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Lobachevski Illuminated

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Lobachevski Illuminated

Nikolai Ivanovich Lobachevski (1792–1856)

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Introduction

Through the ostensibly infallible process of logical deduction, Euclid of Alexandria (ca. 300 B.C.) derived a colossal body of geometric facts from a bare minimum of genetic material: five *postulates* — five simple geometric assumptions that he listed at the beginning of his masterpiece, the *Elements*. That Euclid could produce hundreds of unintuitive theorems from five patently obvious assumptions about space, and, still more impressively, that he could do so in a manner that precluded doubt, sufficed to establish the *Elements* as mankind's greatest monument to the power of rational organized thought. As a logically impeccable, tightly wrought description of space itself, the *Elements* offered humanity a unique anchor of definite knowledge, guaranteed to remain eternally secure amidst the perpetual flux of existence — a rock of certainty, whose truth, by its very nature, was unquestionable.

This universal, even transcendent, aspect of the *Elements* has profoundly impressed Euclid's readers for over two millennia. In contrast to all explicitly advertised sources of transcendent knowledge, Euclid never cites a single authority and he never asks his readers to trust his own ineffably mystical wisdom. Instead, we, his readers, need not accept anything on faith; we are free and even encouraged to remain skeptical throughout. Should one doubt the validity of the Pythagorean Theorem (*Elements* I.47), for example, one need not defer to the reputation of "the great Pythagoras". Instead, one may satisfy oneself in the manner of Thomas Hobbes, whose first experience with Euclid was described by John Aubrey, in his *Brief Lives*, in the following words.

> He was 40 years old before he looked on Geometry; which happened accidentally. Being in a Gentleman's library, Euclid's *Elements* lay open, and 'twas the 47 *E. Libri I.* He read the Proposition. *By G* —, says he, (he would now and then sweare an emphaticall Oath by way of emphasis) *this is impossible*! So he reads the Demonstration of it, which referred him back to such a Proposition; which proposition he read. That referred him back to another, which he also read. *Et sic deincips* and so on that at last he was demonstratively convinced of that truth. This made him in love with Geometry.

The *Elements* was an educational staple until the early twentieth century. So long as reading it remained a common experience among the educated, Euclid's name was synonymous with demonstrable truth[1]. It is not an exaggeration to assert that Euclid was the envy of both philosophy

[1] At the very least, the demonstrations in the *Elements* were acknowledged as the strongest possible sort of which the rational mind is capable. "It is curious to observe the triumph of slight incidents over the mind: — What incredible weight they have in forming and governing our opinions, both of men and things — that trifles, light as air, shall waft a belief into the soul, and plant it so immovably within it – that Euclid's demonstrations, could they be brought to batter it in breach, should not all have power to overthrow it." (Laurence Sterne, *The Life and Opinions of Tristram Shandy.* Book IV, Ch. XXVII)

and theology. In his *Meditations*, Descartes went so far as to base his certainty that God exists on his certainty that Euclid's 32nd proposition is true. This was but a single instance out of many in which theology has tried to prop itself up against the rock of mathematics. Euclid's *Elements*, for all its austerity, appeals to a deep-seated human desire for certainty. This being the case, any individual with the impertinence to challenge Euclid's authority was certain to inspire reactions of both incredulity and scorn.

But how exactly *can* one challenge Euclid's authority? Euclid asks us to accept nothing more than five postulates, and all else follows from pure logic. Therefore, if there is anything to challenge in the *Elements*, it can only be in the postulates themselves. The first four seem almost too simple to question. Informally, they describe the geometer's tools: a straightedge, a compass, and a consistent means for measuring angles. The fifth postulate, however, is of a rather different character:

> That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

This is Euclid's famous *parallel postulate*, so called because it forms the basis for his theory of parallels, which, in turn, forms the basis for nearly everything else in geometry. Modern geometry texts almost invariably replace this postulate with an alternative, to which it is logically equivalent: *given a line and a point not on it, there is exactly one line that passes through the point and does not intersect the line*. Particularly when expressed in this alternate form, the parallel postulate does strike most as "self-evident", and thus beyond question for any sane individual. It would seem, therefore, that Euclid has no significant weaknesses; his geometry is *the* geometry — impregnable, inevitable, and eternal.

The timeless, almost icy, perfection that characterizes Euclid's work made it not only a logical masterpiece, but an artistic one as well. In this latter aspect, commentators often singled out the parallel postulate as the unique *aesthetic* flaw in the *Elements*. The problem was that the parallel postulate seemed out of place: it read suspiciously like a theorem — something that Euclid should have *proved* from his earlier postulates, instead of adjoining it to their ranks. This structural incongruity — a postulate that "should be" a theorem — disturbed many mathematicians from antiquity to the 19th century. We may safely presume that Euclid tried and failed to prove the postulate as a theorem. We *know* that Euclid's followers and admirers tried to do as much, hoping to perfect their master's work by polishing away this one small but irritating blemish. Many believed that they had succeeded.

Records of flawed "proofs" rarely survive, as there generally seems no reason to preserve them, so the astonishing number of alleged proofs of the parallel postulate that have come down to us should serve to indicate just how much attention was given to this problem. Proclus, a 5th-century neo-Platonic philosopher, who wrote an extensive commentary on the first book of the *Elements*, describes two attempts: one by Posidonius (2nd century B.C.), the other by Ptolemy (the 2nd-century A.D. author of the *Almagest*, the Bible of geocentric astronomy). Both arguments, Proclus points out, are inadmissible because they contain subtle flaws. After detailing these flaws, Proclus proceeded to give his own proof, thus settling the matter for once and all — or so he thought. Proclus' proof, for all his critical acumen, was just as faulty as those he had criticized.

We have flawed proofs by Aghanis (5th century) and Simplicius (6th-century), two Byzantine scholars. Many others by medieval Islamic mathematicians have survived, including attempts by al-Jawhari and Thabit ibn Qurra in the 9th century, al-Haytham and Omar Khayyám in the 11th, and Nasir-Eddin al-Tusi in the 13th. There are even a few specimens from medieval Europe, such as those conceived by Vitello in the 13th century and Levi ben Gerson in the 14th. A veritable horde of later Europeans left purported proofs of the postulate (to cite just a few examples: Christopher Clavius in 1574, Pietro Antonio Cataldi in 1604, Giovanni Alfonso Borelli in 1658, Gerolamo Saccheri in 1733, Louis Bertrand in 1788, and Adrien Marie Legendre, who published many attempts between 1794 and 1832). Indeed, in 1763, G.S. Klügel wrote a dissertation examining no less than twenty-eight unsound "proofs" of the postulate. Interestingly, most would-be postulate provers followed Proclus in explicitly criticizing one or more of their predecessors' attempts before giving their own flawed "proof to end all proofs".[2]

Adhering to long-standing custom, Nikolai Ivanovich Lobachevski (1792–1856) began many of his own works on the subject by criticizing the alleged proofs of his immediate predecessor, Legendre. However, instead of forging the chain's next link, Lobachevski suggested that the chain be discarded altogether. He insisted that the parallel postulate *cannot* be proved from Euclid's first four postulates. In this sense, Lobachevski was a great defender of Euclid: he felt that Euclid was fully justified in assuming the parallel postulate as such; indeed, he believed that Euclid had no other way to obtain it.

In another sense, Lobachevski believed that Euclid was wholly *unjustified* in assuming the parallel postulate, for we cannot be certain that it accurately describes the behavior of lines in physical space. Euclidean tradition declares that it does, but the universe is not obliged to respect humanity's traditional beliefs about space, even those codified by its great authority, Euclid of Alexandria. Lobachevski considered the validity of the parallel postulate an empirical question, to be settled, if possible, by astronomical measurements.

Unorthodoxy quickly led to heresy: proceeding from the assumption that the parallel postulate does *not* hold, Lobachevski began to develop a *new* geometry, which he called *imaginary geometry*[3], whose results contradicted Euclid's own. He first described this strange new world on February 24, 1826, in a lecture at the University of Kazan. His first written publication on the subject dates from 1829. Several others followed, and after a decade of failed attempts to convince his fellow Russians of the significance of his work, he published accounts of it in French (in 1837) and German (in 1840), hoping to attract attention in Western Europe. He found none. By the time that he wrote *Pangeometry* (1855), he was blind (he had to dictate the book), exhausted, and embittered. He died the following year.[4]

In fact, although Lobachevski never knew it, his work did find one sympathetic reader in his lifetime: Karl Friedrich Gauss (1777–1855), often classed with Isaac Newton and Archimedes as one of the three greatest mathematicians who have ever lived. Gauss shared Lobachevski's convictions regarding the possibility of an alternate geometry, in which the parallel postulate does

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[2] For detailed descriptions of many alleged proofs of the postulate, consult Rosenfeld (Chapter 2) or Bonola (Chapters 1 and 2).

