
The Method of Increments.

The Second Part. [IIId]

LEMMA XI. [page 102]

The subtangent is given for a logarithmic curve. For if the number is y , then the

logarithm z and the subtangent c are related according to $\frac{\dot{y}}{y} = \frac{\dot{z}}{c}$.

This has been explained by others in various places.

[We may note that this equation can be written in modern terms as $dy/dz = y/c$. Hence the gradient at any point on the curve is proportional to the ordinate y and with the subtangent c constant. This integrates to give a function of the form $y = e^{z/c}$, or $\ln y = z/c$. Taylor fitted the base 10 logarithmic curve to his data.]

HYPOTHESIS I.

The density of air is in proportion to the applied weight [i. e. air pressure.]

This is confirmed by experiments [i. e. Boyle's Law.]

HYPOTHESIS II.

The force of gravity varies inversely as the square of the distance from the centre of the earth.

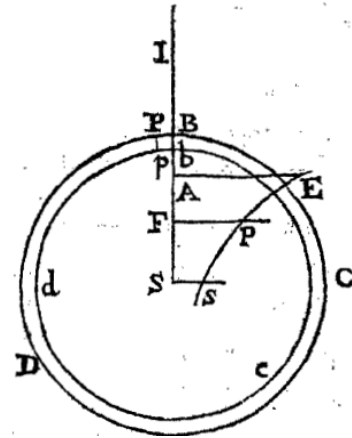
This is a postulate from [Newton's] *Philosophiae Naturalis Principia Mathematica*.

[p. 103]

PROP. XXVI. PROB. XXI.

To find the density of the atmosphere.

Let S be the centre of the earth, and the surfaces of two concentric spheres centred on the earth are represented in the air by the circles BCD and *bcd*, described with centre S. The surface BCD sustains the pressure of the column of air, the base of which is on the same surface and with an altitude BI equal to the height of all the air above the point B : (by Prop. 20. *Book 2. Princip. Math.*) Thus the pressure at a given part BP of the surface is as the column of air of the whole height BI pressing on the base. Let *bp* = BP, and the difference of the pressures on the bases BP and *bp*, is as the weight of air pressing on the given base, with the distance B*b* to be included between the altitudes SB and S*b*. Also let SB = *x*, and with the distance B*b* taken immeasurably small, let



$Bb = \dot{x}$, and *y* the density of the air at B, and *a* shall be the given distance from the centre S, at which the acceleration of gravity is taken as = 1, and the density *d*. Then the

quantity of air in the interval BP*pb* is $\dot{x}y$ (truly varying as the magnitude of the density,) and thus the force of gravity at B is $\frac{aa}{xx}$ (by *Hyp.* 2); the weight of air between B and *b* is

thus as $\frac{aa\dot{x}y}{xx}$ (that is, as the quantity of matter expressed as a weight.) Therefore by

descending towards the centre of the earth, the increment in pressure will be as $\frac{aa\dot{x}y}{xx}$.

[p. 104] But the density *y* is in proportion to the pressure (by *Hypoth.* 1.) whereby

\dot{y} is as $\frac{a^2\dot{x}y}{x^2}$. Let [the constant of proportionality] *c* be given for the curve, then

$$\dot{y} = -\frac{a^2\dot{x}y}{cx^2}, \text{ or } \frac{\dot{y}}{y} = -\frac{a^2\dot{x}}{cx^2}.$$

Let SA = *a* and SF = $\frac{aa}{x} = z$ [note the inversion of the abscissa about *a* in order that the

equation can be reduced to that of the logarithm function, for as *x* increases, *z* decreases, and thus the graph does not represent the actual change in pressure or density with distance from the centre of the earth.]; and erect normals to A and F proportional to *d* and *y*; which hence can be designated by these, as AE = *d* and FP = *y*, and EP is the curve that

the point P always touches. Then $\frac{-aa\dot{x}}{xx} = \dot{z}$, and hence, $\frac{\dot{y}}{y} = \frac{\dot{z}}{c}$. It follows that EF is the

logarithmic curve, the subtangent of which is *c* (by *Lem.* 11.) And thus, in a table of logarithms, if the logarithms [of the numbers] are taken proportional [to the abscissa] AF,

that is from $a - \frac{aa}{x}$, then the numbers at the locations B [i. e. the antilogs.] are as the densities. *Q.E.I.*

SCHOLIUM.

If the weight of all the air above B is p (by this *Prop.*) then p is to \dot{p} as y is to \dot{y} . But $\dot{p} = \frac{aa\dot{x}y}{xx}$ (by this *Prop.*) Whereby if $p = \frac{aay}{xx}A$, [essentially the standard formula ' $p = \rho gh$ '] that is, if p is a column of air of the same mean density y and with gravity $\frac{aa}{xx}$, and with the altitude A , then the air at B satisfies the relation : y to \dot{y} as A to \dot{x} , [or, $\dot{y}/\dot{x} = dy/dx = y/A$; but from above, $\frac{\dot{y}}{y} = -\frac{a^2\dot{x}}{cx^2}$; hence $dy/dx = -\frac{a^2y}{cx^2}$, or y/A ,] that is, as c to $\frac{a^2\dot{x}}{x^2}$ (by this *Prop.*) Hence $c = \frac{a^2}{x^2}A$. Thus c is given, with the altitude A found from Torricelli's experiment.

Moreover, by a certain experiment performed by Hawkesbee, the average density of the air is agreed upon to be to the density of water as nearly 1 to 820. [p. 105.] Also the density of water to the density of mercury is as 1 to $13\frac{1}{2}$. Whereby the density of air to the density of mercury is as 1 to 11070. Also the mean barometric height is 30 inches. Whereby if the point B is taken on the surface of the earth, with a being taken as the radius of the earth, [$x = a$] the altitude is A , (and hence the subtangent c , from $c = \frac{a^2}{x^2}A$) = 332100 inches or 27675 English feet. Also the radius of the earth is 20995444. Whereby for the radius of the earth I write 1, giving $c = \frac{27675}{20995444}$, or with sufficient accuracy in the smaller numbers by $c = \frac{1}{760}$. Hence, with regard to common logarithms, of which the log. of 10 is 1:

Part of an English foot	0.00001569
1000 English paces	0.082856 ...
Radius of the earth	329.47....

[Thus, a scale is chosen in which the lengths of interest lie between 1 and 10, corresponding to logs between 0 and 1. This sort of complication does not occur with natural logarithms, which lay in future, and Taylor had to make do with a set of base 10 logs. Thus, the unit chosen is approximately equal to 12 miles, for which the part corresponding to the English foot is 0.00001569. Thus, c is approximately 0.43 on this scale, corresponding to the 'half-thickness' t of the atmosphere, and most of the density of the atmosphere is accomodated within a few such t values. The interested reader should compare Taylor's analysis with that of the pressure or particle density of the isothermal atmosphere, as derived in elementary texts, see e.g. *The Feynman Lectures on Physics*, Vol. I, 40-1 ;]

It is agreed from this proposition, that the density of air extended to an infinite distance from the centre of the earth, is a finite quantity; for that is set out by the ordinate of the logarithm Ss continued to the centre of the earth. Hence if the elastic force of the air is so great, so that on arriving at this level of rarefaction, the density is still proportional to the compression, then the atmosphere of the earth truly is extended to infinity, and the amount of air in the system of the world will be truly infinite; as that shall be greater than the whole infinite distance taken with the given density Ss . But natural forces do not extend to infinity; whereby it is more probable that the elastic force of the air, upon reaching a certain step in rarefaction, thereafter continues to decrease, and thus the density thereafter constantly decreases in a ratio less than the smallest of weight, and the atmosphere in agreement with that is retained within finite limits, and these perhaps quite close.

HYPOTHESIS III.

Rays of light are agreed upon to consist of small corpuscles, and the refraction of light is by the mutual attraction between the corpuscles and the bodies in the refracting medium, and this attraction decreases greatly with the ratio of the distances between the bodies to such an extent that it is not perceptible, except when they are nearly in contact; and the refraction is in proportion to the density of the refracting body for the remaining parts of the rays. [p. 106.]

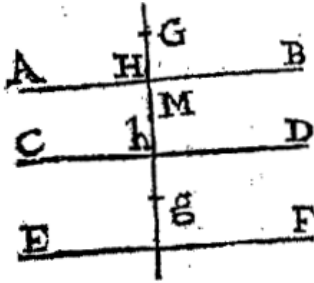
All this is abundantly proven in *Newton's* book on Optics.

