow$ ii)", we obtain that $C_{x_0} \cap sB_X$ is weakly compact for all $s \geq 0$. Clearly, this implies ii).

ii)$\Longrightarrow$ i) and ii)$\Longrightarrow$ iv) are trivial.

It remains to show iv)$\Longrightarrow$ iii). Choose any $x_0 \in C$ with $\|x_0\| < r$ and let $\delta_0 = r - \|x_0\|$. Then $C \cap B(x_0, \delta_0) \subset C \cap rB_X$. Since $C \cap rB_X$ is weakly compact, we have iv)$\Longrightarrow$ iii).

A subset A of a Banach space X is said to be proximinal if the metric projection $P_A : X \to A$ is nonempty set-valued at each point of X.

Lemma 2.2 Let C be a closed convex subset of a Banach space X. If $C \cap rB_X$ is proximinal in X for every $r > 0$ then C is proximinal in X.

Proof Given $u \in X$, let $\alpha = d(u, C) \equiv \inf_{v \in C} \|u - v\|$. Then for every $\beta > \alpha + \|u\|$, we have

$$P_C(u) \equiv \{x \in C : \|u - x\| = d(u, C)\} = \{x \in C \cap \beta B_X : \|u - x\| = d(u, C)\}$$
$$= \{x \in C \cap \beta B_X : \|u - x\| = d(u, C \cap \beta B_X)\}.$$

Since $C \cap \beta B_X$ is proximinal in X, $P_C(u) \neq \emptyset$.

We say that a subset A of a Banach space X is isometrically contained in another Banach space Y if there is a linear isometry $T : \overline{\text{span}}(A) \to Y$. In this case, we call the space Y a superspace of A. A is called proximinal in every superspace provided for every Banach space Y, if $T : \overline{\text{span}}(A) \to Y$ is a linear isometry then TA is proximinal in Y. In this case, A is also called super proximinal. The following result is a localized version of Pollul's theorem [18].

Theorem 2.3 A closed convex subset C of a Banach space is super proximinal if and only if C is locally weakly compact.

Proof Sufficiency. Suppose that Y is a Banach space and $T : X \equiv \overline{\text{span}}(C) \to Y$ is a linear isometry. Since C is convex and locally weakly compact in X, TC is also convex and locally weakly compact in Y. Indeed, it immediately follows from the following fact: for every $x \in C$ and $r > 0$, $T(C \cap B(x, r)) = TC \cap B(Tx, r)$. Applying Proposition 2.1, we obtain that $TC \cap rB_Y$ is weakly compact, and which, in turn, entails that $T(C \cap rB_X) = TC \cap rB_Y$ is proximinal in Y for all $r > 0$. By Lemma 2.2, TC is proximinal in Y.

Necessity. Suppose, to the contrary, that C is not locally weakly compact in $X \equiv \overline{\mathrm{span}}C$. Then again by Proposition 2.1 there exists $r > 0$ such that $D \equiv C \cap rB_X$ is not weakly compact. By James' theorem there exists a functional $x^* \in X^*$ with $\|x^*\| = 1$ such that it does not attain the supremum on D. Let

$$a = \sup_{x\in D}\langle x^*, x\rangle, \quad b = \sup_{x\in D}\|x\|, \quad Y = X\times\mathbb{R}, \tag{2.1}$$

and define a norm $\|(\cdot,\cdot)\|$ for $(x,t) \in Y$ by

$$\|(x,t)\| = \max\{\|x\|, |\langle x^*, x\rangle + t|\}. \tag{2.2}$$

Then, with respect to the norm, Y is a Banach space satisfying

$$\|(x,0)\| = \max\{\|x\|, |\langle x^*, x\rangle|\} = \|x\|, \quad \forall\, x \in X. \tag{2.3}$$

So X is linearly isometric to $X \times \{0\} \subset Y$.