[3] By the end of his life, he preferred the name *pangeometry*, for reasons that will become clear by the end of *The Theory of Parallels*. Other common adjectives for Lobachevski's geometry are *non-Euclidean* (used by Gauss), *hyperbolic* (introduced by Felix Klein), and *Lobachevskian* (used by Russians).

[4] His French paper of 1837, *Géométrie Imaginaire*, appeared in August Crelle's famous journal, *Journal für die Reine und Angewandte Mathematik* (Vol. 17, pp. 295–320). His German publication of 1840 was *The Theory of Parallels*; its full title is *Geometrische Untersuchungen zur Theorie der Parallelinien* (*Geometric Investigations on the Theory of Parallels*). Lobachevski wrote two versions of *Pangeometry*, one in French and one in Russian.

not hold. He reached these conclusions earlier than Lobachevski, but abstained, very deliberately, from publishing his opinions or investigations. Fearing that his ideas would embroil him in controversy, the very thought of which Gauss abhorred, he confided them only to a select few of his correspondents, most of them astronomers. When Gauss read an unfavorable review of Lobachevski's *Theory of Parallels*, he dismissed the opinions of the reviewer, hastened to acquire a copy of the work, and had the rare pleasure of reading the words of a kindred, but more courageous, spirit. Gauss was impressed; he even sought out and read Lobachevski's early publications in Russian. To H.C. Schumacher, he wrote in 1846, "I have not found anything in Lobachevski's work that is new to me, but the development is made in a different way from the way I had started and, to be sure, masterfully done by Lobachevski in the pure spirit of geometry."

True to his intent, Gauss' radical thoughts remained well-hidden during his lifetime, but within a decade of his death, the publication of his correspondence drew the attention of the mathematical world to non-Euclidean geometry. Though the notion that there could be two geometries did indeed generate controversy, the fact that Gauss himself endorsed it was enough to convince several mathematicians to track down the works of the unknown Russian whom Gauss had praised so highly. Unfortunately, Lobachevski reaped no benefit from this interest; he was already dead by that time, as was the equally obscure Hungarian mathematician, János Bolyai (1802–1860), whose related work also met with high praise in Gauss' correspondence.

Bolyai had discovered and developed non-Euclidean geometry independently of both Lobachevski and Gauss. He published an account of the subject in 1832, but it had essentially no hope of finding an audience: it appeared as an appendix to a two-volume geometry text, written by his father, Farkas Bolyai, in Latin. Farkas Bolyai, who had known Gauss in college, sent his old friend a copy of his son's revolutionary studies. Gauss' reply — that all this was already known to him — so discouraged the young János, that he never published again, and even ceased communicating with his father, convinced that he had allowed Gauss to steal and take credit for his own discoveries. Father and son were eventually reconciled, but Bolyai was doubly disheartened some years later to learn that his own *Appendix* could not even claim the honor of being the first published account of non-Euclidean geometry: Lobachevski's earliest Russian paper antedated it by several years.

As mathematicians began to re-examine the work of Lobachevski and Bolyai, translating it into various languages, extending it, and grappling with the philosophical problems that it raised, they changed the very form of the subject in order to assimilate it into mainstream mathematics. By 1900, non-Euclidean geometry remained a source of wonder, but it had ceased to be a controversial subject among mathematicians, who were now describing it in terms of differential geometry, projective geometry, or Euclidean "models" of the non-Euclidean plane. These developments and interpretations helped mathematicians domesticate the somewhat nightmarish creatures that Lobachevski and Bolyai had loosed upon geometry. Much was gained, but something of great psychological importance was also lost in the process. The tidy forms into which the subject had been pressed scarcely resembled the majestic full-blooded animal that Lobachevski and Bolyai had each beheld, alone, in the deep dark wild wood.

Today, in 2011, the vigorous beast is almost never seen in its original habitat. Just as we give toy dinosaurs and soft plushy lions to children, we give harmless non-Euclidean toys, such as the popular Poincaré disc model, to mathematics majors. We take advanced students to the zoo of differential geometry and while we are there, we pause — briefly, of course — to point out a captive specimen of hyperbolic geometry, sullenly pacing behind bars of constant negative curvature.

If we are to understand the meaning of non-Euclidean geometry — to understand why it wrought such important changes in mathematics — we must first recapture the initial fascination and even the horror that mathematicians felt when confronted with the work of Lobachevski and Bolyai. This, however, is difficult. The advent of non-Euclidean geometry changed the mathematical landscape so profoundly that the pioneering works themselves were obscured in the chaos of shifting tectonic plates and falling debris. Mathematical practices of the early 19th century are not the same as those of the early 21st. The gap of nearly two centuries generally precludes the possibility of a sensitive reading of Lobachevski's works by a modern reader. This book is an attempt to rectify the situation, by supplying the contemporary reader with all of the tools necessary to unlock this rich, beautiful, but generally inaccessible world. But where does one start?

Gauss left us nothing to work with. Bolyai's *Appendix* is out of the question; his writing is often terse to the point of incomprehensibility. Lobachevski is far clearer, but he too makes heavy demands on his readers. Perhaps we should read his earliest works? In 1844, Gauss described them (in a letter to C.L. Gerling) as "a confused forest through which it is difficult to find a passage and perspective, without having first gotten acquainted with all the trees individually." At the other chronological extreme, Lobachevski's final work, *Pangeometry*, is inappropriate for beginners since it merely summarizes the elementary parts of the subject, referring the reader to *The Theory of Parallels*, his German book of 1840, for proofs. *Pangeometry* does make a logical second book to read, but the book that it leans upon, *The Theory of Parallels*, remains the best point of ingress for the modern mathematician.

Accordingly, the following pages contain a new English version of *The Theory of Parallels*, together with mathematical, historical, and philosophical commentary, which will expand and explain Lobachevski's often cryptic statements (which even his contemporaries failed to grasp), and link his individual propositions to the related work of his predecessors, contemporaries, and followers. Resituated in its proper historical context, Lobachevski's work should once again reveal itself as an exciting, profound, and revolutionary mathematical document.

The complete text of Lobachevski's *Theory of Parallels* appears **twice** within the pages of this book. In the appendix, it appears as a connected whole, in its first English translation since Halsted's in 1891. In the body of the book, the complete text appears a second time, but broken into more than 100 pieces; I have woven my illumination around these hundred-odd pieces. Lest there be any confusion as to whose voice is speaking at any given place in the book, Lobachevski's words have been printed in red, while everything else is printed in black.

A Note to the Reader

> "Begin at the beginning," the King said, very gravely,
> "and go on till you come to the end: then stop."
> —*Alice in Wonderland,* Chapter 12

Although following the King of Hearts' advice may be the most rigorous way to read *Lobachevski Illuminated*, it is hardly the only way. Beginning at the beginning is always a sensible idea, but one need not feel compelled to "go on" through the details of each and every auxiliary proof that I provide along the way. Many readers, first-time readers in particular, will simply want an overview of Lobachevski's accomplishments and methods. If you are such a person, then you should feel free to skip any technical proofs that threaten to divert you from the main narrative thread.

Ideally, one should at least read the *statements* of the propositions that I prove in the notes. What one chooses to do with them will then vary from reader to reader. Some will want to try coming up with their own proofs. Others will simply read mine. Still others will take the statements on faith and move on, confident in the knowledge that the proofs are there, patiently waiting, should they ever need to be consulted. All of these are reasonable approaches.

Of course, the further one travels into the counterintuitive non-Euclidean countryside, the less confidence one will have in dismissing anything as "obvious", "trivial", or "a mere technicality". If you have never left the Euclidean world before, then be forewarned: you are about to embark on a thoroughly disorienting (but strangely exhilarating) journey. Some readers will be more comfortable taking one tiny step at a time into this new land, mapping the terrain slowly and carefully, proving everything in detail, until even its most alien features take on a kind of unexpected familiarity. Others will charge boldly ahead, skipping many proofs, eager to reach the dark heart of the matter as quickly as possible; they will get there, of course, but are likely to find themselves so thoroughly befuddled that they will almost certainly want to go back and carefully retrace their steps so as to make some retrospective sense of the strange sights they have beheld. Again, both approaches are fine, and are ultimately a matter of individual psychology. An approach somewhere between these two extremes is probably best.

Of course, there will be some readers who will want considerably *more* technical detail than I've provided. In particular, some may desire a rigorously-argued Hilbert-style examination of the foundations of geometry. Others won't stop even there, and will want to dig into the primal matter of logic itself. But as with its readers, so a book's author must draw the line somewhere.

I have laid out the meal. Fall to it and eat. Only you can decide what to put on your plate.

Acknowledgements

Like everything else in the universe, *Lobachevski Illuminated* is a product of chance. Had any of my stone-age ancestors died before begetting offspring, I never would have existed. You might not have either. At any rate, you would not be reading *Lobachevski Illuminated*. If describing the full web of coincidences that ultimately led to this book would be tantamount to narrating the history of the universe, the least I can do is to tip my hat to a few people who loom particularly large in the book's genesis.

