FIXED-SLASH AND FLOATING-SLASH RATIONAL ARITHMETIC

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1. Introduction and Summary.

A finite precision rational number system provides for representation of a collection of rational numbers subject to limitations on numerator and denominator magnitude. In fixed-point and floating-point radix number systems only rationals of the form i/β^j, where β is the base, can be realized. In contrast, a finite precision rational number system will allow representation of practically all simple fractions encountered in applications.

In this preliminary report we first propose two types of finite precision rational number systems which we term fixed-slash and floating-slash systems [2]. We then consider the conversion (rounding) problem, that is, the determination of a number satisfying the numerator and denominator constraints approximating a given non representable real value. We show that the rounding problem is solvable by an efficient procedure, which we term mediant conversion, that derives from the theory of continued fractions.

2. Fixed-Slash and Floating-Slash Systems.

A fixed-slash rational number system denotes a finite precision rational system where the precision limitation is imposed by separately bounding the numerator and denominator size. The numerator and denominator terms may each be stored in fixed length fields in a standard radix polynomial format as shown in Fig. 1a.

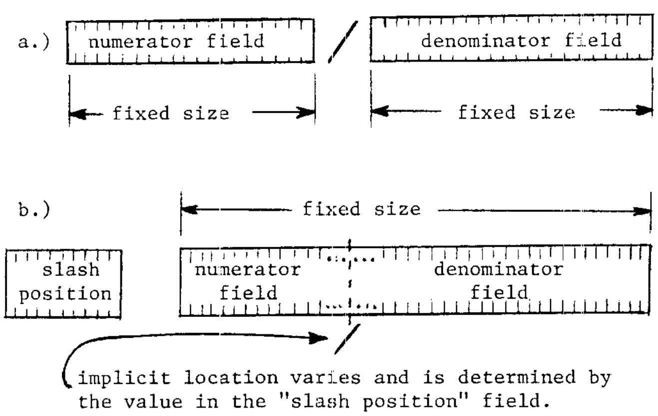

Figure 1: Formats for realization of a (a) fixed-slash rational number system and of a (b) floating-slash rational number system.

A. Svoboda has proposed that the numerator and denominator could each be encoded in residue format, which provides an interesting alternative for realizing fixed-slash rational arithmetic.

An alternative form of precision limitation for a finite precision rational system is derived by limiting the sum of the number of digits in the radix representations of the numerator and denominator. This floating-slash rational number system may be realized by programming the position of the slash in a fixed length field containing the numerator digits to the left of the slash position and the denominator digits to the right as shown in Fig. 1b. The programmed position of the floating slash as the numerator increases may be allowed to move outside the actual fixed field size, denoting that the numerator is scaled up by an appropriate power of the base with implicit denominator unity. This provides for large magnitude representation just as is available in floating-point radix format. The interpretation for the floating slash moving outside the field as the denominator increases is to scale up the denominator size by powers of the base with implicit numerator unity. This extended magnitude range feature of a floating-slash system still allows for the exact reciprocal of any non-zero rational of the system to be in the system.

3. Mediant Conversion.

An essential ingrediant in the computer realization of finite precision rational arithmetic is an efficient procedure for determining an appropriate "approximate fraction" to represent a rational number whose numerator and/or denominator terms exceed the precision limitation. Fortunately, both a theoretical and algorithmic solution to this problem is obtained from classical number theory. For example, suppose we wish to approximate 277/642 = .43146... by a fraction limited to at most two decimal digits each in numerator and denominator. First note that 27/64 = .42187... and 28/64 = .43750... are both poor approximations in comparison to 22/51 = .43137..., which differs by only one unit in the fourth decimal place from 277/642 = .43146... . To obtain this superior approximation the Euclidean Algorithm is utilized to obtain the continued fraction expansion of 277/642 as shown in Fig. 2.

