COMPLETENESS PROPERTIES IN THE COMPACT-OPEN TOPOLOGY ON FANS

GARY GRUENHAGE AND GLENN HUGHES

ABSTRACT. It is an open problem to characterize those spaces X for which $C_k(X)$, the space of real-valued continuous functions on X with the compact-open topology, has various completeness properties, in particular, the Baire property. We investigate completeness properties of $C_k(X)$ for a class of spaces X having intermediate topologies between the metric and sequential fans. We obtain necessary and sufficient conditions on these X for $C_k(X)$ to be Baire, and show that, except for the sequential fan whose function space is completely metrizable, these $C_k(X)$, while they can be Baire, are never hereditarily Baire or Choquet (a property also known as weakly α-favorable).

1. INTRODUCTION

It is an unsolved problem to find an internal characterization of those spaces X for which the space $C_k(X)$ of all continuous real-valued functions on X with the compact-open topology is a Baire space. Gruenhage and Ma in [GM] show that for locally compact or first-countable X, $C_k(X)$ is Baire if and only if X has the Moving Off Property (abbreviated MOP; see the next section for definitions). It is not known if this characterization is valid for all completely regular spaces X. Other relevant known facts are that $C_k(X)$ is metrizable iff X is hemicompact and $C_k(X)$ is completely metrizable iff X is a hemicompact k-space [MN].

If M is the so-called metric fan, then M is metrizable, non-locally compact, and non-hemicompact, so M does not have the MOP and $C_k(M)$ is not Baire or metrizable. On the other hand, if S_ω denotes the sequential fan, then S_ω is a hemicompact k-space, hence $C_k(S_\omega)$ is completely metrizable. In this paper we use filters $\mathfrak{u}$ on ω to define spaces $S_\mathfrak{u}$ which are fans whose topologies lie between the metric fan and the sequential fan. We will show that the MOP also characterizes Baireness of $C_k(S_\mathfrak{u})$. We also characterize hemicompactness and the k-space property on $S_\mathfrak{u}$ by properties on the filter $\mathfrak{u}$, and show that if $C_k(S_\mathfrak{u})$ is Baire, then $C_k(S_\mathfrak{u})$ must be metrizable. We also show that for a free filter $\mathfrak{u}$ that $C_k(S_\mathfrak{u})$ is never hereditarily Baire or Choquet. Finally we give examples of filters $\mathfrak{u}$ for which $C_k(S_\mathfrak{u})$ has combinations of properties that are not disallowed by our results; in particular, we prove that if $\mathfrak{u}$ is a free ultrafilter, then $C_k(S_\mathfrak{u})$ is Baire (and, by afore-mentioned results, metrizable, but not hereditarily Baire or Choquet), and if $\mathfrak{u}$ is isomorphic to the co-nowhere-dense filter on the rationals, then $C_k(S_\mathfrak{u})$ is metrizable but not Baire.

All spaces in this paper are assumed to be completely regular.

2010 *Mathematics Subject Classification.* 54E52, 54C35.

Key words and phrases. Compact-open topology, Baire Property, Moving Off Property, hereditarily Baire, Choquet.

2. Background Definitions and Results

We say that a family $\mathcal{A}$ of sets *move off* a family $\mathcal{B}$ of sets if for any $B \in \mathcal{B}$ there exists an $A \in \mathcal{A}$ such that $A \cap B = \emptyset$. A necessary and sufficient condition on X for the space $C_p(X)$ of all continuous real-valued functions on X with the topology of pointwise convergence to be Baire was given independently by E. K. van Douwen (unpublished; see page 34 in [vD]) and E.G. Pytkeev [Py]. It is equivalent to the following.

Theorem 2.1. *$C_p(X)$ is Baire if and only if any collection $\mathcal{F}$ of finite subsets of X that move off the finite subsets of X contains an infinite strongly discrete subcollection.*

Here, a collection $\mathcal{G}$ of subsets of X is *discrete* if each point of X has a neighborhood meeting at most one element of $\mathcal{G}$, and is *strongly discrete* if for each $G \in \mathcal{G}$ there is an open superset U_G of G such that $\{U_G : G \in \mathcal{G}\}$ is discrete.

As mentioned in the introduction, it is unknown if there is a property on X which is necessary and sufficient for the compact-open topology $C_k(X)$ to be Baire. Gruenhage and Ma [GM] conjectured that the above characterization for $C_p(X)$ can be naturally generalized via the following property: A space X is said to have the *Moving Off Property (MOP)* if any collection $\mathcal{K}$ of compact subsets of X that move off the compact subsets of X contains an infinite strongly discrete subcollection.

Conjecture [GM]**.** *$C_k(X)$ is Baire if and only if X has the Moving Off Property.*

It is often helpful to use a game characterization of the Baire property when working with the compact-open topology. Recall that the *Choquet game* on X, denoted $\mathrm{Ch}(X)$, is a game with two players E and NE.[1] On move 0, E chooses a non-empty open set U_0 and NE responds with an non-empty open set $V_0 \subseteq U_0$. On move $n > 0$, E chooses a non-empty open set $U_n \subseteq V_{n-1}$ and NE responds with a non-empty open set $V_n \subseteq U_n$. E wins if $\bigcap\{U_i : i \in \omega\} = \emptyset$. NE wins otherwise. The space X is said to be *Choquet* if NE has a winning strategy in $\mathrm{Ch}(X)$.[2]

Theorem 2.2. [Ox] *X is Baire if and only if E has no winning strategy in $\mathrm{Ch}(X)$.*

A game characterization of the Moving Off Property was discussed in [GM]. Consider the game $G_{K,L}(X)$ with two players K and L. On move 0, K chooses a compact set K_0 and L responds with a compact set L_0 such that $L_0 \cap K_0 = \emptyset$. On move n, K plays compact set K_n with no restrictions and L responds with compact L_n such that $L_n \cap L_i = \emptyset$ all $i < n$ and $L_n \cap K_n = \emptyset$ for all $i \leq n$. K wins if $\{L_i : i \in \omega\}$ is strongly discrete. L wins otherwise.

Recall that a space X is a *q-space* if each $x \in X$ has a sequence of neighborhoods $U_0, U_1, \ldots$ such that $x_n \in U_n$ for all n implies that the set $\{x_n\}_{n \in \omega}$ has a cluster point. Clearly every locally compact space and every first countable space is a q-space.

Theorem 2.3. [GM]

(a) If $C_k(X)$ is Baire, then X has the Moving Off Property.
(b) If X is a q-space which has the Moving Off Property, then X must be locally compact.

[1] This game is also referred to as the *Banach-Mazur* game with players β and α taking on the roles of E and NE, respectively

[2] This property is also known as *weakly α-favorable*.

(c) If X is a q-space, then $C_k(X)$ is Baire if and only if X has the Moving Off Property.
(d) If $C_k(X)$ is Choquet then K has a winning strategy in $G_{K,L}(X)$.
(e) A space X has the Moving Off Property if and only if L has no winning strategy in $G_{K,L}(X)$.

Proving the converse of (a) would establish the Gruenhage-Ma conjecture. It is also not known if the converse to (d) holds; it does for locally compact spaces, a result essentially due to D.K. Ma [Ma].