Hats off to Richard Mitchell at UC-Santa Cruz. His profound yet playful knowledge of geometry and history, his collection of outrageously complicated polyhedra, his wry skepticism and sense of humor all suggested to me an approach to life and to mathematics far saner than that which seems to engulf the typical harried academic[5]. When, as a master's student at UCSC, I was becoming thoroughly disoriented by the nested abstractions of modern mathematics, Richard encouraged me to delve into the subject's history as a means of recovering the basic intuitions that motivated such abstractions in the first place. I acknowledge you, O Richard!

At the University of Montana, where I wrote the dissertation that became the basis of this book, I was blessed with two excellent advisors, Greg St. George and Karel Stroethoff. Over the years, I have met hundreds of professionals who publish mathematics papers and teach mathematics for a living. The number of actual *mathematicians* I've met, however, I can count on one hand. True mathematicians are a rare breed; I feel privileged to have worked with Karel, who is one of them. I worked with Greg from the moment I arrived at UM. Had I not found such a humane advisor, to whom I could speak as friend to friend rather than as apprentice to master, I probably would have quit the doctoral program. Greg had the wisdom to let me pursue whatever mathematical topic I felt interested in, however unfashionable, which we would then discuss at our informal weekly meetings – meetings that, as often as not, evolved into discussions of languages, philosophy, and literature. He was never one to respect the artificial boundaries between "disciplines", for which failing I extend my humble thanks.

And in a final flourish . . . thanks to Gerald Alexanderson, editor of the MAA's Spectrum series, for his generous support of this book; to Llyd Wells for his friendship and post-seminar whiskey; to Reb Hastrev for his sparkling wit and iconoclastic spirit; to Shannon Michael for this and that; to Professor Q.A. Wagstaff for his luminous discourse on modern education; to Bernard Russo for his curious tract, *Group Interaction Diagrams*; to Edward Lear, whom it is pleasant to know; to Lucretius and Ovid; to Samuel Beckett; to Edwin Arlington Robinson; to Emily

[5] My brain has often been gripped by visions of university professors as members of a weird chain gang condemned to perform meaningless tasks over and over again, but instead of breaking large rocks into smaller rocks, these prisoners endlessly write and publish papers that no one will ever read.

Dickinson; to the broad-shouldered son of Ariston; to Flann O'Brien; to Bohuslav Martinů; to György Ligeti; to Percival Bartlebooth for the watercolors; to my dear Uncle Toby for the lovely map of Namur; to Tom Lehrer for two portraits of Lobachevski; to V. Mora, for demonstrating that last names (in ancient Greek) are often prophetic; to N. Artemiadis, for his letter of explanation; to Vanni Fucci, for the delicious figs; to Anna Livia Plurabelle, whose leaves are drifting from her; and to Bartleby, because he is so very good.

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Appendix: Nicolai Ivanovich Lobachevski's Theory of Parallels

In geometry, I have identified several imperfections, which I hold responsible for the fact that this science, apart from its translation into analysis, has taken no step forward from the state in which it came to us from Euclid. I consider the following to be among these imperfections: vagueness in the basic notions of geometric magnitudes, obscurity in the method and manner of representing the measurements of such magnitudes, and finally, the crucial gap in the theory of parallels. Until now, all mathematicians' efforts to fill this gap have been fruitless. Legendre's labors in this area have contributed nothing. He was forced to abandon the one rigorous road, turn down a side path, and seek sanctuary in extraneous propositions, taking pains to present them—in fallacious arguments—as necessary axioms.

I published my first essay on the foundations of geometry in the "Kazan Messenger" in the year 1829. Hoping to provide an essentially complete theory, I then undertook an exposition of the subject in its entirety, publishing my work in installments in the "Scholarly Journal of the University of Kazan" in the years 1836, 1837, and 1838, under the title, "New Principles of Geometry, with a Complete Theory of Parallels". Perhaps it was the extent of this work that discouraged my countrymen from attending to its subject, which had ceased to be fashionable since Legendre. Be that as it may, I maintain that the theory of parallels should not forfeit its claim to the attentions of geometers. Therefore, I intend here to expound the essence of my investigations, noting in advance that, contrary to Legendre's opinion, all other imperfections, such as the definition of the straight line, will prove themselves quite foreign here and without any real influence on the theory of parallels. Lest my reader become fatigued by a multitude of theorems whose proofs present no difficulties, I shall list here in the preface only those that will actually be required later.

1) *A straight line covers itself in all its positions.* By this, I mean that a straight line will not change its position during a rotation of a plane containing it if the line passes through two fixed points in the plane.

2) *Two straight lines cannot intersect one another in two points.*

3) *By extending both sides of a straight line sufficiently far, it will break out of any bounded region. In particular, it will separate a bounded plane region into two parts.*

4) *Two straight lines perpendicular to a third will never intersect one another, regardless of how far they are extended.*

5) *When a straight lines passes from one side to the other of a second straight line, the lines always intersect.*

6) *Vertical angles, those for which the sides of one angle are the extensions of the other, are equal. This is true regardless of whether the vertical angles lie in the plane or on the surface of a sphere.*

7) *Two straight lines cannot intersect if a third line cuts them at equal angles.*

8) *In a rectilinear triangle, equal sides lie opposite equal angles, and conversely.*

9) *In rectilinear triangles, greater sides and angles lie opposite one another. In a right triangle, the hypotenuse is greater than either leg, and the two angles adjacent to it are acute.*

10) *Rectilinear triangles are congruent if they have a side and two angles equal, two sides and their included angle equal, two sides and the angle that lies opposite the greatest side equal, or three sides equal.*

11) *If a straight line is perpendicular to two intersecting lines, but does not lie in their common plane, then it is perpendicular to all straight lines in their common plane that pass through their point of intersection.*

12) *The intersection of a sphere with a plane is a circle.*

13) *If a straight line is perpendicular to the intersection of two perpendicular planes and lies in one of them, then it is perpendicular to the other plane.*

14) *In a spherical triangle, equal angles lie opposite equal sides, and conversely.*

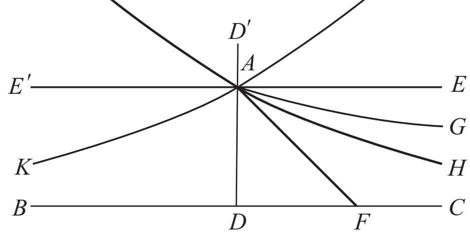
15) *Spherical triangles are congruent if they have two sides and their included angle equal, or one side and its adjacent angles equal.*

Explanations and proofs shall accompany the theorems from now on.

Proposition 16

In a plane, all lines that emanate from a point can be partitioned into two classes with respect to a given line in the same plane; namely, those that cut the given line and those that do not cut it.

The boundary-line separating the classes from one another shall be called a *parallel* to the given line.



From point A (see figure at right), drop the perpendicular AD to the line BC, and erect the perpendicular AE upon it. Now, either all of the lines entering the right angle $\angle EAD$ through A will, like AF in the figure, cut DC, or some of these lines will not cut DC, resembling the perpendicular AE in this respect. The uncertainty as to whether the perpendicular AE is the only line that fails to cut DC requires us to suppose it possible that there are still other lines, such as AG, which do not cut, no matter how far they are extended.

At the transition from the cutting lines such as AF to the non-cutting lines such as AG, one necessarily encounters a parallel to DC. That is, one will encounter a boundary line AH with the property that all the lines on one side of it, such as AG, do not cut DC, while all the lines on the other side of it, such as AF, do cut DC.

The angle $\measuredangle HAD$ between the parallel AH and the perpendicular AD is called the *angle of parallelism*; we shall denote it here by $\Pi(p)$, where $p = AD$. If $\Pi(p)$ is a right angle, then the extension AE' of AE will be parallel to the extension DB of the line DC. Observing the four right angles formed at point A by the perpendiculars AE, AD, and their extensions AE' and AD', we note that any line emanating from A has the property that either it or its extension lies in one of the two right angles facing BC. Consequently, with the exception of the parallel EE', all lines through A will cut the line BC when sufficiently extended.

If $\Pi(p) < \pi/2$, then the line AK, which lies on the other side of AD and makes the same angle $\measuredangle DAK = \Pi(p)$ with it, will be parallel to the extension DB of the line DC. Hence, under this hypothesis we must distinguish directions of parallelism.

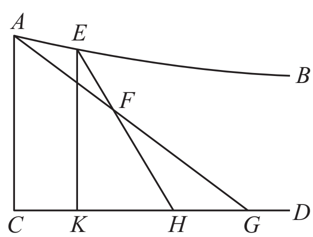
Among the other lines that enter either of the two right angles facing BC, those lying between the parallels (i.e. those within the angle $\measuredangle HAK = 2\Pi(p)$) belong to the class of cutting-lines. On the other hand, those that lie between either of the parallels and EE' (i.e. those within either of the two angles $\measuredangle EAH = \pi/2 - \Pi(p)$ or $\measuredangle E'AK = \pi/2 - \Pi(p)$) belong, like AG, to the class of non-cutting lines.