Let $J : X \to Y$ be the natural embedding. Then JX is a closed hyperplane of Y. We want to show that JC is not proximinal in Y. Note $JD = JC \cap rB_Y$ and

$$d(u, JC) = \min\{d(u, JD), d(u, JC \setminus JD)\}, \quad \forall u \in Y. \tag{2.4}$$

Let $u = (0, a+b)(\in X \times \mathbb{R})$. Since $\|Jx\| = \|x\| > r \geq b$ for all $x \in C \setminus D$,

$$\|u - Jx\| = \|(0, a+b) - (x,0)\| = \max\{\|x\|, |a+b-\langle x^*, x\rangle|\} \geq \|x\| > r \geq b. \tag{2.5}$$

On the other hand, let $\{x_n\} \subset D$ be such that $\langle x^*, x_n\rangle \to a$. Then

$$d(u, JD) = \inf_{x\in D}\|(0, a+b) - (x,0)\| = b. \tag{2.6}$$

Therefore, (2.4), (2.5) and (2.6) together imply that the image $P_{JC}(u)$ of the point u under the metric project P_{JC} is contained in JD.

We will show that $P_{JD}(u)$ is empty and this results a contradiction. For every $x \in D$, we have

$$\|u - Jx\| = \|(0, a+b) - (x,0)\| = \max\{\|x\|, |a+b-\langle x^*, x\rangle|\}$$

$$= a + b - \langle x^*, x\rangle > b = d(u, JD).$$

Corollary 2.4 A bounded closed convex set C of a Banach space X is super proximinal if and only if C is weakly compact in X.

Corollary 2.5 (Pollul) A Banach space X is proximinal in all of its superspaces if and only if it is reflexive.

The following result follows from Theorem 2.3 and Proposition 2.1. It says that global super proximinality of a closed convex set is equivalent to local super proximinality.

Corollary 2.6 Let C be a nonempty closed convex set of a Banach space X. Then the following statements are equivalent.

i) C is super proximinal;

ii) $C \cap B(x,r)$ is super proximinal for some $x \in C$ and $r > 0$;

iii) $C \cap rB_X$ is super proximinal for all $r > 0$.

An (uniform, resp.) embedding from a subset A of a Banach space X to another Banach space Y is a one-to-one mapping $f : A \to Y$ such that both f and $f^{-1} : f(A) \to A$ are (uniformly, resp.) continuous. In this case, we also say that f is an (uniform, resp.) isomorphism from A to $f(A)$. An embedding f is said to be a linear embedding if there is a linear operator (not necessarily continuous) $T : X \to Y$ such that $T|_A = f$. The well-known Davis-Figiel-Johnson-Pelzyński lemma ([7]) deduces that a closed bounded convex subset K of a Banach space is weakly compact if and only if it can be linearly (uniformly) embedded into a reflexive space, i.e. there is a reflexive space Y and a linear mapping $T : \overline{\text{span}}K \to Y$ such that $T|_K$ is a uniform embedding (see, for instance, [5] and [6]). The following theorem says that it is valid again for unbounded sets.

Theorem 2.7 A closed convex set C of a Banach space is super proximinal if and only if it is linearly isomorphic to a subset of a reflexive space.

Proof Without loss of generality we can assume $0 \in C$.

Sufficiency. Let $X = \overline{\text{span}}(C)$. Due to Theorem 2.3 and Proposition 2.1, it suffices to show $C \cap rB_X$ is weakly compact for some $r > 0$. Let Y be a reflexive Banach space and $T : X \to Y$ be a linear (not necessarily continuous) operator such that $T|_C : C \to TC$ is an isomorphism. TC is locally weakly compact since Y is reflexive. Thus, $T^{-1}|_{TC\cap B_Y} : TC \cap B_Y \to C$ is a linear embedding, and which implies that there is $r > 0$ such that $C \cap rB_X \subset T^{-1}(TC \cap B_Y)$. By Lemma 2.4 of [5], $T^{-1}(TC \cap B_Y)$ is weakly compact. Therefore, C is locally weakly compact.

Necessity. Since C is super proximinal, by Theorem 2.3, it is locally weakly compact. by Theorem 2.6 of [5], there is a reflexive Banach space Y and a linear operator $T : X \to Y$ such that $T|_{C\cap B_X}$ is a uniform isomorphism from $C \cap B_X$ to a subset A of Y. Note $C = \cup_n n(C \cap B_X)$ and linearity of T. Clearly, T_C is an isomorphism from C to $\cup_n nA$.