LEMMA XII.

If there are many similar media, constructed from distinct parallel planes in turn, and in which the forces of attraction are barely sensitive to change with distance; then the velocity of a corpuscles travelling through the media, when it reaches a different medium, will not have changed, and if it is not be transmitted as before, then it is directly incident with its first velocity on the forces of attraction of the new medium, and now it changes direction.

[It is an experimental fact that refraction only occurs at the surface between media, and there is no further effect on the ray on being transmitted through the medium, apart from dispersion and absorption, setting any optical activity aside : a puzzling fact for an investigator such as Newton trying to pin the phenomenon down to dynamic interactions between particles. The electro-magnetic theory of the nature of light lay some 150 years into the future with Maxwell's equations, and the best that could be done at the time had of course already been accomplished by Huygens, who had understood and solved the problem in terms of waves in a phenomenological manner, many years previously.....

Thus, Taylor's work here is interesting more from a mathematical perspective, as he shows how the calculus can be extended to solve difficult problems ; in the present case, however, the physics is quite wrong. Light beams do not fall out of the sky under the influence of gravity, and Taylor should have known that!]



Any two contiguous media are distinguished by parallel planes and represented by the parallel lines AB, CD, and EF; the normal GHbg is drawn to these crossing AB and CD in H and *h*. Hence take HG & *hg* equal to the distances in which the attraction of the medium ABDC is defined and begins. Then since the action of the media is uniform, some motion is added to the body entering from G towards M, only the same contrary action being taken by the same medium arriving at *g*,

and the same eventuates for the light body passing through the rest of the media. It remains therefore that all the change of the motion when the body reaches *g*, arises only from the action of the medium CDEF, in which it now changes direction.

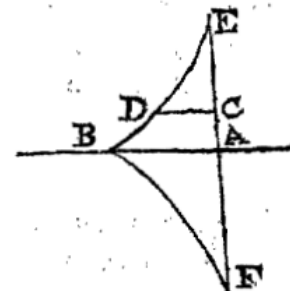
COROLLARIUM.

Hence the motion of light in some medium is always the same, either that which it had in the first medium, or passing through another medium, (by *Hypoth.* 3).

LEMMA XIII.

If in the given distances of mediums, the attractive forces are as the densities of these, then the speed of light is in the square root ratio of the densities. [p. 107.]

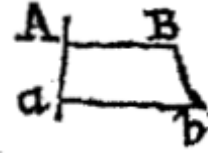
Let AB be the surface of adjacent media at the parts F. Draw EAF normal to AB, and at the point C on the line EA erect the normal CD proportional to the force of attraction of the medium at C; let EDBFE be the whole area described by the ordinate CD. Then the increment in the square of the velocity of the particle crossing through the whole region of attraction of the medium EF, as the whole area EDBF (by *Prop.* 39 & 40 *Lib. I. Princip. Math.*) But since by hypothesis, the attractions in the given distances are as the densities of the mediums, hence the whole areas EDBFE are as the same densities; and thus the increment of the square of the velocities is as the density of the medium. And thus if the square of the given velocity *in vacuo*, before entering into the interval EF, is to the incrementum of the same in passing through this interval, as the given quantity to the density in one case, is always the same square of the given velocities given *in vacuo* to the increment of this kind, as likewise given to the density : and thus jointly the square of the velocity after passing through the space EF is to the square of the velocity given *in vacuo*, as the density added to the given density to the given density; and hence the velocities are themselves in this ratio of the square root ; and thus the velocity in the medium is always in the square root ratio of the density in addition to what is given. *Q.E.D.*



[A basic error in this kind of analysis, which is that on entering a denser medium, and therefore being attracted more, the corpuscles should speed up : but in actuality, the light

slows down with the increased refractive index, an effect that is accomodated using Huygens' wave theory.]

COROLLARY.

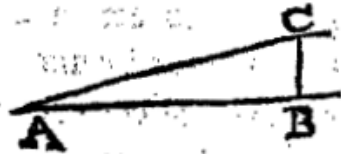


Hence in the in passage of light through a medium of unequal density, such as the atmosphere of the earth, the accelerating force is as the fluxion of the density applied to the fluxion of the distance between the distances. For let Aa be a line in the direction of which the density is variable, and the ordinates AB, ab are as the accelerating forces at A and a , and let Bb be the curve that the point B describes. Then the area $ABba$ is as the increment of the square of the velocity of the particle going from A to a , (by *Prop. 39 Lib. I. Princip. Math.*) that is, as the increment [p. 108.] of the density (by this *Lemma*). Whereby with the distance Aa diminished indefinitely, the accelerating force AB is as the fluxion of the applied density to Aa , that is, to the fluxion of the distance between the densities. [i. e. the force varies as dn/dx , where n is the density and x the abscissa.]

SCHOLIUM.

In an experiment performed by *Hawkesbee* [a person attached to the Royal Society who did experiments of the fellows], it was found that the sine of the angle of refraction of light incident from a vacuum on air at the surface of the earth, to the sine of the angle of incidence is as 999736 ad 1000000. Hence the velocity of light in vacuo to the velocity of light in air at the surface of the earth is in this ratio (by *Prop. 95 Book I. Princip. Math.*) Therefore the quantity 1 is given, and the density of air on the surface of the earth is represented by d : then (by this *Lemma*) we have :

$$1 : \sqrt{1+d} :: 999736 : 1000000, \text{ and thus } d = 0.00052828.$$



In general, if the density of the air is designated by y , and the right angled triangle ABC is set up, the base of which AB is to the hypotenuse AC , as the sine of the angle of refraction to the sine of the angle

of incidence from the vacuum, for the given base AB the perpendicular BC will be as \sqrt{y} .

26.) Hence $\frac{Qq}{QB}$, that is the fluxion of the angle FGH =

$$\frac{a^2by\dot{x}\sqrt{1+d}}{2cx^2\times 1+y\sqrt{x^2\frac{1+d}{1+y}b^2}} \text{ or } \frac{a^2by\dot{x}}{2cx^2\times 1+y\sqrt{\frac{1+d}{1+y}x^2-b^2}}.$$

Thus the fluent of this expression is found by the method of inverse fluxions, and will give the angle FGB. But the angle GBS is given by the value of the perpendicular SQ; and the angle SDG is right; thus the angle DSB is given. Thus, from the given distance SB (= x), with the density in B (= y), and the angle SAD, the position of the line SB is given; and the point B; that is, the figure of the refracted ray ABC is given. *Q.E.I.*

But this fluxion $\frac{a^2by\dot{x}}{2cx^2\times 1+y\sqrt{\frac{1+d}{1+y}x^2-b^2}}$ cannot be reduced to a fluent in a finite number of

terms. Whereby a series is sought for the computation of atmospheric refraction in astronomical usage, which is suitable for the calculation to be carried out by an approximation. Therefore so that this fluxion can be rewritten in the simplest possible terms, for x write $\frac{aa}{z}$, and the fluxion becomes :

$$\frac{-by\dot{z}}{2c\times 1+y\sqrt{\frac{1+d}{1+y}\frac{a^4}{z^2}-b^2}}, \text{ or } \frac{-yz\dot{z}}{2c\times 1+y\sqrt{\frac{1+d}{1+y}\frac{a^4}{z^2}-z^2}}, \text{ i. e. (with the sign ignored) } \frac{yz\dot{z}}{2c\times 1+y\sqrt{\frac{1+d}{1+y}tt-z^2}}, \text{ also by}$$

considering [p. 111.] $y = \frac{yz}{c}$. But on the surface of the earth, where $y = d$, this fluxion is

$$\frac{yz\dot{z}}{2c+2cd\times\sqrt{tt-zz}};$$

and at an infinite disance (where the fluxion also vanishes,) it will differ from this form by less than one part in a thousand. Whereby by ignoring small parts of

this kind, it is possible with care to take $\frac{yz\dot{z}}{2c+2cd\times\sqrt{tt-zz}}$ for the fluxion itself. Thus by

setting aside $\frac{1}{2c+2cd}$, I seek the fluent of $\frac{yz\dot{z}}{\sqrt{tt-zz}}$ with the help of Prop. 11, as follows.

For $\sqrt{tt-zz}$ write x, and $\dot{x} = \frac{-zz\dot{z}}{x}$, and the proposed fluxion is either $\frac{yz\dot{z}}{x}$ or $-y\dot{x}$.