Similarly, on the other side of the line EE', the extensions AH' and AK' of AH and AK are parallel to BC; the others are cutting-lines if they lie in the angle $\measuredangle K'AH'$, but are non-cutting lines if they lie in either of the angles $\measuredangle K'AH'$ or $\measuredangle H'AE'$.

Consequently, under the presupposition that $\Pi(p) = \pi/2$, lines can only be cutting-lines or parallels. However, if one assumes that $\Pi(p) < \pi/2$, then one must admit two parallels, one on each side. Furthermore, among the remaining lines, one must distinguish between those that cut and those that do not cut. Under either assumption, the distinguishing mark of parallelism is that the line becomes a cutting line when subjected to the smallest deviation toward the side where the parallel lies. Thus, if AH is parallel to DC, then regardless of how small the angle $\measuredangle HAF$ may be, the line AF will cut DC.

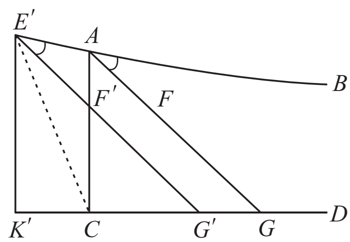
Proposition 17

A straight line retains the distinguishing mark of parallelism at all its points.



Let AB be parallel to CD, with AC perpendicular to the latter. We shall examine two points, one chosen arbitrarily from the line AB and one chosen arbitrarily from its extension beyond the perpendicular.

Let E be a point on that side of the perpendicular in which AB is parallel to BC. From E, drop a perpendicular EK to CD, and draw any line EF lying within the angle $\measuredangle BEK$. Draw the line through the points A and F. Its extension must intersect CD (by TP 16) at some point G. This produces a triangle ΔACG, which is pierced by the line EF. This line, by construction, cannot intersect AC; nor can it intersect AG or EK a second time (TP 2). Hence, it must meet CD at some point H (by TP 3).



Now let E' be a point on the extension of AB, and drop a perpendicular $E'K'$ to the extension of the line CD. Draw any line $E'F'$ with the angle $\measuredangle AE'F'$ small enough to cut AC at some point F'. At the same angle of inclination towards AB, draw a line AF; its extension will intersect CD

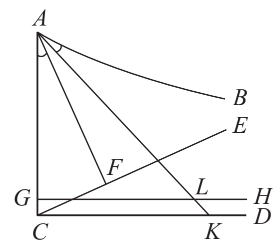
(by TP 16) at some point G. This construction produces a triangle ΔAGC, which is pierced by the extension of line $E'F'$. This line can neither cut AC a second time, nor can it cut AG, since $\measuredangle BAG = \measuredangle BE'G'$ (by TP 7). Thus, it must meet CD at some point G'.

Therefore, regardless of which points E and E' the lines EF and $E'F'$ emanate from, and regardless of how little these lines deviate from AB, they will always cut CD, the line to which AB is parallel.

Proposition 18

Two parallel lines are always mutually parallel.

Let AC be perpendicular to CD, a line to which AB is parallel. From C, draw any line CE making an acute angle $\measuredangle ECD$ with CD. From A, drop the perpendicular AF to CE. This produces a right triangle ΔACF, in which the hypotenuse AC is greater than the side AF (TP 9).

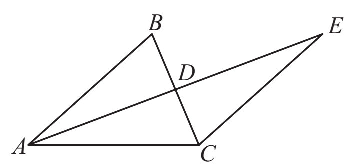


If we make $AG = AF$ and lay AF upon AG, the lines AB and FE will assume positions AK and GH in such a way that $\measuredangle BAK = \measuredangle FAC$. Consequently, AK must intersect the line DC at some point K (TP 16), giving rise to a triangle ΔAKC. The perpendicular GH within this triangle must meet the line AK at some point L (TP 3). Measured along AB from A, the distance AL determines the intersection point of the lines AB and CE. Therefore, CE will always intersect AB, regardless of how small the angle $\measuredangle ECD$ may be. Hence, CD is parallel to AB (TP 16).

Proposition 19

In a rectilinear triangle, the sum of the three angles cannot exceed two right angles.

Suppose that the sum of the three angles in a triangle is $\pi + \alpha$.

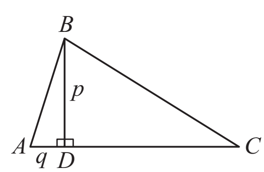


Bisect the smallest side BC at D, draw the line AD, make its extension DE equal to AD, and draw the straight line EC. In the congruent triangles ΔADB and ΔCDE (TP 16 and TP 10), we have $\measuredangle ABD = \measuredangle DCE$ and $\measuredangle BAD = \measuredangle DEC$. From this, it follows that the sum of the three angles in ΔACE must also be $\pi + \alpha$. We note additionally that $\measuredangle BAC$, the smallest angle of ΔABC (TP 9), has been split into two parts of the new triangle ΔACE; namely, the angles $\measuredangle EAC$ and $\measuredangle AEC$.

Continuing in this manner, always bisecting the side lying opposite the smallest angle, we eventually obtain a triangle in which $\pi + \alpha$ is the sum of the three angles, two of which are smaller than $\alpha/2$ in absolute magnitude. Since the third angle cannot exceed π, α must be either zero or negative.

Proposition 20

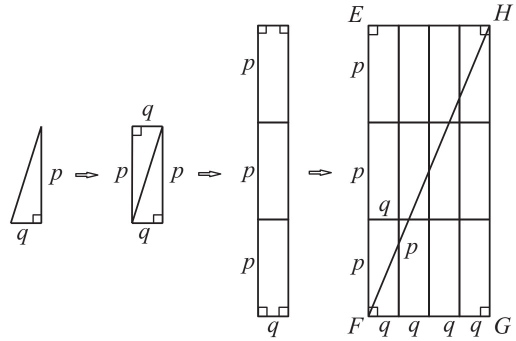
If the sum of the three angles in one rectilinear triangle is equal to two right angles, the same is true for every other triangle.



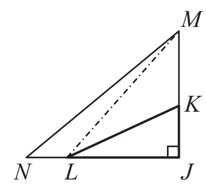
If we suppose that the sum of the three angles in triangle ΔABC is equal to π, then at least two of its angles, A and C, must be acute.

From the third vertex, B, drop a perpendicular p to the opposite side, AC. This will split the triangle ΔABC into two right triangles. In each of these, the angle sum will also be π: neither angle sum can exceed π (TP 19), and the fact that the right triangles comprise triangle ΔABC ensures that neither angle sum is less than π.

In this way, we obtain a right triangle whose legs are p and q; from this we can obtain a quadrilateral whose opposite sides are equal, and whose adjacent sides are perpendicular. By repeated application of this quadrilateral, we can construct another with sides np and q, and eventually a quadrilateral $EFGH$, whose adjacent sides are perpendicular, and in which $EF = np$, $EH = mq$, $HG = np$, and $FG = mq$, where m and n can be any whole numbers. The diagonal FH of such a quadrilateral divides it into two congruent right triangles, ΔFEH and ΔFGH, each of which has angle sum π.



The numbers m and n can always be chosen so large that any given right triangle ΔJKL can be enclosed within a right triangle ΔJMN, whose arms are $NJ = np$ and $MJ = mq$, when one brings their right angles into coincidence. Drawing the line LM yields a sequence of right triangles in which each successive pair shares a common side.

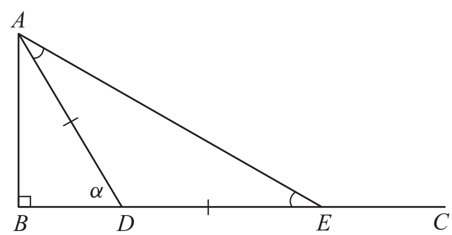


The triangle ΔJMN arises as the union of the triangles ΔNML and ΔJML. The angle sum exceeds π in neither of these; it must, therefore, equal π in each case in order to make the composite triangle's angle sum equal to π. Similarly, the triangle ΔJML consists of the two triangles ΔKLM and ΔJKL, from which it follows that the angle sum of ΔJKL must equal π.

In general, this must be true of every triangle since each triangle can be cut into two right triangles. Consequently, only two hypotheses are admissible: the sum of the three angles either equals π for all rectilinear triangles, or is less than π for all rectilinear triangles.

Proposition 21

From a given point, one can always draw a straight line that meets a given line at an arbitrarily small angle.