McCoy and Ntantu studied completeness properties on $C_k(X)$ and many results were obtained. Recall that a space X is said to be *hemicompact* if there exists a sequence $\{K_i : i \in \omega\}$ which dominates the compact sets; i.e. if K is compact then there is a $j \in \omega$ such that $K \subseteq K_j$.

Theorem 2.4. [MN]

(a) $C_k(X)$ is metrizable if and only if X is hemicompact.
(b) $C_k(X)$ is completely metrizable if and only if X is a hemicompact k-space.

3. Fans

A *fan* is the set $(\omega \times \omega) \cup \{\infty\}$ with a topology where points in $\omega \times \omega$ are isolated and the subspace topology on $\{i\} \times \omega \cup \{\infty\}$ is a convergent sequence for each $i \in \omega$. If $H \subseteq \omega$ then let $P_H = \{(i,j) \in \omega \times \omega : i \in H\} \cup \{\infty\}$. In particular if $n \in \omega$ the set $P_{\{n\}}$ is a homeomorphic to a convergent sequence with its limit point and is called a *blade of the fan.* If X is a fan and $A \subseteq X$ we will use the notation $\pi(A) = \{i \in \omega : \text{there exists } j \in \omega \text{ such that } (i,j) \in A\}$. So $P_H = \pi^{-1}(H) \cup \{\infty\}$.

Classical examples of fans are the metric fan M and the sequential fan S_ω. A basis for ∞ in M are all sets of the form $B(n) = \{(a,b) \in \omega \times \omega : b \geq n\} \cup \{\infty\}$ for $n \in \omega$. And a basis for ∞ in S_ω are all sets of the form $B(f) = \{(a,b) \in \omega \times \omega : b \geq f(a)\} \cup \{\infty\}$, where $f \in \omega^\omega$. Note that the topology on S_ω is finer than M. We will consider fans with intermediate topologies between M and S_ω.

For future reference, we note here that S_ω is hemicompact (witnessed by $\{P_F : F \subset \omega, |F| < \omega\}$), while M is not. The former is easy to check. For the latter, suppose $K_0, K_1, K_2, \ldots$ are compact subsets of M. For each n, $\omega \times \{n\}$ is closed discrete in M, so we may choose a point $a_n \in \omega$ such that $(a_n, n) \notin K_n$. Then $\{(a_n, n) : n \in \omega\} \cup \{\infty\}$. is compact and not contained in any K_n.

For each filter $\mathfrak{u}$ on ω we define $S_\mathfrak{u}$ to be the fan for which

$$\{\langle f, A, n\rangle : f \in \omega^\omega, n \in \omega, A \in \mathfrak{u}\},$$

is a local base at ∞ where

$$\langle f, A, n\rangle = \{(a,b) \in \omega \times \omega : (a \in A \Rightarrow b \geq n) \wedge (a \notin A \Rightarrow b \geq f(a)\}.$$

We will call such spaces *filter-fans*. If $\mathfrak{u}$ is a (free) ultrafilter, then we will refer to $S_\mathfrak{u}$ as a (resp. free) *ultrafilter-fan.*

The metric and sequential fans are special cases of filter-fans. If $\mathfrak{u}$ is the co-finite filter then $S_\mathfrak{u} = M$ and if $\mathfrak{u}$ is a fixed ultrafilter then $S_\mathfrak{u} = S_\omega$.

We observe that, except for mentioning S_ω, we may as well restrict ourselves to considering free filters. Suppose $\cap\mathfrak{u} = A \neq \emptyset$. If A is cofinite, then $S_\mathfrak{u}$ is the metric fan. Suppose A is finite. If $A \in u$, then $S_\mathfrak{u} = S_\omega$. If $A \notin u$, then $S_\mathfrak{u} = S_\mathfrak{v}$, where $\mathfrak{v}$ is the free filter generated by $\mathfrak{u} \cup \{\omega \setminus A\}$. Finally, suppose that A is infinite and co-infinite. If $A \in \mathfrak{u}$, then $S_\mathfrak{u}$ is homeomorphic to the quotient space of the topological sum of the metric fan and S_ω which identifies the two non-isolated points. Finally, if $A \notin \mathfrak{u}$, then $S_\mathfrak{u}$ is homeomorphic to the corresponding quotient space of the topological sum of the metric fan and $S_\mathfrak{v}$, where $\mathfrak{v}$ is isomorphic to the filter $\mathfrak{u}$ restricted to $\omega \setminus A$.

In the following section we study the properties of filter fans and their function spaces. Among other things, we will show that the Gruenhage-Ma conjecture holds for this class of spaces.

4. The Compact-Open Topology on Fans

We have noted that the metric fan is not hemicompact while S_ω is hemicompact, and that the family $\mathcal{K} = \{P_F : F \subseteq \omega, |F| < \omega\}$ is a dominating family of compact subsets of S_ω. The result below shows that a filter-fan is hemicompact if and only if it doesn't contain a copy of the metric fan, and furthermore, if a filter-fan is hemicompact then the above set $\mathcal{K}$ of compact sets is a dominating family. Hemicompactness of the filter-fan $S_\mathfrak{u}$ is also characterized by an internal property of the filter $\mathfrak{u}$, and by the space $\omega \cup \{\mathfrak{u}\}$, where ω is the set of isolated points and a neighborhood of $\mathfrak{u}$ has the form $F \cup \{\mathfrak{u}\}$, where $F \in \mathfrak{u}$.

Recall that, given a filter $\mathfrak{u}$ on ω, a subset $A \subseteq \omega$ is called $\mathfrak{u}$-*positive* if $A \cap F \neq \emptyset$ for all $F \in \mathfrak{u}$. Equivalently, A is $\mathfrak{u}$-positive if and only if $A^c \notin \mathfrak{u}$.

Proposition 4.1. *Let $\mathfrak{u}$ be a free filter on ω. Then the following are equivalent*

(i) *$S_\mathfrak{u}$ is hemicompact.*
(ii) *$S_\mathfrak{u}$ doesn't contain a copy of the metric fan.*
(iii) *There is no infinite $A \subseteq \omega$ such that A is almost contained in every filter member; i.e. there is no infinite $A \subseteq \omega$ such that $|A \setminus F| < \omega$ for all $F \in \mathfrak{u}$.*
(iv) *For all infinite $J \subseteq \omega$ there is an infinite subset $A \subseteq J$ such that A is not $\mathfrak{u}$-positive.*[3]
(v) *The space $\omega \cup \{\mathfrak{u}\}$ has no non-trivial convergent sequences.*
(vi) *The family $\{P_F : F \subseteq \omega$ finite$\}$ is a dominating family of compact sets.*
(vii) *$C_k(S_\mathfrak{u})$ is metrizable.*

Proof. The equivalence of (iii), (iv) and (v) is clear.

Statement (vi) immediately implies (i). According to Theorem 2.4 statements (i) and (vii) are equivalent.

We show (i) implies (ii). Suppose $S_\mathfrak{u}$ is hemicompact but contains a copy Y of the metric fan. Since $\infty \in Y$ it follows that Y is closed and therefore Y is hemicompact, contrary to the fact that the metric fan is not hemicompact. Therefore (i) implies (ii).