3. On super strong proximinality

In this section, we consider super strong proximinality of closed convex sets. As a result, we give Narayana's theorem a localized version: A closed convex set of a Banach space is super strongly proximinal if and only if it is locally compact. To begin with, we restate the notion of strong proximinality of convex sets as follows.

A closed convex set C of a Banach space X is said to be *strongly proximinal* provided the metric projection $P_C : X \to C$ is nonempty set-valued everywhere in X and satisfies that for every $\varepsilon > 0$ and every $x \in X$ there is $\delta > 0$ such that

$$P_C(x, \delta) \subset P_X(x) + \varepsilon B_X, \tag{3.1}$$

where

$$P_C(x; \delta) = \{u \in C : \|x - u\| < d(x; C) + \delta\}.$$

Please note the difference between (3.1) and the upper semicontinuity of the projection P_C. C is called strongly proximinal in every superspace provided for every Banach space Y, if $T : \overline{\text{span}}(C) \to Y$ is a linear isometry then TC is strongly proximinal in Y. In this case, C is also called super strongly proximinal.

Let X ba a Banach space and Y is a superspace of X. It is known that if B_X is strongly proximinal in Y, then X is strongly proximinal in Y(see, for instance, [2]). The following lemma is a localized version of this result.

Lemma 3.1 Let C be a closed convex subset of a Banach space X. If $C \cap rB_X$ is strongly proximinal in X for every $r > 0$ then C is strongly proximinal in X.

Proof Given $y \in X$, let $\alpha = d(y, C)$. Then for any fixed $\gamma > 0$,

$$\alpha = \inf\{\|y - x\| : x \in C, \|y - x\| \leq \alpha + \gamma\}.$$

Now let $\beta = \alpha + \gamma + \|y\|$. Then $P_C(y, \gamma) = P_{C \cap \beta B_X}(y, \gamma)$. Since $C \cap \beta B_X$ is strongly proximinal in X, there exists $\gamma > \delta > 0$ such that $P_{C \cap \beta B_X}(y, \delta) \subset P_{C \cap \beta B_X}(y) + \varepsilon B_X$. $P_C(y, \delta) = P_{C \cap \beta B_X}(y, \delta)$ and $P_C(y) = P_{C \cap \beta B_X}(y)$ together entail $P_C(y, \delta) \subset P_C(y) + \varepsilon B_X$.

We have the following result whose proof is similar to that of Proposition 2.1.

Proposition 3.2 Let C be a nonempty closed convex set of a Banach space X. Then the following statements are equivalent.

i) C is locally compact;
ii) $C \cap rB_X$ is compact for all $r > 0$;
iii) $C \cap B(x_0, r_0)$ is compact for some $x_0 \in C$ and $r_0 > 0$;
iv) there is $r > \inf_{x \in C} \|x\|$ such that $C \cap rB_X$ is compact.

The following theorem is a localization of Narayana's Theorem [17]. The idea of its proof is based on Narayana's procedure [17]. We will use a basic sequence to construct a norm on $X \times \mathbb{R}$. We refer the readers to [8] for classic renorming techniques.

Theorem 3.3 Let C be a closed convex subset of a Banach space X. Then C is super strongly proximinal if and only if C is locally compact.

Proof Sufficiency. Suppose Y is a Banach space isometrically containing C. Without loss of generality, we can assume that C itself is a subset of Y. By Proposition 3.2, for every $r > 0$, $C \cap rB_Y$ is compact. Therefore, $C \cap rB_Y$ is strongly proximinal in Y. Due to Lemma 3.1, C is strongly proximinal in Y.

Necessity. Suppose that C is super strongly proximinal, and suppose, to the contrary, that there exists $r > 0$ such that $D \equiv C \cap rB_X$ is not compact. By Theorem 2.3, D is weakly compact. Therefore, there exist $\theta > 0$ and $\{u_n\} \subset D, u \in D$ such that

$$u_n \xrightarrow{w} u; \ \|u_n - u\| \geq \theta, \ \forall n \in \mathbb{N}.$$