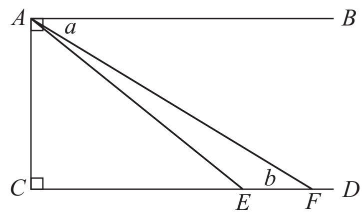


From the given point A, drop the perpendicular AB to the given line BC; choose an arbitrary point D on BC; draw the line AD; make $DE = AD$, and draw AE. If we let $\alpha = \measuredangle ADB$ in the right triangle ΔABD, then the angle $\measuredangle AED$ in the isosceles triangle ΔADE must be less than or equal to $\alpha/2$ (TP 8 & 19)[1]. Continuing in this manner, one eventually obtains an angle ∡AEB that is smaller than any given angle.

[1] I have corrected an apparent misprint occurring in Lobachevski's text and perpetuated in Halsted's 1891 translation of *TP*. In these sources, Lobachevski cites TP 20 at this point, rather than TP 19. This makes little sense; TP 20 relates the

Proposition 22

If two perpendiculars to the same straight line are parallel to one another, then the sum of the three angles in all rectilinear triangles is π.



Let the lines AB and CD be parallel to one another and perpendicular to AC. From A, draw lines AE and AF to points E and F chosen anywhere on the line CD such that $FC > EC$. If the sum of the three angles equals $\pi - \alpha$ in the right triangle ΔACE and $\pi - \beta$ in triangle ΔAEF, then it must equal $\pi - \alpha - \beta$ in triangle ΔACF, where α and β cannot be negative. Further, if we let $a = \measuredangle BAF$ and $b = \measuredangle AFC$, then $\alpha + \beta = a - b$.

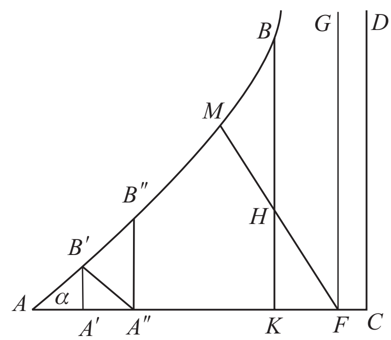
By rotating the line AF away from the perpendicular AC, one can make the angle a between AF and the parallel AB as small as one wishes; one reduces the angle b by the same means. It follows that the magnitudes of the angles α and β can be none other than $\alpha = 0$ and $\beta = 0$.

From what we have seen thus far, it follows either that the sum of the three angles in all rectilinear triangles is π, while the angle of parallelism $\Pi(p) = \pi/2$ for all lines p, or that the angle sum is less than π for all triangles, while $\Pi(p) < \pi/2$ for all lines p. The first hypothesis serves as the foundation of the ordinary geometry and plane trigonometry.

The second hypothesis can also be admitted without leading to a single contradiction, establishing a new geometric science, which I have named Imaginary Geometry, which I intend to expound here as far as the derivation of the equations relating the sides and angles of rectilinear and spherical triangles.

Proposition 23

For any given angle α, there is a line p such that $\Pi(p) = \alpha$.



Let AB and AC be two straight lines forming an acute angle α at their point of intersection A. From an arbitrary point B' on AB, drop a perpendicular $B'A'$ to AC. Make $A'A'' = AA'$, and erect a perpendicular $A''B''$ upon A''; repeat this construction until reaching a perpendicular CD that fails to meet AB. This must occur, for if the sum of the three angles equals π-a in triangle $\Delta AA'B'$, then it equals $\pi - 2a$ in triangle $\Delta AB'A''$, and is less than $\pi - 2a$ in $\Delta AA''B''$ (TP 20); if the construction could be repeated indefinitely, the sum would eventually become negative, thereby demonstrating the impossibility of the perpetual construction of such triangles.

The perpendicular CD itself might have the property that all other perpendiculars closer to A cut AB. At any rate, there is a perpendicular FG at the transition from the cutting-perpendiculars to the non-cutting-perpendiculars that does have this property. Draw any line FH making an acute angle with FG and lying on the same side of it as point A. From any point H of FH, drop a perpendicular HK to AC; its extension must intersect AB at some point; say, at B. In this way,

angle sum of one triangle to the angle sums of all triangles – an issue having scarcely anything to do with the present proposition's modest concerns. - SB the construction yields a triangle ΔAKB, into which the line FH enters and must, consequently, meet the hypotenuse AB at some point M. Since the angle $\measuredangle GFH$ is arbitrary and can be chosen as small as one wishes, FG is parallel to AB, and $AF = p$. (TP 16 and 18).

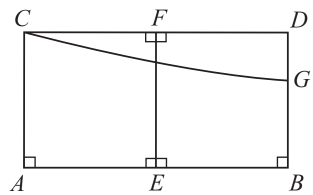
It is easy to see that with the decrease of p, the angle α increases, approaching the value $\pi/2$ for $p = 0$; with the increase of p, the angle α decreases, approaching ever nearer to zero for $p = \infty$.

Since we are completely free to choose the angle that shall be assigned to the symbol Π(p) when p is a negative number, we shall adopt the convention that $\Pi(p) + \Pi(-p) = \pi$, an equation which gives the symbol a meaning for all values of p, positive as well as negative, and for $p = 0$.

Proposition 24

The farther parallel lines are extended in the direction of their parallelism, the more they approach one another.

Upon the line AB, erect two perpendiculars $AC = BD$, and join their endpoints C and D with a straight line. The resulting quadrilateral $CABD$ will have right angles at A and B, but acute angles at C and D (TP 22[2]). These acute angles are equal to one another; one can easily convince oneself of this by imagining laying the quadrilateral upon itself in such a way that the line BD lies upon AC, and AC lies upon BD. Bisect AB. From the midpoint E, erect the line EF perpendicular to AB; it will be perpendicular to CD as well, since the quadrilaterals $CAEF$ and $FEBD$ coincide when one is laid on top of the other in such a way that FE remains in the same place.



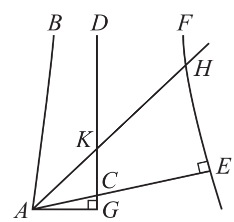
Consequently, the line CD cannot be parallel with AB. On the contrary, the line from point C that *is* parallel to AB, which we shall call CG, must incline toward AB (TP 16), cutting from the perpendicular BD a part $BG < CA$. Since C is an arbitrary point of the line CG, it follows that the farther CG is extended, the nearer it approaches AB.

Proposition 25

Two straight lines parallel to a third line are parallel to one another.

We shall first assume that the three lines AB, CD, and EF lie in one plane.

If one of the outer lines, say AB, and the middle line, CD, are parallel to the remaining outer line, EF, then AB and CD will be parallel to one another. To prove this, drop a perpendicular AE from any point A of AB to EF; it will intersect CD at some point C (TP 5), and the angle $\measuredangle DCE$ will be acute (TP 22). Drop a perpendicular AG from A to CD; its foot G must fall on the side of C that forms an acute angle with AC (TP 9). Every line AH drawn from A into angle $\measuredangle BAC$ must cut EF, the parallel to AB, at some point H, regardless of how small the angle $\measuredangle BAH$ is taken. Consequently, the line CD, which enters the triangle ΔAEH, must cut the line AH at



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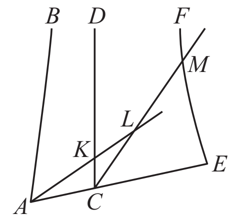
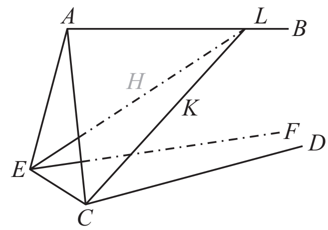
[2] This refers to Lobachevski's declaration at the end TP 22 that he would work in imaginary geometry from that point forward. Had he carried out this construction earlier, he would not have been able to deduce that the angles at C and D were acute; in neutral geometry, they could be either acute or right. - SB

some point K, since it is impossible for it to leave the triangle through EH. When AH is drawn from A into the angle $\measuredangle CAG$, it must cut the extension of CD between C and G in the triangle ΔCAG. From the preceding argument, it follows that AB and CD are parallel (TP 16 and 18).

If, on the other hand, the two outer lines, AB and EF, are both parallel to the middle line CD, then every line AK drawn from A into the angle $\measuredangle$BAE will cut the line CD at some point K, regardless of how small the angle $\measuredangle BAK$ is taken. Draw a line joining C to an arbitrary point L on the extension of AK. The line CL must cut EF at some point M, producing the triangle ΔMCE. Since the extension of the line AL into the triangle ΔMCE can cut neither AC nor CM a second time, it must cut EF at some point H. Hence, AB and EF are mutually parallel.