To show that (ii) implies (iii), suppose that there is an infinite $J \subseteq \omega$ such that J is almost contained in every filter member. Consider the set $Y = P_J$ with the subspace topology. We claim that Y is a copy of the metric fan. Suppose $\langle f, A, n\rangle \cap P_J$ is a basic open set around ∞ in Y. Then $J \setminus A = J'$ is finite. Let $m = \max\{f(a) : a \in J'\} \cup \{n\}$. Then $\langle f, A, n\rangle \cap S_J \subseteq \{(j, k) \in J \times \omega : k \geq m\}$. It follows that each open set in Y contains a metric-fan open set. The converse is clear.

It remains to prove (iii) implies (vi). Suppose that $\{P_F : F \subseteq \omega$ finite$\}$ is not a dominating family of compact sets. Let $K \subseteq S_\mathfrak{u}$ be compact but not contained in any P_F. Let $J = \{i \in \omega : K \cap P_i \neq \emptyset\}$. By assumption $|J| = \omega$. We will show

[3] Equivalently, the co-ideal is *tall* (see, e.g., [Mat]).

that J is almost contained in every filter element. Suppose there is a filter element $A' \in \mathfrak{u}$ such that $|J \setminus A'| = \omega$. For each $i \in J$ let $a_i \in \omega$ such that $(i, a_i) \in K$, and let $B = \{(i, a_i) : i \in J \setminus A'\}$. Then it is easy to check that $\infty \notin \mathrm{cl}(B)$. So B is an infinite closed discrete subset of K, contradicting that K is compact. So J is an infinite subset of ω which is almost contained in every filter element. □

If the Gruenhage-Ma conjecture fails in the class of all completely regular spaces, in view of the fact that it does hold for locally compact spaces, it may still be reasonable to conjecture that it holds more generally for all k-spaces. So we consider when $S_\mathfrak{u}$ is a k-space. Since compact subsets of $S_\mathfrak{u}$ are countable, hence metrizable, k-ness of $S_\mathfrak{u}$ is equivalent to it being Fréchet (i.e., $x \in \overline{A} \setminus A$ implies that there is a sequence of points in A converging to x). It is not hard to see that if $S_\mathfrak{u}$ is Fréchet, then $\omega \cup \{\mathfrak{u}\}$ must be Fréchet as well. It turns out that Fréchetness of $\omega \cup \{\mathfrak{u}\}$ is not sufficient, but a little stronger property is.

Recall that a space X is said to be *strongly Fréchet* if for any sequence of sets $A_0 \supseteq A_1 \supseteq A_2 \cdots$ and any $x \in \bigcap\{\overline{A}_i : i \in \omega\}$, there exists a sequence (a_i) that converges to x such that $a_i \in A_i$ for each $i \in \omega$.[4]

Proposition 4.2. *Suppose $\mathfrak{u}$ is a free filter on ω. $S_\mathfrak{u}$ is Fréchet (or a k-space) if and only if $\omega \cup \{\mathfrak{u}\}$ is strongly Fréchet.*

Proof. Suppose $S_\mathfrak{u}$ is Fréchet. We need to check that $\omega \cup \{\mathfrak{u}\}$ is strongly Fréchet at $\mathfrak{u}$. Let $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ be a sequence such that $\mathfrak{u} \in \bigcap\{\overline{A}_i : i \in \omega\}$. This implies that each A_i is $\mathfrak{u}$-positive. For each $i \in \omega$ let $D_i = A_i \times \{i\}$. Let $D = \bigcup\{D_i : i \in \omega\}$. Then $\infty \in \overline{D}$. Let $((n_i, m_i))$ be a sequence of points in D that converge to ∞. We may assume $m_i < m_{i+1}$. For each $j \in \omega$ let $a_j = \min\{n_i : m_i \geq j\}$. It is easy to chech that $a_j \in A_j$ for each $j \in \omega$ and (a_j) limits to $\mathfrak{u}$. Therefore $\omega \cup \{\mathfrak{u}\}$ is strongly Fréchet.

To show the converse suppose that $\omega \cup \{\mathfrak{u}\}$ is strongly Fréchet. Aiming to show that $S_\mathfrak{u}$ is a Fréchet, suppose $D \subseteq S_\mathfrak{u}$ and $\infty \in \overline{D}$. Let $A_0 = \pi(D)$. For all $i > 0$ let $A_i = \pi(D \setminus \omega \times \{0, 1, \cdots, i-1\})$. Then $A_0, A_1, A_2, \cdots$ is a decreasing sequence of $\mathfrak{u}$-positive sets. Let (a_i) be a sequence that converges to $\mathfrak{u}$ such that $a_i \in A_i$ for each $i \in \omega$. For each $i \in \omega$ let $b_i \in \omega$ such that $b_i \geq i$ and $(a_i, b_i) \in D$. Then the sequence $((a_i, b_i))$ converges to ∞, hence $S_\mathfrak{u}$ is Fréchet. □

Proposition 4.1 and Proposition 4.2 immediately give the following corollary.

Corollary 4.3. *Suppose $\mathfrak{u}$ is a free filter on ω. If $S_\mathfrak{u}$ is a k-space then $S_\mathfrak{u}$ is not hemicompact and $C_k(S_\mathfrak{u})$ is not metrizable.*

Corollary 4.4. *Suppose $\mathfrak{u}$ is a free filter on ω. Then $C_k(S_\mathfrak{u})$ is not completely metrizable.*

Proof. This follows immediately by Theorem 2.4 and from the fact that $S_\mathfrak{u}$ can not be both a k-space and hemicompact. □

We will see later that for a filter-fan $S_\mathfrak{u}$, $C_k(S_\mathfrak{u})$ can be Baire, but is never Choquet or hereditarily Baire.

---

[4]This property has also been called *countably bi-sequential*[Mic], and is also equivalent to Fréchet together with the property α_4 as defined by Arhangel'ski[Ar1, Ar2].

Definition 1. *If $\mathfrak{u}$ is a filter on ω define a game $G(\mathfrak{u})$ with two players P_1 and P_2 as follows. P_1 chooses a finite subset $A_0 \subseteq \omega$ and then P_2 chooses a finite subset $B_0 \subseteq \omega$. On play $n > 0$, P_1 choose a finite subset A_n such that $A_n \cap A_i = \emptyset$ and $A_n \cap B_i = \emptyset$ for all $i < n$, and P_2 chooses a finite subset $B_n \subseteq \omega$ (with no restrictions). P_1 wins if $\bigcup\{A_i : i \in \omega\}$ is $\mathfrak{u}$-positive. P_2 wins otherwise.*

Proposition 4.5. *Let $\mathfrak{u}$ be a free filter on ω. Then P_2 has no winning strategy in $G(\mathfrak{u})$, and if $\mathfrak{u}$ is an ultrafilter, then $G(\mathfrak{u})$ is undetermined.*

Proof. It is easy to show that $G(\mathfrak{u})$ is equivalent, in terms of the existence of winning strategies, to the game $G'(\mathfrak{u})$ in which P_2 has to play by the same rules as P_1, i.e., $B_n \cap A_i = \emptyset$ for all $i \leq n$, and $B_n \cap B_i = \emptyset$ for all $i < n$. Then it follows that neither player can have a winning strategy in which the union of his chosen sets lies in the filter $\mathfrak{u}$; for if he did, his opponent could essentially employ the same strategy to also force the union of his chosen sets to also be in $\mathfrak{u}$, yielding a pair of disjoint filter members. Also, in either game, P_1 or P_2 can always guarantee that $\bigcup_{n\in\omega} A_n \cup B_n = \omega$ by adding $\{n\}$ to his/her play in round n if $\{n\}$ hasn't already been covered.