Also, $y = \frac{yz}{c}$. Thus following Prop. 11. let $\dot{v} = y\dot{z}$, $s = \frac{z}{x}$, $\dot{w} = \dot{z} = \frac{-xx\dot{x}}{z}$. Then by taking the fluxions, and these by continued applicaton to

\dot{w} , we have $\dot{s} = \frac{z^2}{x^3} + \frac{1}{x} = \frac{tt}{x^3}$, $\ddot{s} = \frac{3z^3}{x^5} + \frac{3z}{x^3} = \frac{3ttz}{x^5}$, $\dddot{s} = \frac{15z^4}{x^7} + \frac{18z^2}{x^5} + \frac{3}{x^3} = \frac{15ttz^2}{x^7} + \frac{3tt}{x^5}$, and

so on. Moreover it can be agreed from the form of the terms, that if n is the distance of some term \dot{s} , \ddot{s} , \dddot{s} , &c., from the first term s, then \dot{s} can be expressed (that is \dot{s} , if n is 1, \ddot{s} , if n is 2, &c.) either by a series of the form :

$s = A \frac{z^{n+1}}{x^{2n+1}} + B \frac{z^{n-1}}{x^{2n-1}} + C \frac{z^{n-3}}{x^{2n-3}} + D \frac{z^{n-5}}{x^{2n-5}} + \&c.$, or by a series [p. 112.] of the form :

$s = tt \times A \frac{z^{n+1}}{x^{2n+1}} + B \frac{z^{n-1}}{x^{2n-1}} + C \frac{z^{n-3}}{x^{2n-3}} + D \frac{z^{n-5}}{x^{2n-5}} + \&c.$ Moreover, I investigate the

coefficients A, B, C, D, &c. in the first of these series in this manner, for by our notation explained in the introduction, the values of the coefficients before n are called $\overset{\vee}{n}, \hat{n}, \overset{\text{III}}{n}$, &c., and $\underset{\vee}{n}, \underset{\wedge}{n}, \underset{\text{III}}{n}$, &c. are the values of the same after n. By taking the fluxions of the

series, first in x, then in z, and the terms being continually produced by application to $\overset{\cdot}{w}$, will be [the original text, written in a slightly different manner, is included for comparison]:

$$s = \overline{2n+1} A \frac{z^{\overline{n+1}}}{x^{\overline{2n+1}}} + [\overline{2n-1} B + \overline{n+1} A] \frac{z^{\overline{n-1}}}{x^{\overline{2n-1}}} + [\overline{2n-3} C + \overline{n-1} B] \frac{z^{\overline{n-3}}}{x^{\overline{2n-3}}} + [\overline{2n-5} D + \overline{n-3} C] \frac{z^{\overline{n-5}}}{x^{\overline{2n-5}}} + \&c.$$

$$s = \overline{2n+1} A \frac{z^{\overline{n+1}}}{x^{\overline{2n+1}}} + \overline{2n-1} B \left\{ \frac{z^{\overline{n-1}}}{x^{\overline{2n-1}}} + \overline{2n-3} C \left\{ \frac{z^{\overline{n-3}}}{x^{\overline{2n-3}}} + \overline{2n-5} D \left\{ \frac{z^{\overline{n-5}}}{x^{\overline{2n-5}}} + \&c. \right. \right. \right. + \overline{n+1} A \left\{ \frac{z^{\overline{n-1}}}{x^{\overline{2n-1}}} + \overline{n-1} B \left\{ \frac{z^{\overline{n-3}}}{x^{\overline{2n-3}}} + \overline{n-3} C \left\{ \frac{z^{\overline{n-5}}}{x^{\overline{2n-5}}} + \&c. \right. \right. \right.$$

Hence the new A is $\overline{2n+1} A$. Hence it is agreed that A is to be formed by the continued multiplication of the terms 1, 3, 5, 7, &c. the last and largest of which is $2n-1$. In what follows, write m in place of $2n-1$, and $A = \overline{m} A$.

Likewise for the second term $B = \overline{m} B + \overline{n} A$. If it is possible for B to be produced from A by multiplication and division, let $B = \frac{Q}{R} A$. Then $\overline{m} B = \frac{Q}{R} \overline{m} A = \frac{Q}{R} \overline{m} A$. Hence eliminating B & $\overline{m} B$ from the first equation, and with A set to zero at the same time, then

$$\frac{\overline{m} Q}{R} = \frac{\overline{m} Q}{R} + \overline{n}, \text{ that is } \frac{\overline{m} Q}{R} + \frac{\overline{m} Q}{R} = \frac{\overline{m} Q}{R} + \overline{n}. \text{ In order that this equation can be reduced to}$$

the simplest terms, I put [p. 113.] $\frac{\overline{m} Q}{R} = \frac{\overline{m} Q}{R}$, that is $\frac{\overline{m}}{R} = \frac{\overline{m}}{R}$. Hence it becomes

$$\frac{\overline{m} Q}{R} = \overline{n}. \text{ But } \frac{\overline{m}}{R} \text{ is the new value of } \frac{\overline{m}}{R}, \text{ and hence } Q = \overline{n}, \text{ and by summing the}$$

quantities $Q = \frac{nn}{2} + p$. But B should be equal to 0 when $n = 0$; and thus $p = 0$, and

$$Q = \frac{nn}{2}. \text{ Hence } B = \frac{nn}{2m} A. \text{ Hence } B = \frac{mn}{n} B.$$

The third term is $C = mC + nB$. I put $C = \frac{Q}{R} B$, and $C = \frac{Q}{R} B$, that is

$$\frac{Qmn}{Rn} B = \frac{mQ}{R} B + nB, \text{ or } \frac{Qmn}{nR} + \frac{mn}{nR} Q = \frac{mQ}{R} + n. \text{ Put } \frac{mn}{nR} = \frac{m}{R}, \text{ that is,}$$

$$\frac{mnn}{R} = \frac{mnn}{R}, \text{ in order that } \frac{mn}{nR} Q = n. \text{ But } \frac{mnn}{R} \text{ is the new value of } \frac{mnn}{R}. \text{ Hence}$$

$$R = mnn, \text{ and thus; } Q = nnn, \text{ hence; } Q = \frac{nnnm}{4}. \text{ Thus, } C = \frac{nn}{4m} B, \text{ and } C = \frac{mn}{n} C.$$

The fourth term is $D = mD + nC$. Thus in the same way it is found that

$$D = \frac{nn}{6m} C, \text{ and } D = \frac{mn}{n} C. \text{ Moreover from the terms now put in place the manner in which}$$

the rest are formed can be agreed upon. Hence if now for all the terms with their signs A,

$$B, C, \&c \text{ can be written, and } s = 1.3.5 \dots \overline{2n-1} \frac{z^{n+1}}{x^{2n+1}} + \frac{nn}{2m} \frac{xx}{zz} A + \frac{nn}{4m} \frac{xx}{zz} B + \&c.$$

that is :

$$s = 1.3.5 \dots \overline{2n-1} \frac{z^{n+1}}{x^{2n+1}} + \frac{\overline{n+1}.n}{2 \times \overline{2n-1}} \frac{xx}{zz} A + \frac{\overline{n-1}. \overline{n-2}}{4 \times \overline{2n-3}} \times \frac{xx}{zz} B + \frac{\overline{n-3}. \overline{n-4}}{6 \times \overline{2n-5}} \times \frac{xx}{zz} C + \&c.$$

And in the same way the coefficients in the other series can be found, as

$$s = 1.3.5 \dots \overline{2n-1} \frac{tz^{n-1}}{x^{2n+1}} + \frac{\overline{n-1}. \overline{n-2} xx}{2 \times \overline{2n-1} zz} A + \frac{\overline{n-3}. \overline{n-4} xx}{4 \times \overline{2n-3} zz} B + \frac{\overline{n-5}. \overline{n-6} xx}{6 \times \overline{2n-5} zz} C + \&c.$$

For indeed if now for the distance m of any term $s, s, s, \&c.$ from the term s , for n I

write $-m, s$ can also be expressed by the same series. Moreover in this case the coefficient of the first term can be found, as we did in in Prop. 12. For it ought to be $2n - 1$, that is, $-2m - 1$, the largest of the factors of 1, 3, 5, 7, &c. in that coefficient. Thus here it is possible to write the coefficient :

$$\frac{\overline{2n-1} \dots \overline{5}.3.1. -1. -3. -5. \&c.}{-1. -3. -5. \&c.} \text{ that is, } \frac{-\overline{2m-1}. -\overline{2m-3}. -\overline{2m-5}. \&c.}{-1. -3. -5. -7 \&c.}.$$