Suppose now that two parallels, AB and CD, lie in two planes whose line of intersection is EF. From an arbitrarily chosen point E of EF, drop a perpendicular EA to one of the parallels, say to AB. From the foot of this perpendicular, A, drop a new perpendicular, AC, to CD, the other parallel. Draw the line EC joining E and C, the endpoints of this perpendicular construction. The angle $\measuredangle BAC$ must be acute (TP 22), so the foot G of a perpendicular CG dropped from C to AB will fall on that side of AC in which the lines AB and CD are parallel. The line EC, together with any line EH that enters angle $\measuredangle AEF$ (regardless of how slightly EH deviates from EF), determines a plane. This plane must cut the plane of the parallels AB and CD along some line CK. This line cuts AB somewhere – namely, at the very point L common to all three planes, through which the line EH necessarily passes as well. Thus, EF is parallel to AB. We can establish the parallelism of EF and CD similarly.[3]

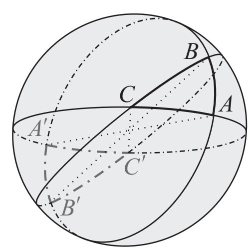


Therefore, a line EF is parallel to one of a pair of parallels, AB and CD, if and only if EF is the intersection of two planes, each containing one of the parallels, AB and CD. Thus, two lines are parallel to one another if they are parallel to a third line, even if the lines do not all lie in one plane. This last sentence could also be expressed thus: the lines in which three planes intersect must all be parallel to one another if the parallelism of two of the lines is established.

Proposition 26

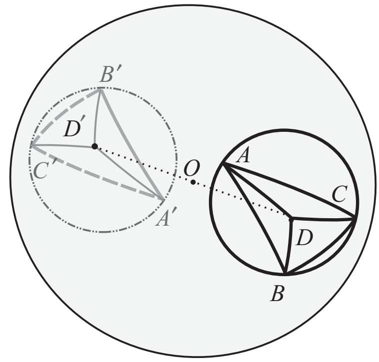
Antipodal spherical triangles have equal areas.

By antipodal triangles, I mean those triangles that are formed on opposite sides of a sphere when three planes through its center intersect it. It follows that antipodal triangles have their sides and angles in reverse order.



By antipodal triangles, I mean those triangles that are formed on opposite sides of a sphere when three planes through its center intersect it. It follows that antipodal triangles have their sides and angles in reverse order.

[3] For the sake of clarity, I have taken the liberty of changing the names of some of the points in this passage: the points that I have called H, K, and L are *all* called H in Lobachevski's original. $-SB$



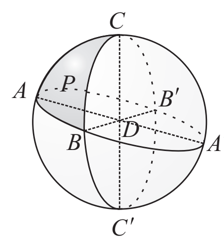
The corresponding sides of antipodal triangles ΔABC and $\Delta A'B'C'$ are equal: $AB = A'B'$, $BC = B'C'$, $CA = C'A'$. The corresponding angles are also equal: those at A, B, and C equal those at A', B', and C' respectively.

Consider the plane passing through the points A, B, and C. Drop a perpendicular to it from the center of the sphere, and extend this perpendicular in both directions; it will pierce the antipodal triangles in antipodal points, D and D'. The distances from D to the points A, B, and C, as measured along great circles of the sphere, must be equal, not only to one another (TP 12), but also to the distances $D'A'$, $D'B'$, and $D'C'$ on the antipodal triangle (TP 6). From this, it follows that the three isosceles triangles that surround D and comprise the spherical triangle ΔABC are congruent to the corresponding isosceles triangles surrounding D' and comprising $\Delta A'B'C'$.

As a basis for determining when two figures on a surface are equal, I adopt the following postulate: two figures on a surface are equal in area when they can be formed by joining or detaching equal parts.

Proposition 27

A trihedral angle equals half the sum of its dihedral angles minus a right angle.

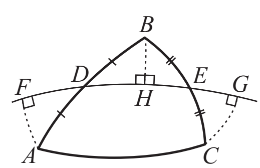


Let ΔABC be a spherical triangle, each of whose sides is less than half a great circle. Let A, B, and C denote the measures of its angles. Extending side AB to a great circle divides the sphere into two equal hemispheres. In the one containing ΔABC, extend the triangle's other two sides through C, denoting their second intersections with the great circle by A' and B'. In this way, the hemisphere is split into four triangles: ΔABC, $\Delta ACB'$, $\Delta B'CA'$, and $\Delta A'CB$, whose sizes we shall denote by P, X, Y, and Z respectively.

Clearly, $P + X = B$, and $P + Z = A$. Moreover, since the size Y of the spherical triangle $\Delta B'CA'$ equals that of its antipodal triangle $\Delta ABC'$ [TP 26], it follows that $P + Y = C$. Therefore, since $P + X + Y + Z = \pi$, we conclude that $P = \frac{1}{2}(A + B + C - \pi)$.

*

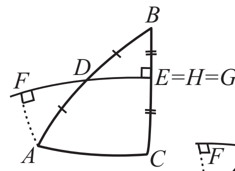
It is also possible to reach this conclusion by another method, based directly upon the postulate on equivalence of areas given above [in TP 26].

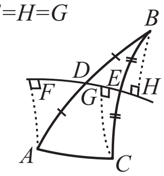


In the spherical triangle ΔABC, bisect the sides AB and BC, and draw the great circle through D and E, their midpoints. Drop perpendiculars AF, BH, and CG upon this circle from A, B, and C.

If H, the foot of the perpendicular dropped from B, falls between D and E, then the resulting right triangles ΔBDH and ΔAFD will be congruent, as will ΔBHE and ΔEGC (TP 6 & 15). From this, it follows that the area of triangle ΔABC equals that of the quadrilateral $AFGC$.

If H coincides with E (see figure at left), only two equal right triangles will be produced, ΔAFD and ΔBDE. Interchanging them establishes the equality of area of triangle ΔABC and quadrilateral $AFGC$.



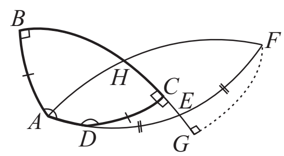


Finally, if H falls outside triangle ΔABC (see figure at right), the perpendicular CG must enter the triangle. We may then pass from triangle ΔABC to quadrilateral $AFGC$ by adding triangle $\Delta FAD \cong \Delta DBH$ and then taking away triangle $\Delta CGE \cong \Delta EBH$.

Since the diagonal arcs AG and CF of the spherical quadrilateral $AFGC$ are equal to one another (TP 15), the triangles ΔFAC and ΔACG are congruent to one another (TP 15), whence the angles $\measuredangle FAC$ and $\measuredangle ACG$ are equal to one another. Hence, in all the preceding cases, the sum of the three angles in the spherical triangle equals that of the two equal, non-right angles in the quadrilateral.

Therefore, given any spherical triangle whose angle sum is S, there is a quadrilateral with two right angles of the same area, each of whose other two angles equals $S/2$.

Let $ABCD$ be such a quadrilateral, whose equal sides AB and DC are perpendicular to BC, and whose angles at A and D each equal $S/2$. Extend its sides AD and BC until they meet at E; extend AD beyond E to F, so that $EF = ED$, and then drop a perpendicular FG upon the extension of BC. Bisect the arc BG, and join its midpoint H to A and F with great circle arcs.



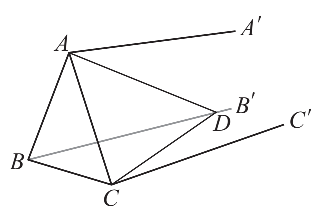
The congruence of the triangles ΔEFG and ΔDCE (TP 15) implies that $FG = DC = AB$. The right triangles ΔABH and ΔHGF are also congruent, since their corresponding arms are equal. From this it follows that the arcs AH and AF belong to the same great circle. Thus, the arc AHF is half a great circle, as is the arc $ADEF$. Since $\measuredangle HFE = \measuredangle HAD = S/2 - \measuredangle BAH = S/2 - \measuredangle HFG = S/2 - \measuredangle HFE - \measuredangle EFG = S/2 - \measuredangle HAD - \pi + S/2$, we conclude that $\measuredangle HFE = \frac{1}{2}(S - \pi)$. Equivalently, we have shown that $\frac{1}{2}(S - \pi)$ is the size of the spherical lune $AHFDA$, which in turn equals the size of the quadrilateral ABCD; this last equality is easy to see, since we may pass from one to the other by first adding the triangles ΔEFG and ΔBAH, and then removing triangles that are congruent to them: ΔDCE and ΔHFG.

Therefore, $\frac{1}{2}(S - \pi)$ is the size both of the quadrilateral $ABCD$, and of the spherical triangle, whose angle sum is S.

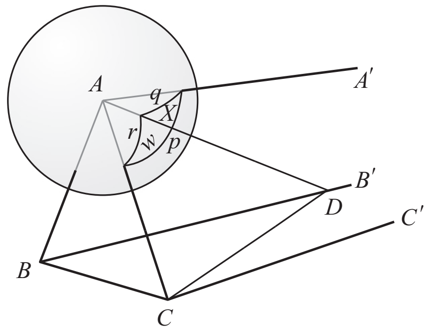
Proposition 28

If three planes intersect one another along parallel lines, the sum of the three resulting dihedral angles is equal to two right angles.