Now to prove the proposition. By the above comment, a winning strategy for P_2 in $G'(\mathfrak{u})$ would give one in which the union of P_2's sets are in $\mathfrak{u}$; hence P_2 has no winning strategy in either game. For ultrafilters, $\mathfrak{u}$-positive sets are in $\mathfrak{u}$, so we get a similar contradiction if we assume a winning strategy for P_1. □

Hemicompactness of the space $S_\mathfrak{u}$ is desirable since it allows the use of a simple family of dominating compact sets. The following two lemmas will be of use.

Lemma 4.6. *Suppose $\mathfrak{u}$ is a free filter. If P_1 has no winning strategy in $G(\mathfrak{u})$ then $S_\mathfrak{u}$ is hemicompact.*

Proof. Suppose $S_\mathfrak{u}$ is not hemicompact. We will show that P_1 has a winning strategy in $G(\mathfrak{u})$. By Proposition 4.1 there is an infinite set $A \subseteq \omega$ which is almost contained in every filter element. If P_1 chooses points from A, then P_1 will win the game $G(\mathfrak{u})$. □

Lemma 4.7. *Suppose $\mathfrak{u}$ is a free filter on ω. If $S_\mathfrak{u}$ has the Moving Off Property then $S_\mathfrak{u}$ is hemicompact.*

Proof. Suppose $S_\mathfrak{u}$ is not hemicompact. Then by Proposition 4.1, $S_\mathfrak{u}$ contains a closed copy M of the metric fan. Since the metric fan is a non-locally compact metric space, it follows that M doesn't have the Moving Off Property by Theorem 2.3(b). Since the Moving Off Property is hereditary under closed sets it follows that $S_\mathfrak{u}$ doesn't have the Moving Off Property. □

Corollary 4.8. *Suppose $\mathfrak{u}$ is a free filter on ω. If $C_k(S_\mathfrak{u})$ is Baire, then $S_\mathfrak{u}$ is hemicompact and $C_k(S_\mathfrak{u})$ is metrizable.*

Proof. Suppose $C_k(S_\mathfrak{u})$ is Baire. It follows from Theorem 2.3(a) that $S_\mathfrak{u}$ has the Moving Off Property. By Lemma 4.7 $S_\mathfrak{u}$ is hemicompact. Therefore by Theorem 2.4, $C_k(S_\mathfrak{u})$ is metrizable. □

The converse of the above corollary is not true, as is shown by Example 5.3 in the next section.

We now show that there is a strong connection between the games $G(\mathfrak{u})$ and $G_{K,L}(S_\mathfrak{u})$.

Proposition 4.9. *Suppose* $\mathfrak{u}$ *is a free filter.*

(i) P_1 *has a winning strategy in* $G(\mathfrak{u})$ *if and only if* L *has a winning strategy in* $G_{K,L}(S_\mathfrak{u})$.

(ii) P_2 *has a winning strategy in* $G(\mathfrak{u})$ *if and only if* K *has a winnings strategy in* $G_{K,L}(S_\mathfrak{u})$.

Proof. We will begin by showing (i). Suppose σ is a winning strategy for P_1 in $G(\mathfrak{u})$ but L doesn't have a winning strategy in $G_{K,L}(S_\mathfrak{u})$. This implies by Lemma 4.7 that $S_\mathfrak{u}$ is hemicompact. Define a strategy τ for L in $G_{K,L}(S_\mathfrak{u})$ as follows.

$G(\mathfrak{u})$	$G_{K,L}(S_\mathfrak{u})$
$A_0 = \sigma(\emptyset)$	K_0
$B_0 = \pi(K_0)$	$L_0 = \tau(K_0) = A_1 \times \{0\}$
$A_1 = \sigma(B_0)$	K_1
$B_1 = \pi(K_1)$	$L_1 = \tau(K_0, K_1) = A_2 \times \{1\}$
$A_2 = \sigma(B_0, B_1)$	$\vdots$

To interpret the chart: A_0 is P_1's first play using σ in $G(\mathfrak{u})$, and K_0 is K's first play in $G_{K,L}(S_\mathfrak{u})$. Then we let $B_0 = \pi(K_0)$ be P_2's response, which is finite since $S_\mathfrak{u}$ is hemicompact, and consider P_1's reply A_1 to this play. Then let $\tau(K_0) = L_0 = A_1 \times \{0\}$ be L's response to K_0, etc.

Since σ is a winning strategy for P_1 it follows that $\bigcup\{A_i : i \in \omega\}$ is $\mathfrak{u}$-positive. Hence if $F \in \mathfrak{u}$ then $\{i \in \omega : A_i \cap F \neq \emptyset\}$ is infinite. Suppose $U = \langle f, F, n\rangle$ is a basic open set around ∞ in $S_\mathfrak{u}$. By the previous observation $\{i > n : A_i \cap F \neq \emptyset\}$ is infinite. Therefore $\{i : L_i \cap U \neq \emptyset\}$ is infinite. It follows that $\{L_i : i \in \omega\}$ is not a strongly discrete family. Therefore τ is a winning strategy for L in $G_{K,L}(S_\mathfrak{u})$, a contradiction.

On the other hand suppose σ is a winning strategy for L in $G_{K,L}(S_\mathfrak{u})$. By Lemma 4.6, $S_\mathfrak{u}$ is hemicompact. Define a strategy τ for P_1 in $G(\mathfrak{u})$ as follows.

$G(\mathfrak{u})$	$G_{K,L}(S_\mathfrak{u})$
$A_0 = \tau(\emptyset) = \{0\}$	$K_0 = P_{B_0 \cup A_0}$
B_0	$L_0 = \sigma(K_0)$
$A_1 = \tau(B_0) = \pi(L_0)$	$K_1 = P_{B_1 \cup A_1}$
B_1	$L_1 = \sigma(K_0, K_1)$
$A_2 = \tau(B_0, B_1) = \pi(L_1)$	$K_2 = P_{B_2 \cup A_2}$
B_2	$\vdots$

That is, start with $\tau(\emptyset) = \{0\} = A_0$, let B_0 be P_2's response, then let $K_0 = P_{B_0 \cup A_0}$ be K's first play in $G_{K,L}(S_\mathfrak{u})$, and if L responds with L_0, let $\tau(B_0)$ be $A_1 = \pi(L_0)$, etc.