Without loss of generality we assume that $u = 0$ and $D \subset B_X$. Applying Bessaga-Pelczynski selection principle [3] we can obtain a basic sequence $\{x_n\} \subset \{u_n\}$. Let $Z = \overline{span}\{x_n\}$ and let $\{x_n^*\} \subset Z^*$ be the corresponding coordinate functionals of $\{x_n\}$. Next, let $Y = X \times \mathbb{R}$,

$$e = (0, 1), \ e_n = (-x_n, 1 - \frac{1}{n}) \in X \times \mathbb{R}, \ n \in \mathbb{N},$$

and

$$A = \{e\} \cup \{e_n, n \in \mathbb{N}\}.$$

Finally, let $\|\cdot\|$ be the Minkowski functional generated by $\overline{\mathrm{co}}(B_X \times \{0\} \cup \pm A)$. Then $\|\cdot\|$ is a norm on Y and the norm topology and the product topology of $Y = X \times \mathbb{R}$ are compatible and with respect to the norm

$$B_Y = \overline{\mathrm{co}}(B_X \times \{0\} \cup \pm A).$$

It is easy to see

$$B_Y \cap (X \times \{0\}) = B_X \times \{0\},$$

or, equivalently,

$$\|x\| = \|(x, 0)\|, \ \forall x \in X.$$

Let $J : X \to Y$ be the natural embedding from X to Y. Then $JX = X \times \{0\}$ and $JC = C \times \{0\}$. In the following, we often blur the distinction between X and JX. Please note that we will finish the proof if we show the following two conditions.

(1) $d(e, JC) = 1, P_{JC}(e) = \{0\}$;

and

(2) $P_{JC}(e, \delta) \not\subseteq P_{JC}(e) + \frac{\theta}{2} B_Y$ for all $\delta > 0$.

Proof of (1). In fact, (1) is equivalent to $B_Y \cap (e - JC) = \{e\}$. Let $f = (-c, 1) \in B_Y \cap (e - JC)$ with $c \in C$. We will show $c = 0$.

Since $B_Y = \overline{\text{co}}(JB_X \cup \pm A)$, there exists $\{f_n\} \subset \text{co}(JB_X \cup \pm A)$, such that $f = \lim_{n\to\infty} f_n$. For every $n \in \mathbb{N}$, let

$$f_n = (z_n, \alpha_n) = \lambda_n u_n + \mu_n v_n + \eta_n w_n,$$

where

$$z_n \in X, \alpha_n \in \mathbb{R}, u_n \in JB_X, v_n \in \text{co}(A), w_n \in \text{co}(-A);$$

and

$$\lambda_n, \mu_n, \eta_n \in \mathbb{R};\ 0 \le \lambda_n, \mu_n, \eta_n \le 1,\ \lambda_n + \mu_n + \eta_n = 1.$$

Note

$$\alpha_n \le \mu_n \le 1;\ z_n \to -c,\ \alpha_n \to 1.$$

We have

$$\mu_n \to 1,\ \lambda_n \to 0,\ \eta_n \to 0.$$

Therefore, $v_n \to f$.

Without loss of generality we can assume $\{f_n\} \subset \text{co}(A)$. Then for every $n \in \mathbb{N}$, there exist

$$\beta_n \ge 0, \lambda_{n,k} \ge 0;\ \text{with}\ \ \beta_n + \sum_{k\ge 1} \lambda_{n,k} = 1,$$

such that

$$f_n = (z_n, \alpha_n) = \beta_n e + \sum_{k\ge 1} \lambda_{n,k} e_k.$$

Therefore,

$$\alpha_n = \beta_n + \sum_{k\ge 1} \lambda_{n,k}(1 - 1/k).$$

So that

$$1 - \alpha_n = \sum_{k\ge 1} \lambda_{n,k}/k.$$

Consequently, for every k,

$$0 \le \lambda_{n,k} \le k(1 - \alpha_n).$$

Since $\alpha_n \to 1$, $\lambda_{n,k} \to 0$ for every $k \in \mathbb{N}$.

On the other hand, it follows from

$$z_n = \sum_{k\ge 1} \lambda_{n,k}(-x_k) = \sum_{k\ge 1} (-\lambda_{n,k}) x_k \to (-c),$$

$c \in Z = \overline{\text{span}}\{x_n\}$ and

$$\langle x_k^*, c\rangle = \lim_{n\to\infty} \langle x_k^*, -z_n\rangle = \lim_{n\to\infty} \lambda_{n,k} = 0,\ \text{for every}\ k \in \mathbb{N}.$$

Hence, $c = 0$ and $f = (0, 1) = e$, i.e. $B_Y \cap (e - JC) = \{e\}$.