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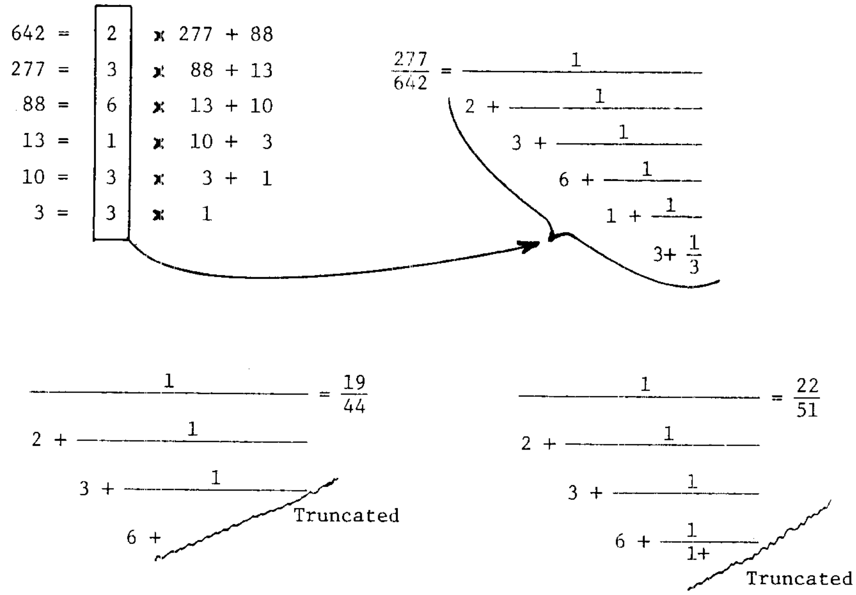

Figure 2. Use of the Euclidean Algorithm to determine a continued fraction expansion and two successive convergents to 277/642.

The truncated continued fractions can be evaluated as shown in Fig. 2, and provide "best" approximations [1] to the original value 277/642. These truncated continued fractions are termed convergents, and an efficient forward computation procedure is available [3] for computing the succession of convergents.

Let $x = a_1 + 1/(a_2 + 1/(a_3 + \dots))$. Then the convergents $c_n = p_n/q_n$ to x are determined by the following recursive equations.

$$p_1 = a_1 \qquad p_2 = a_2 a_1 + 1$$

$$q_1 = 1 \qquad q_2 = a_2 \tag{1}$$

$$p_n = a_n p_{n-1} + p_{n-2} \quad \text{for} \quad n \geq 3,$$

$$q_n = a_n q_{n-1} + q_{n-2} \quad \text{for} \quad n \geq 3.$$

For the data of Fig. 2 we have:

n	1	2	3	4	5	6	7
a_n	0	2	3	6	1	3	3
p_n	0	1	3	19	22	85	277
q_n	1	2	7	44	51	197	642

The largest indexed convergent satisfying the precision limitations of the finite precision rational system is termed the mediant conversion value for the original real value. Thus 22/51 is the desired result for our numeric example. It can be shown that the process of mediant conversion provides an "optimal rounding" in the sense of Kulisch for both fixed-slash and floating-slash rational systems.

4. Conclusion.

The structure of fixed-slash and floating-slash rational number systems, the theory of mediant conversion, and the foundations of finite-precision rational number systems in general are being developed by this author. These investigations are at too premature a state to ascertain how much influence such numeric systems may have on practical computation, but it should be clear that greater knowledge of the number theoretic foundation of rational arithmetic can provide insight and possibly invaluable procedural tools to the arithmetic computer architect.

References

1. Hardy, G. H. and Wright, E. M., An Introduction to the Theory of Numbers, Clarendon Press, Oxford, 1954.

2. Matula, D. W., Number theoretic foundations of finite precision arithmetic, in Applications of Number Theory to Numerical Analysis, W. Zaremba, ed., Academic Press, New York, 1972, 479-489.

3. Pettofrezzo, A. J. and Byrkit, D. R., Elements of Number Theory, Prentice Hall, Englewood Cliffs, 1970.