Moreover it happens that it is possible to find n negative numbers, and m positive numbers between

the numbers 1, 2, 3, &c. and all the factors $-\overline{2m-1}. -\overline{2m-3}. -\overline{2m-5}. \&c.$ to be carried

in the similar factors in the denominator [p. 115]. Thus it remains that in this case the coefficient of the first term is $\frac{1}{-1.-3.-5.....-2m-1}$, and

$$s = \frac{z^{-m+1}}{-1.-3.-5.....-2m+1} x^{-2m+1} + \frac{-m+1.-m}{2.-2m-1} \frac{xx}{zz} A + \frac{-m-1.-m-2}{4.-2m-3} \frac{xx}{zz} B + \&c. \text{ or}$$

$$s = \frac{x^{2m-1}}{-1.-3.-5.....-2m+1} z^{m-1} + \frac{-m+1.m}{2.2m+1} \frac{xx}{zz} A + \frac{-m-1.m+2}{4.2m+3} \frac{xx}{zz} B + \frac{-m-3.m+4}{6.2m+5} \frac{xx}{zz} C + \&c.$$

that is by the first series. Indeed this series is more suitable for finding the fluents

$\dot{s}, \ddot{s}, \dddot{s}, \&c.$, while the other is suited for finding the fluxions $\dot{s}, \ddot{s}, \dddot{s}, \&c.$

Again, $\dot{r} = \dot{y}\dot{z}$, and $\dot{y} = \frac{y\dot{z}}{c}$. Thus by taking pure fluents,

$$r = cy, \dot{r} = c^2 \dot{y}, \ddot{r} = c^3 \ddot{y}, \&c., \text{ likewise by taking the fluxions, } r = y, \dot{r} = \frac{y}{c}, \ddot{r} = \frac{y}{c^2}, \&c.$$

Hence with the signs observed, from these values of

$\dot{s}, \ddot{s}, \dddot{s}, \&c. \dot{s}, \ddot{s}, \dddot{s}, \&c. \dot{r}, \ddot{r}, \dddot{r}, \&c. \dot{r}, \ddot{r}, \dddot{r}, \&c.$ the angle

$$FHG (= \frac{1}{2c+2cd} \times rs - \dot{r}\dot{s} + \ddot{r}\ddot{s} - \&c.) = \frac{1}{2c+2cd} \text{ multiplied by this series :}$$

$$cy \times \frac{z}{x}$$

$$-c^2 y \times \frac{tz}{x^3}$$

$$c^3 y \times \frac{3ttz}{x^5}$$

$$-c^4 y \times \frac{1.3.5ttz^2}{x^7} + \frac{1}{5} \frac{x^2}{z^2} A$$

&c.

[p. 116.] and the angle FGH

$$\left(\frac{1}{2c+2cd} \times -\dot{r}\dot{s} + \ddot{r}\ddot{s} - \ddot{r}\ddot{s} \&c. - P \right) \text{ is equal to } \frac{1}{2c+2cd} \text{ multiplied by this series :}$$

$$y \times x$$

$$\frac{y}{c} \times \frac{x^3}{1.3z} + \frac{-1}{5} \frac{x^2}{z^2} A + \frac{-3}{7} \frac{x^2}{z^2} B + \&c.$$

$$\frac{y}{c^2} \times \frac{x^5}{1.3.5z^2} + \frac{-3}{7} \frac{x^2}{z^2} A + \frac{-5}{9} \frac{x^2}{z^2} B + \&c.$$

&c. -P.

Where P is the value of the same series emerging from z, x, y at the point A.

Also, another series can be found for the angle FGH; truly thus by correcting the fluents $r, \dot{r}, \ddot{r}, \&c.$ as all vanish at the point A, where $z = a$. In order that this can happen,

put $z = a - v$, hence $\dot{z} = -\dot{v}$, and the fluxion of the angle FGH is $\frac{\dot{v}yz}{x}$ by $\frac{1}{2c + 2cd}$.

Hence, by putting $s = \frac{z}{x}$ (as before), $\dot{r} = \dot{v}y$, by considering $\dot{w} = -\dot{v}$, and $\dot{y} = \frac{-\dot{v}y}{c}$.

Thus, $\dot{r} = -c\dot{y}$, and hence $r = cd - cy$, (since d is the value of y at the point A,) and

hence, $\dot{r} = c^2d - cdv - c^2y$, and hence

$$\ddot{r} = c^3d - c^2dv + \frac{cdv^2}{2} - c^3y, \ddot{\dot{r}} = c^4d - c^3dv + \frac{c^2dv^3}{2.3} - c^4y, \text{ and so on.}$$

Thus the angle FGH is equal to $\frac{1}{2c + 2cd}$ multiplied by

$$\begin{aligned} & \overline{cd - cy} \times \frac{z}{x} \\ & - \overline{c^2d + cdv + c^2y} \times \frac{tz}{x^3} \\ & + \overline{c^3d - c^2dv + \frac{cdv^2}{2} - c^3y} \times \frac{3ttz}{x^5} \\ & - \overline{c^4d + c^3dv - \frac{c^2dv^2}{2} + \frac{cdv^3}{2.3} + c^4y} \times \frac{1.3.5ttz^2}{x^7} + \frac{1}{5} \frac{x^2}{z^2} A \end{aligned}$$

& c.

[p. 117.] And hence the sum of the angles FGH and FGH, that is the angle GFH is equal

to $\frac{1}{2c + 2cd}$ multiplied by

$$\begin{aligned} & \overline{cd} \times \frac{z}{x} \\ & - \overline{c^2d + cdv} \times \frac{tz}{x^3} \\ & + \overline{c^3d - c^2dv + \frac{cdv^2}{2}} \times \frac{3ttz}{x^5} \\ & - \overline{c^4d + c^3dv - \frac{c^2dv^2}{2} + \frac{cdv^3}{2.6}} \times \frac{1.3.5ttz^2}{x^7} + \frac{1}{5} \frac{x^2}{z^2} A \end{aligned}$$

& c.

When the angle SAD is small enough, it is convenient to find the angle GFH by this series. But when the angle SAD is too large, the angle FGH is found from the other series.

Other series can be found for the angle FGH, by Prop. 7. For let Q be the fluent of $\frac{-zzy}{x}$, that is of $\dot{x}y$. Then by that Proposition, in which the time x is $x \pm v$, it becomes

$$Q = Q \pm \frac{\dot{Q}}{x}v \pm \frac{\ddot{Q}}{2x}v^2 \pm \frac{\dddot{Q}}{2.3x}v^3 + \&c. \text{ truly for a uniformly flowing } x. \text{ Thus if for } x \text{ the}$$

value is taken at some other given point I, and $x - v$ the value of the same at A, & $x + v$ the value of the same I α , then the value of the fluent at the point A will be

$$Q + \frac{\dot{Q}}{x}v + \frac{\ddot{Q}}{2x}v^2 + \frac{\dddot{Q}}{2.3x}v^3 + \&c. \text{ and the value at the point } \alpha \text{ will be}$$

$$Q - \frac{\dot{Q}}{x}v + \frac{\ddot{Q}}{2x}v^2 - \frac{\dddot{Q}}{2.3x}v^3 + \&c. \text{ Withe the value of which taken from the other value,}$$

the remainder is the fluent part adjoining the line $A\alpha$; and thus, if $SB = \frac{SA^2}{S\alpha}$, the angle is

$$[\text{p. 118.}] = \frac{1}{c+cd} \times \frac{Q}{x} * + \frac{\ddot{Q}}{2.3x}v^3 * + \frac{\dddot{Q}}{2.3.4.5x}v^5 + \&c. \text{ Moreover in this case for } \dot{x} \text{ I}$$

write 1, that is $\dot{z} = \frac{x}{z}$, & $\dot{y} = \frac{-xy}{cz}$. Hence by considering $\dot{Q} = y$,

$$\ddot{Q} = \frac{y}{cz^2} \times \frac{x^2}{c} - \frac{ty}{z},$$

$$\dddot{Q} = \frac{y}{c^2z^4} \times \frac{x^4}{cc} - \frac{6tx^2}{cz} + \frac{3tx^2}{z} \times \frac{1}{z-c} \times \frac{1+5x^2}{z^2}. \&c.$$

SCHOLIUM.