Proof of (2). It suffices to prove

$$P_{JC}(e; 2/n) \equiv \{f \in C : \|e - f\| < 1 + 2/n\} \not\subseteq P_{JC}(e) + (\theta/2)B_Y = (\theta/2)B_Y$$

for all $n \in \mathbb{N}$. With $\{x_n\}$ as the same as in the proof of (1), we first show $(x_n, 0) \in P_{JC}(e; 2/n)$.

Since

$$(1 - 1/n)(e - (x_n, 0)) = (1 - 1/n)(-x_n, 0) + (1 - 1/n)e + 2/n(0, 0),$$

$(1 - 1/n)(e - (x_n, 0)) \in B_Y$. Consequently, $\|e - (x_n, 0)\| \leq (1 - 1/n)^{-1} < (1 + 2/n)$.

$(x_n, 0) \notin \theta/2B_Y$ follows from $\|(x_n, 0)\| = \|x_n\| \geq \theta$. Therefore, $P_{JC}(e; 2/n) \not\subseteq P_{JC}(e) + \theta/2B_Y$.

Corollary 3.4 A bounded closed convex subset C of a Banach space X is super strongly proximinal if and only if C is compact in X.

Corollary 3.5 (Narayana) A Banach space X is strongly proximinal in all of its superspaces if and only if it is finite dimensional.

The following result follows from Theorem 3.3 and Proposition 3.2. It says that global super strong proximinality of a closed convex set is equivalent to local super strong proximinality.

Corollary 3.6 Let C be a nonempty closed convex set of a Banach space X. Then the following statements are equivalent.
i) C is super strongly proximinal;
ii) $C \cap B(x, r)$ is super strongly proximinal for some $x \in C$ and $r > 0$;
iii) $C \cap rB_X$ is super strongly proximinal for all $r > 0$.

Narayana has showed [17] that if X is an infinite-dimensional Banach space, then there exists a superspace Y of X such that the metric projection P_X is not u.s.c. We give it a localized version as follows.

Theorem 3.7 Let C be a closed convex subset of a Banach space X. Then for any superspace Y of C, the projection P_C from Y to C is nonempty set-valued and u.s.c. if and only if C is locally compact.

Proof Sufficiency. Suppose that C is locally compact and Y is a superspace of C. By theorem 3.3, C is strongly proximinal in Y. Theorefore, the metric projection P_C is u.H.s.c. (for upper Hausdorff semi-continuity). On the other hand, because for every $y \in Y, P_C(y) \subset C \cap rB_X$ for sufficiently large $r > 0$, $P_C(y)$ is compact due to Proposition 3.2. Thus, these imply that P_C is u.s.c.

Necessity. Without loss of generality we can assume $0 \in C$. Let $Y = X \times_\infty \mathbb{R}$ and let $J : X \to Y$ be the natural embedding from X to Y. Then $JX = X \times \{0\}$ and $JC = C \times \{0\}$. Likewise, in the following we often blur the distinction between C and JC. By necessity hypothesis, the metric projection P_C from Y to C is u.s.c. on Y. Let $e = (0, 1) \in Y$. Then we will finish the proof by showing that $C \cap B_X = P_C(e)$ is compact.

Suppose that $C \cap B_X$ is not compact. We can choose a sequence $\{x_n\} \subset C \cap B_X$ which has no convergent subsequence such that $\|x_n\| \geq \theta$ for all $n \in \mathbb{N}$ and for some $\theta > 0$. For every n, let $y_n = \frac{(n+1)\cdot\theta}{2n} \cdot \frac{x_n}{\|x_n\|}$. Then $\frac{\theta}{2} < \|y_n\| = \frac{(n+1)\cdot\theta}{2n} \leq \theta$. Then $\{y_n\} \subset C$ since C is convex with $0 \in C$. Obviously, $\{y_n\}$ has no convergent subsequence and $y_n \in P_C(\frac{(n+1)\cdot\theta}{2n}e) \setminus P_C(\frac{\theta}{2}e)$ for every n. Let $U \equiv Y \setminus \{y_n\}$. Then U is an open subset of Y satisfying $P_C(\frac{\theta}{2}e) \subset U$ and $P_C(\frac{(n+1)\cdot\theta}{2n}e) \not\subseteq U$ for every n. This contradicts to the upper semi-continuity of P_C at $\frac{\theta}{2}e$ as long as we note that $\frac{(n+1)\cdot\theta}{2n}e \to \frac{\theta}{2}e$.