Suppose that three planes intersect one another along three parallel lines, AA', BB', and CC' (TP 25). Let X, Y, and Z denote the dihedral angles they form at AA', BB', and CC', respectively. Take random points A, B, and C, one from each line, and construct the plane passing through them. Construct a second plane containing the line AC and some point D of BB'. Let the dihedral angle that this plane makes with the plane containing the parallel lines AA' and CC' be denoted by w.



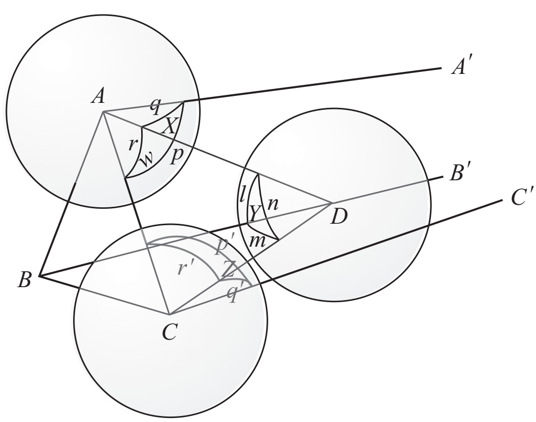
Draw a sphere centered at A; the points in which the lines AC, AD, and AA' intersect it determine a spherical triangle, whose size we shall denote by α, and whose sides we shall denote p, q, and r.



If q and r are those sides whose opposite angles have measures w and X respectively, then the angle opposite side p must have measure $\pi + 2\alpha g w - X$. (TP 27)

Similarly, the intersections of CA, CD, and CC' with a sphere centered at C determine a spherical triangle of size β, whose sides are denoted by p', q', and r', and whose angles are: w opposite q', Z opposite r', and thus, $\pi + 2\beta - w - Z$ opposite p'.

Finally, the intersections of DA, DB, and DC with a sphere centered at D determine a spherical triangle, whose sides, l, m, and n, lie opposite its angles, $w + Z - 2\beta$, $w + X - 2\alpha$, and Y, respectively. Its size, consequently, must be $\delta = \frac{1}{2}(X + Y + Z - \pi) - (\alpha + \beta - w)$.

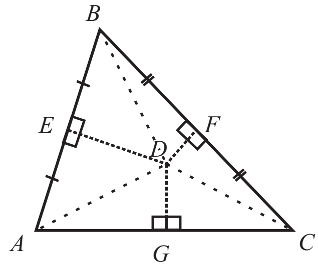


If w decreases toward zero, then α and β will vanish as well, so that $(\alpha + \beta g w)$can be made less than any given number. Since sides l and m of triangle δ will also vanish (TP 21), we can, by taking w sufficiently small, place as many copies of δ as we wish, end to end, along the great circle containing m, without completely covering the hemisphere with triangles in the process. Hence, δ vanishes together with w. From this, we conclude that we must have $X + Y + Z = \pi$.

Proposition 29

In a rectilinear triangle, the three perpendicular bisectors of the sides meet either in a single point, or not at all.

Suppose that two of triangle ABC's perpendicular bisectors, say, those erected at the midpoints E & F of AB and BC respectively, intersect at some point, D, which lies within the triangle. Draw the lines DA, DB, and DC, and observe that the congruence of the triangles ADE and BDE (TP 10) implies that $AD = BD$. For similar reasons, we have $BD = CD$, whence it follows that triangle ADC is isosceles. Consequently, the perpendicular dropped from D to AC must fall upon AC's midpoint, G.



This reasoning remains valid when D, the point of intersection of the two perpendiculars ED and FD, lies outside the triangle, or when it lies upon side AC.

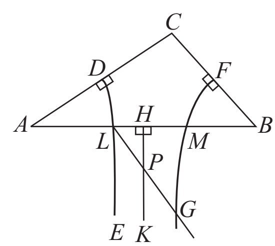
Thus, if two of the three perpendiculars fail to intersect one another, then neither of them will intersect the third.

Proposition 30

In a rectilinear triangle, if two of the perpendicular bisectors of the sides are parallel, then all three of them will be parallel to one another.

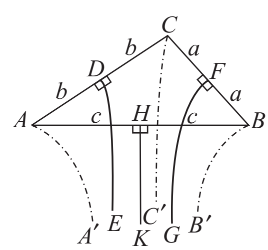
In triangle ΔABC, erect perpendiculars DE, FG, and HK from D, F, and H, the midpoints of the sides. (See the figure.)

We first consider the case in which DE and FG are parallel, and the third perpendicular, HK, lies between them. Let L and M be the points in which the parallels DE and FG cut the line AB. Draw an arbitrary line entering angle ∡BLE through L. Regardless of how small an angle it makes with LE, this line must cut FG (TP 16); let G be the point of intersection. The perpendicular HK enters triangle ΔLGM, but because it cannot intersect MG (TP 29), it must exit through LG at some point P. From this it follows that HK must be parallel to DE (TP 16 & 18) and FG (TP 18 & 25).



In the case just considered, if we let the sides $BC = 2a$, $AC = 2b$, $AB = 2c$, and denote the angles opposite them by A, B, C, we can easily show that

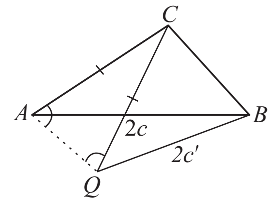
$$A = \Pi(b) - \Pi(c) \quad B = \Pi(a) - \Pi(c) \quad C = \Pi(a) + \Pi(b)$$



by drawing lines AA', BB', CC', from points A, B, C, parallel to HK– and therefore parallel to DE and FG as well (TP 23 & 25).

Next, consider the case in which HK and FG are parallel. Since DE cannot cut the other two perpendiculars (TP 29), it either is parallel to them, or intersects AA'. To assume this latter possibility is to assume that $C > \Pi(a) + \Pi(b)$.

If this is the case, we can decrease the magnitude of this angle to $\Pi(a) + \Pi(b)$ by rotating line AC to a new position CQ (see the figure). The angle at B is thereby increased. That is, in terms of the formula proved above,



$$\Pi(a) - \Pi(c') > \Pi(a) - \Pi(c),$$

where 2c′ is the length of BQ. From this it follows that $c' > c$ (TP 23).

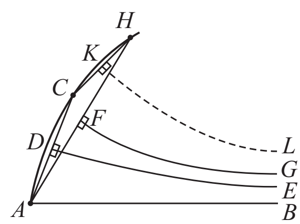
On the other hand, since the angles at A and Q in triangle ΔACQ are equal, the angle at Q in triangle ΔABQ must be greater than the angle at A in the same triangle. Consequently, $AB > BQ$ (TP 9); that is, $c > c'$.

Proposition 31

We define a horocycle to be a plane curve with the property that the perpendicular bisectors of its chords are all parallel to one another.

In accordance with this definition, we may imagine generating a horocycle as follows: from a point A on a given line AB, draw various chords AC of length $2a$, where $\Pi(a) = \measuredangle CAB$. The endpoints of such chords will lie on the horocycle, whose points we may thus determine one by one.

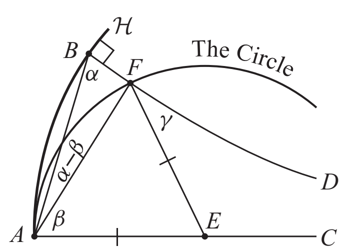
The perpendicular bisector DE of a chord AC will be parallel to the line AB, which we shall call the *axis* of the horocycle. Since the perpendicular bisector FG of any chord AH will be parallel to AB, the perpendicular bisector KL of any chord CH will be parallel to AB as well, regardless of the points C and H on the horocycle between which the chord is drawn (TP 30). For that reason, we shall not distinguish AB alone, but shall instead call *all* such perpendiculars *axes of the horocycle*.



Proposition 32

A circle of increasing radius merges into a horocycle.

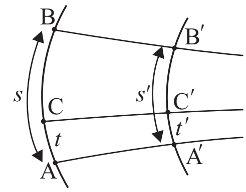
Let AB be a chord of the horocycle. From its endpoints, A and B, draw the two axes AC and BD; these will necessarily make equal angles, $BAC = ABD = \alpha$, with the chord AB (TP 31). From either axis, say AC, select an arbitrary point E to be the center of a circle. Draw an arc of this circle extending from A to F, the point at which it intersects BD. The circle's radius EF will make angle $AFE = \beta$ on one side of the chord of the circle, AF; on the other side, it will make angle $EFD = \gamma$ with the axis BD.



Now, angle γ will decrease if we move F toward B along axis BF while holding the center E fixed (TP 21). Moreover, γ will decrease to zero if we move the center E down axis AC while holding F fixed (TP 21, 22).