We claim that τ is a winning strategy. Assume towards a contradiction that $\bigcup\{A_i : i \in \omega\}$ is not $\mathfrak{u}$-positive. Let $F \in \mathfrak{u}$ such that $F \cap A_i = \emptyset$ for all $i \in \omega$. Since $\infty \notin L_i$ for any $i \in \omega$, it follows that each L_i is a finite subset of $\omega \times \omega$. Furthermore $P_{\pi(L_i)} \cap P_{\pi(L_j)} = \emptyset$ if $i \neq j$. Therefore we can pick a function $f : \omega \to \omega$ that dominates $\bigcup\{L_i : i \in \omega\}$ in the sense that, for any $k \in \omega$, $f(k) > \max\{j : (k, j) \in \bigcup_{i\in\omega} L_i\}$. Then the open set $U = \langle f, F, 0\rangle$ is a basic open set around ∞ that misses each L_i. It follows that $\{L_i : i \in \omega\}$ is a (strongly) discrete family, contrary to the fact that σ is winning for L. This completes the proof of statement (i).

We will now show (ii). Suppose K has a winning strategy σ in $G_{K,L}(S_{\mathfrak{u}})$. We may assume $\infty \in \sigma(\emptyset)$. Since L doesn't have a winning strategy in $G_{K,L}(S_{\mathfrak{u}})$ it follows that $S_{\mathfrak{u}}$ has the Moving Off Property and is hemicompact by Lemma 4.7. Construct a winning strategy τ for P_2 in $G(\mathfrak{u})$ as follows.

$G(\mathfrak{u})$	$G_{K,L}(S_{\mathfrak{u}})$
A_0	$K_0 = \sigma(\emptyset)$
$B_0 = \tau(A_0) = \pi(K_0)$	$L_0 = A_1 \times \{0\}$
A_1	$K_1 = \sigma(K_0, L_0)$
$B_1 = \tau(A_0, A_1) = \sigma(K_1)$	$A_2 \times \{1\}$
A_2	$\vdots$

Since σ is a winning strategy for K in $G_{K,L}(S_{\mathfrak{u}})$ it follows that there is a basic open set $U = \langle f, F, k\rangle$ around ∞ such that $U \cap L_i = \emptyset$ for all $i \in \omega$. Therefore $\{i \in \omega : F \cap A_i \neq \emptyset\} \subseteq \{0, 1, \ldots, k\}$ is finite. So $\bigcup\{A_i : i \in \omega\}$ is not $\mathfrak{u}$-positive.

On the other hand suppose σ is a winning strategy for P_2. Since P_1 doesn't have a winning strategy it follows that $S_{\mathfrak{u}}$ is hemicompact by Lemma 4.6. Define a strategy τ for K in $G_{K,L}(S_{\mathfrak{u}})$ as follows.

$G(\mathfrak{u})$	$G_{K,L}(S_{\mathfrak{u}})$
$A_0 = \pi(L_0)$	$K_0 = \tau(\emptyset) = P_0$
$B_0 = \sigma(A_0)$	L_0
$A_1 = \pi(L_1)$	$K_1 = \tau(L_0) = P_{B_0 \cup A_0}$
$B_0 = \sigma(A_0, A_1)$	L_1
$A_2 = \pi(L_2)$	$K_2 = \tau(L_0, L_1) = P_{B_1 \cup A_1}$
$B_1 = \sigma(A_0, A_1, A_2)$	L_2
$\vdots$	$\tau(L_0, L_1, L_2) = P_{B_2 \cup A_2}$

Since σ is a winning strategy for P_2 in $G(\mathfrak{u})$ it follows that $\bigcup\{A_i : i \in \omega\}$ is not $\mathfrak{u}$-positive. Let $F \in \mathfrak{u}$ such that $F \cap A_i = \emptyset$ for all $i \in \omega$. By similar observations as above we can find a function $f : \omega \to \omega$ that dominates $\bigcup\{L_i : i \in \omega\}$. Let $U = \langle f, F, 0\rangle$. Then $U \cap L_i = \emptyset$ for all $i \in \omega$. It follows that $\{L_i : i \in \omega\}$ is strongly discrete. Hence τ is a winning strategy for K in $G_{K,L}(S_{\mathfrak{u}})$. □

Corollary 4.10. *Suppose $\mathfrak{u}$ is a free filter on ω. Then $C_k(S_{\mathfrak{u}})$ is not Choquet.*

Proof. By Proposition 4.5, P_2 does not have a winning strategy in $G(\mathfrak{u})$. Consequently by Proposition 4.9, K does not have a winning strategy in $G_{K,L}(S_{\mathfrak{u}})$, and so by Theorem 2.3, $C_k(S_{\mathfrak{u}})$ is not Choquet. □

The next corollary, which is immediate from Corollaries 4.8 and 4.10, shows that S_ω is the only space among those we are considering whose function space is completely metrizable.

Corollary 4.11. *Let $\mathfrak{u}$ be a free filter. If $C_k(S_{\mathfrak{u}})$ is Baire, then $C_k(S_{\mathfrak{u}})$ is metrizable but not Choquet.*

We proceed to characterize when $C_k(S_{\mathfrak{u}})$ is Baire. First, we give a characterization on the filter $\mathfrak{u}$ for P_1 not having a winning strategy in $G(\mathfrak{u})$. By Lemma 4.6 and Proposition 4.1, we know if P_1 has no winning strategy in $G(\mathfrak{u})$ then for all infinite $J \subseteq \omega$ there is an infinite subset $A \subseteq J$ such that A is not $\mathfrak{u}$-positive. A strengthening of this will give us our characterization.

Proposition 4.12. *Suppose $\mathfrak{u}$ is a free filter on ω. The following are equivalent.*

(i) P_1 has no winning strategy in $G(\mathfrak{u})$.
(ii) If $\mathcal{F}$ is a collection of finite subsets of ω that moves off the finite sets, then there exists an infinite $\mathcal{F}' \subseteq \mathcal{F}$ such that $\bigcup \mathcal{F}'$ is not $\mathfrak{u}$-positive.

Proof. We will show $(ii) \to (i)$ by contrapositive. Suppose σ is a winning strategy for P_1 in $G(\mathfrak{u})$. Let $\{B_0, B_1, \ldots\}$ denote the finite subsets of ω. Let $A_\emptyset = \sigma(\emptyset)$. For each $i \in \omega$ let $A_i = \sigma(A_\emptyset, B_i)$. If A_s has been defined for all $s \in \omega^{<\omega}$ such that $|s| = n$ then for each $i \in \omega$ let:

$$A_{s^\frown i} = \sigma(A_\emptyset, B_{s(0)}, A_{s(0)}, B_{s(1)}, A_{(s(0),s(1))}, \ldots, B_{s(i)}, A_{s\restriction i+1}, \ldots, B_{s(n-1)}, A_s, B_i)$$

This defines a tree whose branches correspond to plays of the game $G(\mathfrak{u})$. Note if $s \in \omega^\omega$ then $\bigcup\{A_{s\restriction n} : n \in \omega\}$ is $\mathfrak{u}$-positive. We will create a collection $\mathcal{F}$ of finite sets which moves off the finite sets such that if $\mathcal{F}' \subseteq \mathcal{F}$ is infinite, then $\bigcup \mathcal{F}'$ contains the union of $\{A_s \restriction n : n \in \omega\}$ for some sequence $s \in \omega^\omega$; consequently $\bigcup \mathcal{F}'$ is $\mathfrak{u}$-positive.