The radius of curvature of this curve is : $\frac{\overline{2+2y \times c \times SBcub.}}{y \times SQ \times SA \text{ quad.}}$; which is at the point A :

$\frac{\overline{2+2d \times c \times SA}}{d \times SD}$; and when the angle SAD is right, it is $\frac{\overline{2+2d}}{d}c$. Which from the values of c and d (*Schol. Prop. 26. & Schol. Lem. 13.*) as around 5 SA, by considering SA to be the radius of the earth. Hence the curvature of a horizontal ray of light, to the surface of the earth, is to the curvature of the great circle of the earth, as 1 to 5. Moreover the speed of light is to the speed of maximum rotation of a body under gravity in the great circle of

the earth is as around 40000 ad 1. Hence the refractive force of the air to the force of gravity at the surface of the earth is around 320000000 to 1. For in the given inclination of the trajectory to the direction of the centripetal force, these forces are in the ratio composed from the bending and squaring of the velocities.

The End.

METHODUS INCREMENTORUM.

Pars Secunda IId.

[p. 102]

LEMMA XI.

In Curva Logarithmica subtangens datur. Et si sit numerus y , Logarithmus z &

$$\text{Subtangens illa } c, \text{ erit } \frac{\dot{y}}{y} = \frac{\dot{z}}{c}.$$

Hoc ab aliis passim demonstratur.

HYPOTHESIS I.

Densitas Aeris est oneri imposito proportionalis.

Hoc Experimentis confirmatur.

HYPOTHESIS II.

Vis Gravitatis est reciprocis in duplica ratione distantiarum a Centro Terrae.

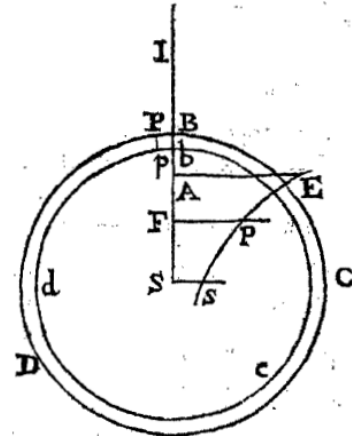
Demonstratur hoc a priori inter *Philosophiae Naturalis Principia Mathematica*.

[p. 103]

PROP. XXVI. PROB. XXI.

Invenire Densitatem Atmosphaerae.

Sit S centrum Terrae, & per circulos BCD, *bcd* centro S descriptos, repraesentur superficies duae sphaericae ipsi Terrae concentricae in Aere descriptae. Tum superficies BCD sustinebit pressionem columnae Aeris, cujus basis est eadem superficies atque altitudo aequatur altitudini BI totius Aeris supra punctum B : (per Prop. 20. *Lib. 2. Princip. Math.*) Pressio itaque in superficiei datam partem BP est ut columna Aeris totius altitudinis BI datae basi insistentis. Sit itaque; $bp = BP$, atque erit differentia pressionum in bases BP & bp , ut pondus aeris datae basi insistentis, inclusi inter altitudines SB & Sb . Sit itaque $SB = x$, atque imminuta



distantia Bb in infinitum, sit $Bb = \dot{x}$, & y densitas Aeris in B, & sit a data distantia a centro S, ad quam distantiam sit gravitas = 1 & densitas d .

Tum quantitas aeris in spatio $BPpb$ erit ut $\dot{x}y$ (nempe ut magnitudo in densitatem,) atque vis gravitatis in B erit $\frac{aa}{xx}$ (per *Hyp.* 2) adeoque; pondus aeris inter B & b erit ut

$\frac{aa\dot{x}y}{xx}$ (hoc est, ut quantitatis materiae in gravitatem.) Descendendo ergo versus centrum

Terrae, erit incrementum pressionis ut $\frac{aa\dot{x}y}{xx}$. [p. 104] Sed est densitas y pressioni

proportionalis (per *Hypoth.* 1.) quare est \dot{y} ut $\frac{a^2\dot{x}y}{x^2}$. Sit ergo c linea data, atq; erit

$$\dot{y} = -\frac{a^2\dot{x}y}{cx^2}, \text{ seu } \frac{\dot{y}}{y} = -\frac{a^2\dot{x}}{cx^2}.$$

Sit $SA = a$ & $SF = \frac{aa}{x} = z$, atq; ad A & F erige normales ipsis d & y proportionales; quae proinde per eas designari possunt, ut sit $AE = d$, & $FP = y$, & sit EP curva quam

punctum P perpetuo tangit. Tum erit $\frac{-aa\dot{x}}{xx} = \dot{z}$, adeoque; $\frac{\dot{y}}{y} = \frac{\dot{z}}{c}$. Unde est curva EP

Logarithmica, cujus subtangens est c (per *Lem.* 11.) Adeoque; si in tabula

Logarithmorum sumantur Logarithmi proportionales ipsis AF, hoc est ipsis $a - \frac{aa}{x}$,

Numeri erunt ut densitates in locus B. *Q.E.I.*

SCHOLIUM.

Si Aeris totius pondus supra B sit p (per hanc Prop.) erit p ad \dot{p} ut y ad \dot{y} . Sed est $\dot{p} = \frac{aa\dot{x}y}{xx}$ (per hanc Prop.) Quare si sit $p = \frac{aay}{xx} A$, hoc est, si p sit columna Aeris ejusdem densitatis y & gravitatis $\frac{aa}{xx}$ ac Aer in B, & altitudinis A, erit y ad \dot{y} ut A ad \dot{x} , hoc est, ut c ad $\frac{a^2\dot{x}}{x^2}$ (per hanc Prop.) Unde sit $c = \frac{a^2}{x^2} A$. Inventa itaque; altitudina A per Experimentum *Torricellianum*, dabitur c .

Sed per Experimentum quoddam ab Hanksbeio factum, constat Aeris densitaatem medioreme esse ad densitatem Aquae ut 1 ad 820 sere. [p. 105.] Est etiam densitas Aequae ad densitatem Mercurii ut 1 ad $13\frac{1}{2}$. Quare est densitas Aeris ad densitatem Mercurii ut 1 ad 11070. Altitudo etiam Barometri mediocris est 30 unc. Quare si punctum B sumatur in superficie Terrae, existente a radio Terrae, erit altitudo A, (adeoque; subtangens c) = 332100 unc. vel 27675 ped. Anglic. Est etiam Radius Terrae pedum 20995444. Quare pro Radio Terrae scripto 1, erit $c = \frac{27675}{20995444}$, vel satis accurate in numeris minoribus $\frac{1}{760}$. Et hinc respectu Logarithmorum communium, in quibus est 1 Log. ipsius 10, est

Pes Anglicanus partium	0.00001569
Mille passus Anglican	0.082856 ...
Radius Terrae	329.47.....

Ex hac Propositione constat, Aeris densitatem, etiam ad distantiam infinitam a Centro Terrae, esse quantitatis finitae; nam exponitur ea per Logarithmicam ordinatam Ss ad Centrum Terrae. Proinde si Aeris vis elastica tanta sit, ut ad hunc gradum raritatis pervento etiam adhuc sit densitas compressioni proportionalis, Atmosphaera Terrae vere in infinitum extenditur, atque erit quantitas Aeris in toto Systemate Mundano verre infinita; utpote quae major sit quam totum spatium infinitum ductum in datam densitatem Ss. Sed vires naturales non in infinitum extenduntur; quare plusquam probabile est, Aeris vim elasticam, postquam ad certum gradum raritatis perventum est, subinde continuo languescere, adeoque densitatem subinde decrescere in ratione continuo minori quam ponderis imminuti, atque Atmosphaeram eo pacto revocari intra limites finitos, eosque fortasse satis arctos.

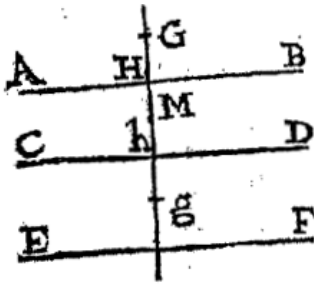
HYPOTHESIS III.

