Bounded Variation and Nowhere Dense Sets.

Definition. The subset M of $\mathbb{R}$ is said to be *nowhere dense* means that if U is a non-empty open set then there is a non-empty open subset V of U that does not intersect M.

[Note: the word “segment” can replace the word “open set” in the above and the definition is equivalent.]

Theorem 1. If M is a nowhere dense then $\overline{M}$ is nowhere dense.

Theorem 2. If M is is an interval $[a, b]$, then M is not the union of countable many nowhere dense sets. [Hint: recall that the monotonic common part of non-empty compact sets is not empty.]

Definition. The subset M of the space X is said to be *perfect* if and only if every point of M is a limit point of M.

Theorem 3. There exists a closed perfect nowhere dense subset of the reals.

Lemma 4. Suppose $f : [a, b] \to \mathbb{R}$ is an increasing function and M is the set of numbers in the domain of f at which f is discontinuous. Then $\{f(x)|x \in M\}$ is nowhere dense in $\mathbb{R}$.

Lemma 4'. Same as lemma 4 except replace “increasing” with “non-decreasing”.

Theorem 5. Suppose $f : [a, b] \to \mathbb{R}$ is an increasing function. Then f is continuous at some point. Furthermore, if $M = \{x|f \text{ is continuous at } (x, f(x))\}$ then M is dense in $[a, b]$

Theorem 6. Suppose $f : [a, b] \to \mathbb{R}$ is a bounded variation function. Then f is continuous at some point. Furthermore, if $M = \{x|f \text{ is continuous at } (x, f(x))\}$ then M is dense in $[a, b]$.