Let $K_0 = \{(0)\}$ and $F_0 = \bigcup\{A_0\}$. Let $K_1 = \{(1), (0,1)\}$ and $F_1 = \bigcup\{A_1, A_{(0,1)}\}$. In general let $K_n = \{s : \text{s is a finite increasing sequence in } \omega \text{ whose last term is } n\}$ and let $F_n = \bigcup\{A_s : s \in K_n\}$. Note if $s \in K_n$ then since $s(n-1) = n$ it follows that A_s is a play in response to P_2 playing B_n, hence $A_s \cap B_n = \emptyset$. Therefore $B_n \cap F_n = \emptyset$. Thus $\mathcal{F} = \{F_i : i \in \omega\}$ moves off the finite sets.

Suppose $\mathcal{F}' \subseteq \mathcal{F}$ is infinite. Then write $\mathcal{F}' = \{F_{n_0}, F_{n_1}, \ldots\}$ where $s = (n_i)_{i\in\omega}$ is an increasing sequence. The sequence $s \restriction (i+1) \in K_{n_i}$ since $s \restriction (i+1)$ is increasing and $s(i) = n_i$. Hence $A_{s\restriction(i+1)} \subseteq F_{n_i}$ for all $i \in \omega$. Therefore $\bigcup\{A_{s\restriction i} : i \in \omega\} \subseteq \bigcup \mathcal{F}'$, and consequently $\bigcup \mathcal{F}'$ is $\mathfrak{u}$-positive.

Thus we have shown that there exists a collection $\mathcal{F}$ of finite sets that move off the finite sets that has the property that if $\mathcal{F}' \subseteq \mathcal{F}$ is infinite then $\bigcup \mathcal{F}'$ is $\mathfrak{u}$-positive. This is the negation of statement (ii).

On the other hand, to show $(i) \to (ii)$, suppose P_1 does not have a winning strategy in $G(\mathfrak{u})$, and suppose $\mathcal{F}$ is a collection of finite subsets of ω that move off the finite subsets of ω. Clearly there is a strategy σ for P_1 such that P_1 always plays a member of $\mathcal{F}$. Since σ can't be winning, there must be a sequence $\mathcal{F}' = \{F_0, F_1, \ldots\}$ of members of $\mathcal{F}$ corresponding to plays by P_1 using σ whose union is not $\mathfrak{u}$-positive. □

Lemma 4.13. *Suppose $\mathfrak{u}$ is a free filter on ω. If P_1 has no winning strategy in $G(\mathfrak{u})$, then $C_k(S_\mathfrak{u})$ is Baire.*

Proof. Suppose P_1 has no winning strategy in $G(\mathfrak{u})$. Then $S_\mathfrak{u}$ is hemicompact by Lemma 4.6. By Proposition 4.1 we have that $\mathcal{K} = \{P_F : F \subseteq \omega, |F| < \omega\}$ dominates the compact subsets of $S_\mathfrak{u}$. We will show that E has no winning strategy in $\mathrm{Ch}(C_k(S_\mathfrak{u}))$.

Recall that a basic open set in $C_k(X)$ has the form $B(f, K, \epsilon) = \{g \in C_k(X) : |g(x) - f(x)| < \epsilon \text{ for all } x \in K\}$, where $f \in C_k(X)$ and K is compact. This is still a base if the compact sets are restricted to members of a dominating family. So in the play of the game, we may assume the players are restricted to choosing basic open sets $B(f, K, \epsilon)$ where $K \in \mathcal{K}$.

Aiming towards a contradiction, suppose that $C_k(S_\mathfrak{u})$ is not Baire. Then E has a winning strategy σ in $\mathrm{Ch}(C_k(S_\mathfrak{u}))$. Let NE choose finite sets $G_0, G_1, \ldots$ as follows. Suppose $A_0 = \sigma(\emptyset) = B(f_0, P_{F_0}, \epsilon_0)$ is E's first play in the Choquet game. Let G_0 be any finite set such that $G_0 \cap F_0 = \emptyset$ and $\{0\} \subseteq G_0 \cup F_0$. Define $B_0 = B(g_0, P_{F_0 \cup G_0}, \epsilon_0/4)$ as NE's first play, where $g_0 \restriction P_{F_0} = f_0$ and $g_0 \restriction S_{F_0^c} = f_0(\infty)$. Then $B_0 \subseteq A_0$, and it therefore legal play by NE. Suppose $A_0, F_0, f_0, B_0, G_0, g_0, \ldots, A_{n-1}, F_{n-1}, f_{n-1}, B_{n-1}, G_{n-1}, g_{n-1}$ have been defined as above, then let

$$A_n = \sigma(A_0, B_0, \ldots, A_{n-1}, B_{n-1}) = B(f_n, P_{F_0 \cup G_0 \cup \cdots \cup F_{n-1} \cup G_{n-1} \cup F_n}, \epsilon_n)$$

where F_n is disjoint from all previous G_i's and F_i's. Let NE pick a finite set G_n disjoint from all previous F_i's and G_i's such that $\{0, \ldots, n\} \subseteq \bigcup\{G_i \cup F_i : i \leq n\}$ and define NE's play at round n as:

$$B_n = B(g_n, P_{F_0 \cup G_0 \cup \cdots \cup F_n \cup G_n}, \epsilon_n/4)$$

where $g_n \restriction P_{F_0 \cup G_0 \cup \cdots \cup F_n} = f_n$ and $g_n \restriction (P_{F_0 \cup G_0 \cup \cdots \cup F_n})^c = f_n(\infty)$.

We claim that if σ is winning strategy, then $\bigcup\{F_i : i \in \omega\}$ is $\mathfrak{u}$-positive. Suppose $\bigcup\{F_i : i \in \omega\}$ is not $\mathfrak{u}$-positive. We will show that there is a continuous function in $\bigcap\{B_i : i \in \omega\}$. Let $F = \bigcup\{F_i : i \in \omega\}$ and $H = \omega \setminus F$. Note $H \in \mathcal{F}$. Define a function $g : S_{\mathcal{F}} \to \mathbb{R}$ by $g(x) = \lim_{i \to \infty} g_i(x)$. We will show that g is continuous.

Let $\epsilon > 0$. Let $n \in \omega$ such that $\epsilon_n < \epsilon/3$. Then for all $x \in S_H$ we have

$$|g_n(x) - g(x)| \leq \sum_{i=0}^{\infty} |g_{n+i}(x) - g_{n+i+1}(x)| < \sum_{i=0}^{\infty} \epsilon_n/2^i = \epsilon_n < \epsilon/3.$$

Let $A = \bigcup\{G_i : i \leq n\}$ which is finite. By definition, for all $b \in H \setminus A$ and for all $i \in \omega$ we have $g_n(b, i) = f_n(\infty)$. For each $a \in A$ let $N_a \in \omega$ such that if $i > N_a$ then $|g_n(a, i) - g_n(\infty)| < \epsilon/3$. Let $N = \max\{N_a : a \in A\}$. Then for all $h \in H$ and all $i > N$ it follows that $|g_n(h, i) - g_n(\infty)| < \epsilon/3$. Consequently for all $h \in H$ and all $i > N$ we have

$$|g(h,i) - g(\infty)| \leq |g(h,i) - g_n(h,i)| + |g_n(h,i) - g_n(\infty)| + |(g_n(\infty) - g(\infty)| < \epsilon$$

For all $i \in F$ we have that $g \restriction P_i$ is continuous, since P_i is compact and the g_i's uniformly converge to g. For each $i \in F$ let $n_i \in \omega$ such that if $m > n_i$ then $|g(i, m) - g(\infty)| < \epsilon$. Define a function $f : \omega \to \omega$ by $f(i) = n_i$ if $i \in F$ and $f(i) = 0$ otherwise. Then for all $x \in \langle f, H, N \rangle$ we have $|g(x) - g(\infty)| < \epsilon$.