Radii Lucis constant ex particulis corporeis, atque Refractio Lucis sit per attractionem mutuum Lucis & Mediorum refringentium, atque haec Attractio decrescit in tam magna ratione distantiarum a corporibus, ut non sit sensibilis, nisi in ipso sere contactu; estque caeteris partibus, ut corporum densitas. [p. 106.]

Haec omnia abunde comprobantur in Libro Opticorum *Newtoni*.

LEMMA XII.

Si sint Media plura similaria, planis parallelis ab invicem distincta, quorum vires attractiones sunt tantum in distantiiis minimis sensibiles; corporis per Media ista transeuntis velocitas, ubi in Medium aliquod pervenerit, eadem erit, ac si per Media jam praeterita non transiisset, sed cum prima sua velocitate directe incidisset in vires attractrices istius Medii, in quo jam versatur.



Distinguantur duo quaevis Media contigua planis per rectas parallelas AB, CD, EF repraesentatis, atq; ducatur istis normalis GHbg occurrens AB & CD in H & b. Sume hinc inde HG & hg aequalis distantiiis in quibus incipit ac definit attractio Medii ABDC. Tum quoniam Mediorum actio est uniformis, quantum motus additur corpori pergenti de G versus M, tantum aufertur per contrariam actionem ejusdem medii eodem perveniente in g; atque idem eveniet in transitu corporis per media

reliqua. Restat ergo ut omnis mutatio motus ubi corpus pervenerit in g, oriatur ex sola actione medii CDEF, in quo jam versatur.

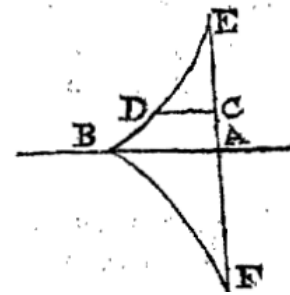
COROLLARIUM.

Hinc motus Lucis in Medio aliquo idem est, sive illud primum inciderit, sive per alia Media jam transierit, (per *Hypoth.* 3).

LEMMA XIII.

Si in datis distantiiis Mediorum Vires attractrices sunt ut ipsorum Densitates, erit Lucis velocitas in Medio in dimidiata ratione densitatis Medii quantitate data auctae. [p. 107.]

Sit AB superficies Medii jacentis ad partes F. Duc EAF ipsi AB normalem, atque ad punctum C in recta EA erige normalem CD proportionalem Medii vi attractrici in C, atq; sit EDBFE area tota quam describit ordinata CD. Tum erit incrementum quadrati velocitatis particulae transeuntis per totam regionem attractionis Medii EF, ut spatium totum EDBF (per *Prop.* 39 & 40 *Lib. I. Princip. Math.*) Sed quoniam, ex hypothesi, Attractiones in datis distantiiis sunt ut Densitates Mediorum, ergo sunt areae integrae EDBFE

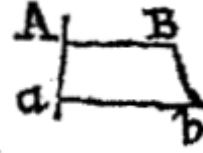


ut eadem densitates; adeoque incrementum quadrati velocitatis est ut Medii densitas. Si itaque quadratum velocitatis datae in vacuo, ante ingressum in spatium EF, sit ad incrementorum ejusdem in transitu per hoc spatium, ut data quantitas ad densitatem in uno casu, erit semper quadratum velocitatis datae in vacuo ad ejusmodi incrementum, ut idem Datum ad densitatem : adeoque conjunctim quadratum velocitatis post transitum per spatium EF erit ad quadratum velocitatis datae in vacuo, ut densitas plus dato ad datum;

& proinde ipsae velocitates erunt in hac ratione dimidiata ; adeoque velocitas in Medio est semper in dimidiata ratione densitatis plus dato. *Q.E.D.*

COROLLARIUM.

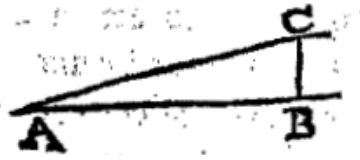
Hinc in transitu Lucis per Medium inaequaliter densum, qualis est Atmosphaera Terrae, Vis acceleratrix est ut fluxio densitatis applicata ad fluxionem distantiae inter densitates. Nam sit *Aa* linea in cujus directione variatur densitas, & sint ordinatae *AB*, ab ut vires acceleratrices in *A* & *a*, & sit *Bb* curva quam describit punctum *B*. Tum erit area *ABba* ut incrementum quadrati velocitatis particulae pergentis de *A* in *a*, (per *Prop. 39 Lib. I. Princip. Math.*) hoc est, ut incrementum [p. 108.] densitatis (per hoc *Lemma*). Quare imminuta distantia *Aa* in infinitum, erit vis acceleratrix *AB* ut fluxio densitatis applicata ad *Aa*, hoc est, ad fluxionem distantiae inter densitates.



SCHOLIUM.

Per experimentiam ab *Haukesbeio* factam, est sinus refractionis Radii Lucis a vacuo incidentis in Aerem ad superficiem Terrae, ad sinum incidentiae ut 999736 ad 1000000. Ergo in hac ratione est Lucis velocitas in vacuo ad ejusdem velocitatem in Aere ad superficiem Terrae (per *Prop. 95 Lib. I. Princip. Math.*) Sit ergo 1 quantitas data, atque repraesentetur densitas Aeris ad superficiem Terrae per *d* : tum (per hoc *Lemma*) erit

$$1 : \sqrt{1+d} :: 999736 : 1000000, \text{ adeoque; } d = 0.00052828.$$



Et in genere, si Aeris densitas designetur per *y*, & constituatur triangulum rectangulum *ABC*, cujus basis *AB* sit ad hypotenusam *AC*, ut sinus refractionis ad sinum incidentiae a vacuo, data basi

AB erit perpendiculum *BC* ut \sqrt{y} .

PROP. XXVII. PROB. XXII.

Invenire Refractionem Lucis per Atmospheram Terrae transeuntis.

Sit *S* centrum Terrae, *ABC* Radius Lucis incurvatus, quem tangant rectae *AG*; *BG* in *A* & *B*, sibi mutuo occurrentes in *G*, atque ad tangentes dimittantur perpendiculares *sD*, *SQ*, & ducantur *SA*, *SB*; atque ad *A* ducatur ipsi *SA* normalis *AE* occurrens *SD* in *E*, & sit *A* punctum in Radio datum, *B* punctum variable. Et sint *SA* = *a*, *SD* = *b*, *SE* = *t* (= $\frac{aa}{b}$), *SB* = *x*, *d* densitas in *A*, *y* densitas in *B*. [p. 109.]

atque fluxio fiet

$$\frac{-by\dot{z}}{2c \times 1 + y \times \sqrt{\frac{1+d}{1+y} \frac{d^4}{z^2} - b^2}}, \text{ vel } \frac{-yz\dot{z}}{2c \times 1 + y \times \sqrt{\frac{1+d}{1+y} \frac{d^4}{z^2} - z^2}}, \text{ h. e. (neglecto signo) } \frac{yz\dot{z}}{2c \times 1 + y \times \sqrt{\frac{1+d}{1+y} tt - z^2}} \text{ existente}$$

[p. 111.] etiam $\dot{y} = \frac{y\dot{z}}{c}$. Sed ad superficiem Terrae, ubi est $y = d$, est haec fluxio

$\frac{yz\dot{z}}{2c + 2cd \times \sqrt{tt - zz}}$; & ad distantiam infinitam (ubi etiam fluxio ipsa evanescit,) differt ab hac forma minus parte millesima. Quare neglectis istiusmodi minutiis, pro Fluxione illa tuto

sumi potest $\frac{yz\dot{z}}{2c + 2cd \times \sqrt{tt - zz}}$. Seposito itaque coefficiente dato $\frac{1}{2c + 2cd}$, fluentem ipsius

$\frac{yz\dot{z}}{\sqrt{tt - zz}}$ quaero ope Propositionis undecimae, ut sequitur.

Pro $\sqrt{tt - zz}$ scribe z , atque erit $\dot{x} = \frac{-z\dot{z}}{x}$, & Fluxio proposita erit $\frac{yz\dot{z}}{x}$ vel etiam

$-y\dot{x}$. Est etiam $\dot{y} = \frac{y\dot{z}}{c}$. Itaque secundum Prop. 11. sit $\dot{v} = y\dot{z}$, $s = \frac{z}{x}$, $\dot{w} = \dot{z} = \frac{-x\dot{x}}{z}$. Tum

capiendo fluxiones, easque continuo applicando ad

\dot{w} , erit $\dot{s} = \frac{z^2}{x^3} + \frac{1}{x} = \frac{tz}{x^3}$, $\ddot{s} = \frac{3z^3}{x^5} + \frac{3z}{x^3} = \frac{3ttz}{x^5}$, $\dddot{s} = \frac{15z^4}{x^7} + \frac{18z^2}{x^5} + \frac{3}{x^3} = \frac{15ttz^2}{x^7} + \frac{3tz}{x^5}$, & sic porro.