Corollary 3.8 (Narayana) Let X be a Banach space. Then for any superspace Y of X, the projection from Y to X is nonempty set-valued and u.s.c. if and only if X is finite dimensional.

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References

[1] P. Bandyopadhyay, Y. Li, Bor-Luh Lin, D. Narayana, Proximinality in Banach spaces, J. Math. Anal. Appl. 341 (2008), no. 1, 309C317

[2] P. Bandyopadhyay, B.L. Lin and T.S.S.R.K. Rao, Ball proximinality in Banach spaces, Banach spaces and their applications in analysis, Walter de Gruyter, Berlin, 2007, 251-264.

[3] C. Bessaga and A. Pelczynski, On bases and unconditional convergence of series in Banach spaces, Studia Math., 17 (1958) 151-164.

[4] E.W. Cheney and D.E. Wulbert, Corrigendum: "The existence and unicity of best approximations". Math. Scand. 27 (1970), 245 (1971).

[5] L. Cheng, Q. Cheng and Z. Luo, On some new characterizations of weakly compact sets in Banach spaces, Studia Math, (201) 2010, no.2, 155-166.

[6] L. Cheng, Q. Cheng, B. Wang and W. Zhang, On super-weakly compact sets and uniformly convexifiable sets, Studia Math. 199 (2010), no.2, 145-169.

[7] W.J. Davis, T. Figiel, W.B.J ohnson and A. Pelczyński, Factoring weakly compact operators, J.Funct.Anal. 17 (1974), 311-327.

[8] R. Deville, G. Godefroy, and V. Zizler, Smoothness and renormings in Banach spaces, Pitman Monographs Surveys Pure Appli. Math., Vol. 64, Longman Sci. Tech., Harlow, 1993.

[9] S. Dutta and D. Narayana, Strongly proximinal subspaces in Banach spaces. Function spaces, 143-152, Contemp. Math., 435, Amer. Math. Soc., Providence, RI, 2007.

[10] S. Dutta, D. Narayana, Strongly proximinal subspaces of finite codimension in C(K). Colloq. Math. 109 (2007), no. 1, 119-128.

[11] S. Dutta and P. Shunmugaraj, Strong proximinality of closed convex sets, J. Approx. Theory, (163) 2011, no. 4, 547-553.

[12] S. Dutta and P. Shunmugaraj, Modulus of strong proximinality and continuity of metric projection. Set-Valued Var. Anal. 19 (2011), no. 2, 271-281.

[13] V. Fonf, J. Lindenstrauss and L. Vesely, Best approximation in polyhedral Banach spaces. J. Approx. Theory 163 (2011), no. 11, 1748-1771.

[14] G. Godefroy and V. Indumathi, Strong proximinality and polyhedral spaces, Rev. Mat. Complut. (14) 2001, no.1, 105–125.

[15] G. Godefroy, V. Indumathi and F. Lust-Piquard, Strong subdifferentiability of convex functionals and proximinality, J. Approx. Theory. 116 (2002), 397–415.

[16] V. Indumathi, Semi-continuity of metric projections in ℓ^∞-direct sums, Proc. Amer. Math. Soc. 133 (2005), no. 5, 1441-1449.

[17] D. Narayana, Strong proximinality and renormings, Proc. Amer. Math. Soc, (134)2006, no.4, 1167-1172.

[18] W. Pollul, Reflexivität und Existenz-Teilräume in der linearen Approximationstheorie. Gesellschaft für Mathematik und Datenverarbeitung, Bonn, Ber. No. 53. Gesellschaft für Mathematik und Datenverarbeitung, Bonn, 1972.

[19] T.S.S.R.K. Rao, Some remarks on proximinality in higher dual spaces, J. Math. Anal. Appl. 328 (2007), no. 2, 1173C1177.

[20] I. Singer, On normed linear spaces which are proximinal in every superspace, J. Approx. Theory, (7)1973, 399-402.