It follows that g is continuous and $g \in \bigcap\{B_i : i \in \omega\}$. Therefore σ isn't a winning strategy. In summary, if σ is a winning strategy for E then $\bigcup\{F_i : i \in \omega\}$ is $\mathfrak{u}$-positive. Therefore if σ is a winning strategy there will be a corresponding winning strategy for P_1 in $G(\mathfrak{u})$, which is a contradiction. Hence E has no winning strategy. It follows that $C_k(S_{\mathfrak{u}})$ is Baire. □

We can summarize the above results by proving the following equivalent conditions for $C_k(S_{\mathfrak{u}})$ being Baire.

Theorem 4.14. *Suppose $\mathfrak{u}$ is a free filter on ω. The following are equivalent.*

(i) *$S_{\mathfrak{u}}$ has the Moving Off Property.*
(ii) *L has no winning strategy in $G_{K,L}(S_{\mathfrak{u}})$.*
(iii) *P_1 has no winning strategy in $G(\mathfrak{u})$.*
(iv) *For any collection $\mathcal{F}$ of finite subsets of ω that moves off the finite sets, there exists an infinite $\mathcal{F}' \subseteq \mathcal{F}$ such that $\bigcup \mathcal{F}'$ is not $\mathfrak{u}$-positive.*
(v) *$C_k(S_{\mathfrak{u}})$ is Baire.*

Proof. (*i*) implies (*ii*) by Theorem 2.3(e). (*ii*) implies (*iii*) by Proposition 4.9. Proposition 4.12 shows (*iii*) and (*iv*) are equivalent. Proposition 4.13 shows (*iii*) implies (*v*). And Theorem 2.3 shows (*v*) implies (*i*). ☐

We will now show for a free filter $\mathfrak{u}$ that $C_k(S_\mathfrak{u})$ is not hereditarily Baire. Recall that the *Strong Choquet Game* on a topological space X, denoted $\mathrm{Ch}^*(X)$, is a game with two player E and NE. On move 0 E chooses a non-empty open set U_0 and a point $p_0 \in U_0$. NE responds with a open set V_0 such that $p_0 \in V_0 \subseteq U_0$. On move $n > 0$ E chooses a non-empty open set $U_n \subseteq V_{n-1}$ and a point $p_n \in U_n$. NE responds with a open set V_n such that $p_n \in V_n \subseteq U_n$. E wins if $\bigcap\{V_n : n \in \omega\} \neq \emptyset$. While the Choquet game $\mathrm{Ch}(X)$ characterizes Baireness of X, Debs shows in [De] the strong Choquet game $\mathrm{Ch}^*(X)$ characterizes hereditary Baireness for many spaces.

Theorem 4.15. [De] *Let X be a regular first-countable space in which every closed set is a G_δ-set. Then the following are equivalent:*

(i) X is hereditarily Baire;
(ii) E has no winning strategy in $Ch^*(X)$.

Proposition 4.16. *Suppose $\mathfrak{u}$ is a free filter on ω. $C_k(S_\mathfrak{u})$ is not hereditarily Baire.*

Proof. If $C_k(S_\mathfrak{u})$ is not Baire, then of course it is not hereditarily so, thus we may suppose $C_k(S_\mathfrak{u})$ is Baire. Then by Corollary 4.8, $C_k(S_\mathfrak{u})$ is metrizable.

We will show that E has a winning strategy in $\mathrm{Ch}^*(C_k(S_\mathfrak{u}))$. By Corollary 4.8 and Proposition 4.1, $\{P_F : |F| < \omega\}$ is a dominating family of compact subsets of $S_\mathfrak{u}$. Thus if E plays the non-empty open set U and the point $f \in U$, we may assume U is of the form $B(f, P_F, \epsilon)$. If NE responds with the basic open set V, then $f \in V$ so we may assume V has the form $B(f, P_{F\cup G}, \delta)$, where $G \cap F = \emptyset$ and $\delta \leq \epsilon$.

Define the strategy σ for E in $\mathrm{Ch}^*(X)$ as follows. Let $F_0 = \{0\}, \epsilon_0 = 1$, and

$$f_0(\mathbf{x}) = \begin{cases} 1 & : \mathbf{x} = (i,j), i \in \omega, j = 0 \\ 0 & : \text{ otherwise} \end{cases}$$

Let $\sigma(\emptyset) = U_0 = B(f_0, P_{F_0}, \epsilon_0))$. Suppose NE responds with $V_0 = \langle f_0, P_{F_0\cup G_0}, \delta_0\rangle$. Let $F_1 = \{\min\{i \in \omega : i \notin F_0 \cup G_0\}\}$, $\epsilon_1 = \delta_0/2$, and

$$f_1(\mathbf{x}) = \begin{cases} f_0(\mathbf{x}) & : \mathbf{x} \in P_{F_0\cup G_0} \\ 1 & : \mathbf{x} = (i,j), \mathbf{x} \notin P_{F_0\cup G_0}, i \in \omega, j \leq 1 \\ 0 & : \text{ otherwise} \end{cases}$$

Let $\sigma(U_0, V_0) = U_1 = B(f_1, P_{F_0\cup G_0\cup F_1}, \epsilon_1)$.

Suppose $U_i, V_i, f_i, F_i, G_i, \epsilon_i$, and δ_i have been defined for all $i \leq k$. Let $F_{k+1} = \{\min\{i \in \omega : i \notin F_0 \cup G_0 \cup \cdots \cup F_k \cup G_k\}\}$, $\epsilon_{k+1} = \delta_k/2$ and

$$f_{k+1}(\mathbf{x}) = \begin{cases} f_k(\mathbf{x}) & : \mathbf{x} \in P_{F_0\cup G_0\cup\cdots\cup F_k\cup G_k} \\ 1 & : \mathbf{x} = (i,j), \mathbf{x} \notin P_{F_0\cup G_0\cdots\cup F_k\cup G_k}, i \in \omega, j \leq k+1 \\ 0 & : \text{ otherwise} \end{cases}$$

Let $\sigma(U_0, V_0, U_1, V_1, \cdots, U_k, V_k) = B(f_{k+1}, P_{F_0\cup G_0\cup\cdots\cup F_k\cup G_k\cup F_{k+1}}, \epsilon_{k+1})$.