Hinc autem per formationes terminorum constat, quod si sit n distantia termini alicujus

\dot{s} , \ddot{s} , \dddot{s} , &c. a termino primo s , exprimetur $\overset{n}{s}$, (hoc est \dot{s} , si n sit 1, \ddot{s} , si n sit 2, &c.)

vel per seriem hujus formae, $\overset{n}{s} = A \frac{z^{n+1}}{x^{2n+1}} + B \frac{z^{n-1}}{x^{2n-1}} + C \frac{z^{n-3}}{x^{2n-3}} + D \frac{z^{n-5}}{x^{2n-5}} + \&c.$ vel [p.

112.] per seriem hujus formae, $\overset{n}{s} = tt \times A \frac{z^{n+1}}{x^{2n+1}} + B \frac{z^{n-1}}{x^{2n-1}} + C \frac{z^{n-3}}{x^{2n-3}} + D \frac{z^{n-5}}{x^{2n-5}} + \&c.$

Coefficientes autem A, B, C, D, &c. in harum serierum prima investigo ad hunc modum,

juxta notationem autem nostram sint $\overset{1}{n}$, $\overset{2}{n}$, $\overset{3}{n}$, &c. valores ipsius n praecedens, &

$\overset{1}{n}$, $\overset{2}{n}$, $\overset{3}{n}$, &c. ejusdem valores subsequentes, ut in introductione explicavimus. Capiendo

fluxiones seriei, primo in x , deinde in z , & terminos prodeuntes continuo applicando ad

$$\overset{n}{s} = \overset{2n+1}{2n+1} A \frac{z^{n+1}}{x^{2n+1}} + [\overset{2n-1}{2n-1} B + \overset{n+1}{n+1} A] \frac{z^{n-1}}{x^{2n-1}} + [\overset{2n-3}{2n-3} C + \overset{n-1}{n-1} B] \frac{z^{n-3}}{x^{2n-3}} + [\overset{2n-5}{2n-5} D + \overset{n-3}{n-3} C] \frac{z^{n-5}}{x^{2n-5}} + \&c.$$

Hinc novum A erit $\overset{2n+1}{2n+1} A$. Unde constat ipsum A formari per continuam multiplicationem terminorum 1, 3, 5, 7, &c. quorum ultimus & maximus sit $2n - 1$. In sequentibus autem vice $2n - 1$ scribe m , atque $A = \overset{m}{m} A$.

Item per terminum secundum est $B = mB + nA$. Si fieri potest ut B producat ab A per multiplicationem & divisionem, sit $B = \frac{Q}{R}A$. Tum sit $B = \frac{Q}{R}A = \frac{Q}{R}mA$. Unde eliminatis B & B ab aequatione priori, & simul evanescente A, erit

$$\frac{mQ}{R} = \frac{mQ}{R} + n, \text{ hoc est } \frac{mQ}{R} + \frac{mQ}{R} = \frac{mQ}{R} + n. \text{ Ut reducat haec aequatio [p. 113.] ad}$$

terminos simplicissimos pono $\frac{mQ}{R} = \frac{mQ}{R}$, hoc est $\frac{m}{R} = \frac{m}{R}$. Unde sit $\frac{mQ}{R} = n$. Sed est

$$\frac{m}{R} \text{ novus valor ipsius } \frac{m}{R}, \text{ indeque } Q = n, \text{ \& sumendo integrales } Q = \frac{nn}{2} + p. \text{ Sed}$$

debet esse $B = 0$ ubi $n = 0$; adeoque; est $p = 0$, atque $Q = \frac{nn}{2}$. Unde sit $B = \frac{nn}{2m}A$. Et

$$\text{hinc } B = \frac{mn}{n}B.$$

Per terminum tertium est $C = mC + nB$. Pono $C = \frac{Q}{R}B$, & sit $C = \frac{Q}{R}B$, hoc est

$$\frac{Qmn}{Rn}B = \frac{mQ}{R}B + nB, \text{ seu } \frac{Qmn}{nR} + \frac{mn}{nR}Q = \frac{mQ}{R} + n. \text{ Pone } \frac{mn}{nR} = \frac{m}{R}, \text{ hoc est,}$$

$$\frac{mnn}{R} = \frac{mnn}{R}, \text{ ut fiat } \frac{mnn}{nR}Q = n. \text{ Sed est } \frac{mnn}{R} \text{ novus valor ipsius } \frac{mnn}{R}. \text{ Unde sit}$$

$$R = mnn, \text{ adeoque; } Q = nnn, \text{ indeque; } Q = \frac{nnnm}{4}. \text{ Unde sit}$$

$$C = \frac{nn}{4m}B, \text{ atque } C = \frac{mn}{n}C.$$

Per terminum quartum est $D = mD + nC$. Unde ad eundem modum invenitur

$$D = \frac{nn}{6m}C, \text{ atque } D = \frac{mnn}{n}. \text{ Ex terminis autem jam apposis satis constat modus formandi}$$

caeteros. Un de si jam pro totis terminis cum suis signis scribantur A, B, C, &c. erit

$$s = 1.3.5....\overline{2n-1} \frac{z^{n+1}}{x^{2n+1}} + \frac{nn}{2m} \frac{xx}{zz} A + \frac{nn}{4m} \frac{xx}{zz} B + \&c.$$

hoc est

$$s = 1.3.5....\overline{2n-1} \frac{z^{n+1}}{x^{2n+1}} + \frac{n+1.n}{2 \times \overline{2n-1}} \frac{xx}{zz} A + \frac{n-1.n-2}{4 \times \overline{2n-3}} \times \frac{xx}{zz} B + \frac{n-3.n-4}{6 \times \overline{2n-5}} \times \frac{xx}{zz} C + \&c.$$

Et ad eundem modum inveniuntur coefficientes in serie altera, ut

$$s^n = \frac{1.3.5\dots.2n-1.tt.z^{n-1}}{x^{2n+1}} + \frac{n-1.n-2.xx}{2 \times 2n-1.zz} A + \frac{n-3.n-4.xx}{4 \times 2n-3.zz} B + \frac{n-5.n-6.xx}{6 \times 2n-5.zz} C + \&c.$$

Quinetiam si jam sit m distantia termini alicujus $s, s, s, \&c.$ a termino s , pro n scripto $-m$ exprimetur etiam s^m per eadem series. In hoc autem casu inveniendus est coefficiens termini primi, ut fecimus in Proportione duodecima. Debet enim esse $2n-1$, hoc est $-2m-1$, maximus factorum $1, 3, 5, 7, \&c.$ in coeffiente illo. Et hic coefficiens sic scribi potest

$\frac{2n-1\dots.5.3.1.-1.-3.-5.\&c.}{-1.-3.-5.\&c.}$ hoc est, $\frac{-2m-1.-2m-3.-2m-5.\&c.}{-1.-3.-5.-7.\&c.}$. Incidente autem n inter numeros negativos, adeoque; & m inter numeros affirmativos $1, 2, 3, \&c.$

omnes factores $-2m-1.-2m-3.-2m-5.\&c.$ in numeratore tolluntur per similes factores [p. 115] in denominatore. Unde relinquitur ut in isto casu sit coefficiens termini

primi $\frac{1}{-1.-3.-5\dots.-2m-1}$, atq; sit

$$s^m = \frac{z^{-m+1}}{-1.-3.-5\dots.-2m+1.x^{-2m+1}} + \frac{-m+1.-m.xx}{2.-2m-1.zz} A + \frac{-m-1.-m-2.xx}{4.-2m-3.zz} B + \&c. \text{ vel}$$

$$s^m = \frac{x^{2m-1}}{-1.-3.-5\dots.-2m+1.z^{m-1}} + \frac{-m+1.m.xx}{2.2m+1.zz} A + \frac{-m-1.m+2.xx}{4.2m+3.zz} B + \frac{-m-3.m+4.xx}{6.2m+5.zz} C + \&c.$$

hoc est per seriem priorem. Et haec series quidem est praestantior ad inveniendas

fluentes $s, s, s, \&c.$ altera autem ad inveniendas fluxiones $\dot{s}, \ddot{s}, \ddot{\ddot{s}}, \&c.$