We will show that $\bigcap U_i = \emptyset$. Let $f = \lim f_i$, where the limit is the pointwise limit. Since $\lim \epsilon_i = 0$ and $\omega = \bigcup(F_i \cup G_i)$ it follows that f is the only candidate for a point in $\bigcap U_i$. We will show that f is not continuous at ∞. Note that $f(\infty) = 0$.

Let $\langle g, A, n\rangle$ be a basic open set around ∞. Since $\mathfrak{u}$ is free, the filter element A is infinite. Let $k \in A$ such that $k \notin F_0 \cup G_0 \cup \cdots \cup F_n \cup G_n$. Then $(k, n) \in \langle g, A, n\rangle$

and $|f(\infty)-f(k,n)|=|0-1|=1$. It follows that f is not continuous and $\bigcap U_i=\emptyset$. We have constructed a winning strategy σ for E in $\mathrm{Ch}^*(C_k(S_{\mathfrak{u}}))$. Since $C_k(S_{\mathfrak{u}})$ is metrizable, it follows from Theorem 4.15 that $C_k(S_{\mathfrak{u}})$ is not hereditarily Baire. □

5. Examples of specific filter fans

Example 5.1. *Let M be the metric fan. Then $C_k(M)$ is neither Baire nor metrizable.*

Proof. As pointed out earlier, $M=S_{\mathfrak{u}}$, where $\mathfrak{u}$ is the cofinite filter. By Proposition 4.1 it follows that M is not hemicompact. By Corollary 4.8, $C_k(M)$ is not Baire. □

Example 5.2. *$C_k(S_\omega)$ is completely metrizable.*

Proof. By a simple argument S_ω is hemicompact. Furthermore it is a k-space since its the quotient image of countably many convergent sequences. The result follows from Theorem 2.4. □

Below is an example that shows that the converse of Corollary 4.8 is not true.

Example 5.3. *Suppose $\mathfrak{u}$ is isomorphic to the co-nowhere-dense filter $\mathfrak{n}$ on the rationals $\mathbb{Q}$; i.e. $A\in\mathfrak{n}$ iff $\mathbb{Q}\setminus A$ is nowhere-dense. Then $S_{\mathfrak{u}}$ is hemicompact and $C_k(S_{\mathfrak{u}})$ is metrizable, but $C_k(S_{\mathfrak{u}})$ is not Baire.*

Proof. It is easy to see that every infinite subset A of $\mathbb{Q}$ contains an infinite nowhere-sense subset. E.g., if no point of A is a limit point of A, then A is nowhere-dense, while if some point x of A is a limit point of A, then any sequence in A converging to x is nowhere-dense. Thus $S_{\mathfrak{u}}$ is hemicompact and $C_k(S_{\mathfrak{u}})$ is metrizable by Proposition 4.1.

It is also easy to see that P_1 has a winning strategy in the game $G(\mathfrak{n})$: in round n, he simply has to choose a rational within $1/2^n$ of q_n, where $\mathbb{Q}=\{q_i\}_{i\in\omega}$. It now follows from Theorem 4.14((v)$\Rightarrow$(ii)) that $C_k(S_u)$ is not Baire. □

Example 5.4. *Suppose $\mathfrak{u}$ is a free ultrafilter on ω. Then $C_k(S_{\mathfrak{u}})$ is Baire and metrizable but not hereditarily Baire or Choquet, hence not completely metrizable.*

Proof. By Proposition 4.5, P_1 has no winning strategy in $G(\mathfrak{u})$, hence $C_k(S_{\mathfrak{u}})$ is Baire by Theorem 4.14((i) $\Longleftrightarrow$ (iii)). The rest is immediate from Corollary 4.11 and Proposition 4.16. □

Our final example shows that a free filter $\mathfrak{u}$ need not be an ultrafilter for $C_k(S_{\mathfrak{u}})$ to be Baire.

Example 5.5. *There is a free filter $\mathfrak{v}$ on ω which is not an ultrafilter such that $C_k(S_{\mathfrak{v}})$ is Baire.*

Proof. Let $\mathfrak{u}$ be a free ultrafilter on ω. Let $\mathfrak{u}_2$ be the filter on two disjoint copies of ω such that $F\in\mathfrak{u}_2$ iff F meets each copy in a member of $\mathfrak{u}$. Then $\mathfrak{u}_2$ is not an ultrafilter because the disjoint copies of ω are both $\mathfrak{u}_2$-positive. It is easy to see that if P_1 had a winning strategy in $G(\mathfrak{u}_2)$, then he would have one in $G(\mathfrak{u})$ too, a contradiction. (The idea is that if F is a play of P_1 in $G(\mathfrak{u}_2)$ using a winning strategy, then the union of the traces of F on the two copies should win for P_1 in $G(\mathfrak{u})$.) So P_1 has no winning strategy in $G(\mathfrak{u}_2)$. Hence if $\mathfrak{v}$ is a filter on ω isomorphic to $\mathfrak{u}_2$, then $\mathfrak{v}$ is not an ultrafilter and $C_k(S_{\mathfrak{v}})$ is Baire by Theorem 4.14. □

References

[Ar1] A. V. Arhangelskii, *Frequency spectrum of a topological space and the classification of spaces*, Dokl. Akad. Nauk. SSSR 206(1972), 265-268. English translation in Soviet Math. Dokl. 13(1972), 1185-1189.

[Ar2] A. V. Arhangelskii, *Frequency spectrum of a topological space and the product operation*, Trudy Mosk. Mat. Obs. 40(1979). English translation in Trans. Moscow Math. Soc. (1981) Issue 2, 163-200.

[De] G. Debs, *Espaces hérédiairment de Baire*, Fund. Math. 129 (1988), 199-206

[vD] E.K. van Douwen, *Collected Papers, vol. I*, edited by J. van Mill, North Holland, Amsterdam, 1994.

[GM] G. Gruenhage and D. Ma, *Baireness of $C_k(X)$ for locally compact X*, Topology Appl. 80(1997), 131-139.

[GG] M. Granado and G. Gruenhage *Baireness of $C_k(X)$ for ordered X*, Comment. Math. Univ. Carolin., 47(2006), 103-111.

[Ma] D.K. Ma, *The Cantor tree, the γ-property, and Baire function spaces*, Proc. Amer. Math. Soc. 119 (1993), no. 3, 903-913.

[Mat] P. Matet and J. Pawlikowski, *Ideals over ω and cardinal invariants of the continuum*, J. Symbolic Logic 63 (1998), no. 3, 1040-1054.

[Mic] E.A. Michael, *A quintuple quotient quest*, General Topology and Appl. 2 (1972), 91-138.

[MN] R.A. McCoy and I. Ntantu *Completeness properties of function spaces*, Topology Appl. 22(1986), no.2, 191-206.

[Ox] J.C. Oxtoby, *Measure and category. A survey of the analogies between topological and measure spaces*, Graduate Texts in Mathematics, Vol. 2. Springer-Verlag, New York-Berlin, 1971.

[Py] E.G. Pytkeev, *The Baire property of spaces of continuous functions*, Math. Zemetki 38(1985), 726-740.

Department of Mathematics, Auburn University, Auburn, AL 36830

E-mail address: garyg@auburn.edu

E-mail address: gsh0002@auburn.edu