Porro est $\dot{r} = \dot{y}\dot{z}$, atque $\dot{y} = \frac{y\dot{z}}{c}$. Unde sumendo fluentes purae sit

$r = cy, \dot{r} = c^2\dot{y}, \ddot{r} = c^3\ddot{y}, \&c.$ item sumendo fluxiones, $r = y, \ddot{r} = \frac{y}{c}, \ddot{\ddot{r}} = \frac{y}{c^2}, \&c.$ Unde

observatis signis, per hos valores ipsorum $\dot{s}, \ddot{s}, \ddot{\ddot{s}}, \&c.$ & $c.s, s, s, \&c.$ & $r, \dot{r}, \ddot{r}, \&c.$ & $\dot{r}, \ddot{r}, \ddot{\ddot{r}}, \&c.$

sit angulus FHG (= $\frac{1}{2c+2cd} \times rs - \dot{r}\dot{s} + \ddot{r}\ddot{s} - \&c.$) = $\frac{1}{2c+2cd}$ in hanc seriem,

$$\begin{aligned}
 &cy \times \frac{z}{x} \\
 &-c^2y \times \frac{tz}{x^3} \\
 &c^3y \times \frac{3ttz}{x^5} \\
 &\hline
 &-c^4y \times \frac{1.3.5ttz^2}{x^7} + \frac{1}{5} \frac{x^2}{z^2} A \\
 &\&c.
 \end{aligned}$$

[p. 116.] atque angulus FGH

$$\left(\frac{1}{2c+2cd} \times -\dot{r}s + r''s - \ddot{r}s \&c. - P \right) \text{ aequalis } \frac{1}{2c+2cd} \text{ in hanc seriem}$$

$y \times x$

$$\frac{y}{c} \times \frac{x^3}{1.3z} + \frac{-1}{5} \frac{x^2}{z^2} A + \frac{-3}{7} \frac{x^2}{z^2} B + \&c.$$

$$\frac{y}{c^2} \times \frac{x^5}{1.3.5z^2} + \frac{-3}{7} \frac{x^2}{z^2} A + \frac{-5}{9} \frac{x^2}{z^2} B + \&c.$$

&c. - P.

Ubi est P valor ejusdem seriei prodiens per ipsorum z, x, y in puncto A.

Potest etiam alia series inveniri pro angulo FGH; nempe ita corrigendo Fluentes

$r, r', r'', \&c.$ ut omnes evanescant in puncto A, ubi est $z = a$. Ut hoc fiat pone $z = a - v$,

unde sit $\dot{z} = -\dot{v}$, & fluxio anguli FGH sit $\frac{\dot{v}yz}{x}$ in $\frac{1}{2c+2cd}$. Posito itaque; $s = \frac{z}{x}$ (ut

prius) erit $\dot{r} = \dot{v}y$, existente $\dot{w} = -\dot{v}$, atque; $\dot{y} = \frac{-\dot{v}y}{c}$. Unde sit $\dot{r} = -c\dot{y}$, adeoque

$r = cd - cy$, (quoniam est d valor ipsius y in puncto A,) indeque $\dot{r} = c^2d - cdv - c^2y$, &

inde $\ddot{r} = c^3d - c^2dv + \frac{cdv^2}{2} - c^3y, \ddot{r} = c^4d - c^3dv + \frac{c^2dv^3}{2.3} - c^4y$, & sic porro.

Unde sit angulus FGH aequalis $\frac{1}{2c+2cd}$ in

$$\overline{cd - cy} \times \frac{z}{x}$$

$$\overline{-c^2d + cdv + c^2y} \times \frac{tz}{x^3}$$

$$\overline{c^3d - c^2dv + \frac{cdv^2}{2} - c^3y} \times \frac{3ttz}{x^5}$$

$$\overline{-c^4d + c^3dv - \frac{c^2dv^2}{2} + \frac{cdv^3}{2.3} + c^4y} \times \frac{1.3.5ttz^2}{x^7} + \frac{1}{5} \frac{x^2}{z^2} A$$

& c.

[p. 117.] Et hinc sit summa angulorum FHG, FGH, hoc est angulus GFH aequalis

$$\frac{1}{2c + 2cd} \text{ in}$$

$$cd \times \frac{z}{x}$$

$$\overline{-c^2d + cdv} \times \frac{tz}{x^3}$$

$$\overline{c^3d - c^2dv + \frac{cdv^2}{2}} \times \frac{3ttz}{x^5}$$

$$\overline{-c^4d + c^3dv - \frac{c^2dv^2}{2} + \frac{cdv^3}{2.6}} \times \frac{1.3.5ttz^2}{x^7} + \frac{1}{5} \frac{x^2}{z^2} A$$

& c.

Ubi angulus SAD est satis parvus, commode invenitur angulus GFH per hanc seriem. Sed ubi est angulus SAD nimis magnus, quaerendus est angulus FGH per seriem alteram.

Potest & alia series inveniri pro angulo FGH, per Propositionem septimam. Sit enim Q

fluens ipsius $\frac{-zz\dot{y}}{x}$, hoc est ipsius $\dot{x}y$. Tum per Propositionem illam, quo tempore x sit

$x \pm v$, fiet $Q = Q \pm \frac{\dot{Q}}{x}v \pm \frac{\ddot{Q}}{2x}v^2 \pm \frac{\dddot{Q}}{2.3x}v^3 + \&c.$ nempe fluente uniformiter x. Unde si

pro x sumatur ipsius valor in puncto aliquo dato I, & sit $x - v$ ejusdem valor in A, & $x + v$

ijusdem valor 1 α , valor fluentis in puncto A erit $Q + \frac{\dot{Q}}{x}v + \frac{\ddot{Q}}{2x}v^2 + \frac{\dddot{Q}}{2.3x}v^3 + \&c.$ &

valor fluentis in puncto α erit $Q - \frac{\dot{Q}}{x}v + \frac{\ddot{Q}}{2x}v^2 - \frac{\dddot{Q}}{2.3x}v^3 + \&c.$ Quo valore dempto a

valore altero, residium erit Fluentis pars adjacens rectae A_a ; adeoque; si sit $SB = \frac{SA^2}{S_a}$,

erit angulus [p. 118.] $= \frac{1}{c+cd} \times \frac{Q}{x} * + \frac{\ddot{Q}}{2.3x} v^3 * + \frac{\ddot{\ddot{Q}}}{2.3.4.5x} v^5 + \&c.$ In hoc casu autem

pro \dot{x} scripto 1, est $\dot{z} = \frac{x}{z}$, & $\dot{y} = \frac{-xy}{cz}$. Unde existente $\dot{Q} = y$, fiunt

$$\ddot{Q} = \frac{y}{cz^2} \times \overline{\frac{x^2}{c} - \frac{tt}{z}},$$

$$\ddot{\ddot{Q}} = \frac{y}{c^2z^4} \times \frac{x^4}{cc} - \frac{6ttx^2}{cz} + \frac{3ttx^2}{z} \times \overline{z - c} \times \overline{\frac{1+5x^2}{z^2}}. \&c.$$

SCHOLIUM.

In hac Curva Radius Curvaturae est $\frac{\overline{2+2y \times c \times SBcub.}}{y \times SQ \times SA quad.}$; quod in puncto A est

$\frac{\overline{2+2d \times c \times SA}}{d \times SD}$; & ubi angulus SAD est rectus, est $\frac{\overline{2+2d}}{d} c$. Quod per valores c & d

(*Schol. Prop. 26. & Schol. Lem. 13.*) est 5 SA, circiter, existente SA Radio Terrae. Proinde curvatura Radii Luminis horixontalis, ad superficiem Terrae, est ad curvatruam Circuli maximi Terrae, ut 1 ad 5. Velocitas autem Luminis est ad velocitatem Corporis revolventis in Circulo maximo Terrae, cum vi Gravitatis, circiter ut 40000 ad 1. Unde est Aeris vis refrangens ad vim Gravitatis in superficie Terrae circiter ut 320000000 ad 1. Nam in data inclinatione Trajectoriarum ad directionem Virium centripetarum, Vires illae sunt in ratione composita Flexurarum & quadratorum Velocitatum.

FINIS.