As γ vanishes, so does $\alpha - \beta$, the angle between AB and AF. Consequently, the distance from point B of the horocycle to point F of the circle vanishes as well. For this reason, one may also call the horocycle a circle of infinite radius.

Proposition 33



Let $AA' = BB' = x$ be segments of two lines that are parallel in the direction from A to A'. If these parallels are axes of two horocycles, whose arcs $AB = s$ and $A'B' = s'$ they delimit, then the equation $s' = se^{-x}$ holds, where e is some number independent of the arcs s, s', and the line segment x, the distance between the arcs s' and s.

Suppose that n and m are whole numbers such that $s : s' = n : m$. Draw a third axis CC' between AA' and BB'. Let $t = AC$ and $t' = A'C'$ be the lengths of the arcs that it cuts from AB and $A'B'$ respectively. Assuming that $t : s = p : q$ for some whole numbers p and q, we have

$$s = (n/m)s' \text{ and } t = (p/q)s.$$

If we divide s into nq equal parts by axes, any one such part will fit exactly mq times into s' and exactly np times into t. At the same time, the axes dividing s into nq equal parts divide s' into nq equal parts as well. From this it follows that

$$t'/t = s'/s.$$

Consequently, as soon as the distance x between the horocycles is given, the ratio of t to t' is determined; this ratio remains the same, no matter where we draw CC' between AA' and BB'.

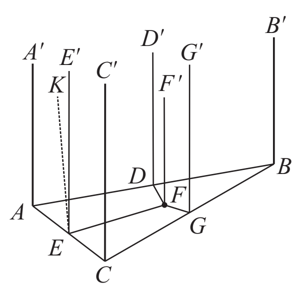
From this, it follows that if we write $s = es'$ when $x = 1$, then $s' = se^{-x}$ for every value of x.

We may choose the unit of length with which we measure x as we see fit. In fact, because e is an undetermined number subject only to the condition $e > 1$, we may, for the sake of computational ease, choose the unit of length so that the number e will be the base of the natural logarithm.

In addition, since $s' = 0$ when $x = \infty$, we observe that, in the direction of parallelism, the distance between two parallels not only decreases (TP 24), but ultimately vanishes. Thus, parallel lines have the character of asymptotes.

Proposition 34

We define a horosphere to be the surface generated by revolving a horocycle about one of its axes, which, together with all the remaining axes of the horocycle, will be an axis of the horosphere.



Any chord joining two points of the horosphere will be equally inclined to the axes that pass through its endpoints, regardless of which two points are taken.

Let A, B, and C be three points on the horosphere, where AA' is the axis of rotation and BB' and CC' are any other axes. The chords AB and AC will be equally inclined toward the axes passing through their endpoints; that is, $\measuredangle A'AB = \measuredangle B'BA$ and $\measuredangle A'AC = \measuredangle C'CA$ (TP 31). The axes BB' and CC' drawn through the endpoints of the third chord BC are, like those of the other chords, parallel and coplanar with one another (TP 25).

The perpendicular DD' erected from the midpoint D of chord AB in the plane of the two parallels AA', BB' must be parallel to the three axes AA', BB', CC' (TP 31, 25). Similarly, the perpendicular bisector EE' of chord AC in the plane of parallels AA', CC' will be parallel to the three axes AA', BB', CC', as well as the perpendicular bisector DD'.

Denote the angle between the plane of the parallels AA', BB' and the plane in which triangle ΔABC lies by $\Pi(a)$, where a may be positive, negative, or zero. If a is positive, draw $DF = a$ in the plane of triangle ΔABC, into the triangle, perpendicular to chord AB at its midpoint D; if a is negative, draw $DF = a$ outside the triangle on the other side of chord AB; if $a = 0$, let point F coincide with D.

All cases give rise to two congruent right triangles, ΔAFD and ΔDFB, whence $FA = FB$. From F, erect FF' perpendicular to the plane of triangle ΔABC.

Because $\measuredangle D'DF = \Pi(a)$ and $DF = a$, FF' must be parallel to DD'; the plane containing these lines is perpendicular to the plane of triangle ΔABC.

Moreover, FF' is parallel to EE'; the plane containing them is also perpendicular to the plane of triangle ΔABC.

Next, draw EK perpendicular to EF in the plane containing the parallels EE' and FF'. It will be perpendicular to the plane of triangle ΔABC (TP 13), and hence to the line AE lying in this plane. Consequently, AE, being perpendicular to EK and EE', must be perpendicular to FE as well (TP 11). The triangles ΔAEF and ΔCEF are congruent, since they each have a right angle, and their corresponding sides about their right angles are equal. Therefore, $FA = FC = FB$.

In isosceles triangle ΔBFC, a perpendicular dropped from vertex F to the base BC will fall upon its midpoint G.

The plane containing FG and FF' will be perpendicular to the plane of triangle ΔABC, and will cut the plane containing the parallels BB', CC' along a line that is parallel to them, GG'. (TP 25).

Since CG is perpendicular to FG, and thus to GG' as well [TP 13], it follows that $\measuredangle C'CG = \measuredangle B'BG$ (TP 23).

From this, it follows that any axis of the horosphere may be considered its axis of rotation.

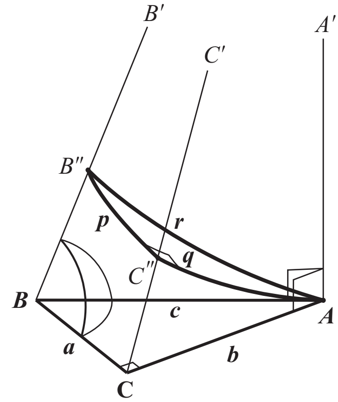
We shall refer to any plane containing an axis of a horosphere as a *principal plane*. The intersection of the principal plane with the horosphere is a horocycle; for any other cutting plane, the intersection is a circle.

Any three principal planes that mutually cut one another will meet at angles whose sum is π (TP 28). We shall consider these the angles of a *horospherical triangle*, whose sides are the arcs of the horocycles in which the three principal planes intersect the horosphere. Accordingly, *the relations that hold among the sides and angles of horospherical triangles are the very same that hold for rectilinear triangles in the ordinary geometry*.

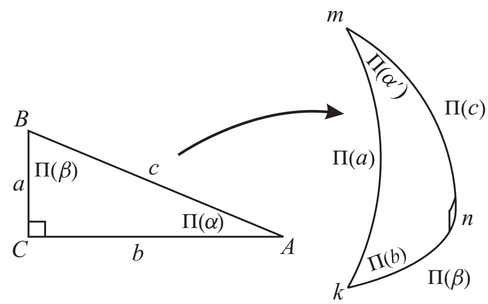
Proposition 35

In what follows, we shall use an accented letter, *e.g.* x', to denote the length of a line segment when its relation to the segment which is denoted by the same, but unaccented, letter is described by the equation $\Pi(x) + \Pi(x') = \pi/2$.

Let ΔABC be a rectilinear right triangle, where the hypotenuse is $AB = c$, the other sides are $AC = b$, $BC = a$, and the angles opposite them are $\measuredangle BAC = \Pi(\alpha)$, $\measuredangle ABC = \Pi(\beta)$. At point A, erect the line AA', perpendicular to the plane of triangle ΔABC; from B and C, draw BB' and CC' parallel to AA'.



The planes in which these parallels lie meet one another at the following dihedral angles: $\Pi(\alpha)$ at AA', a right angle at CC' (TP 11 & 13), and therefore, $\Pi(\alpha')$at BB' (TP 28).



The points at which the lines BB', BA, BC intersect a sphere centered at B determine a spherical triangle Δmnk, whose sides are $mn = \Pi(c)$, $kn = \Pi(\beta)$, $mk = \Pi(a)$, and whose opposite angles are, respectively, $\Pi(b)$, $\Pi(\alpha')$, $\pi/2$.

Thus, the existence of a rectilinear triangle with sides a, b, c and opposite angles $\Pi(\alpha)$, $\Pi(\beta)$, $\pi/2$ implies the existence of a spherical triangle with sides Π(c), $\Pi(\beta)$, Π(a) and opposite angles Π(b), $\Pi(\alpha')$, $\pi/2$.

Conversely, the existence of such a spherical triangle implies the existence of such a rectilinear triangle.

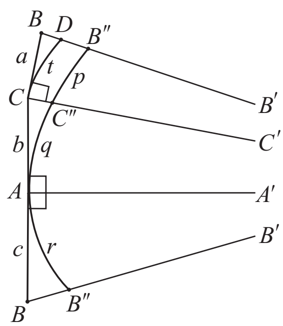
Indeed, the existence of such a spherical triangle also implies the existence of a second rectilinear triangle, with sides a, α', β and opposite angles Π(b′), Π(c), $\pi/2$. Hence, we may pass from a, b, c , α , β to b, a, c, β, α, and to a, α ′, β, b′, c, as well.

If the horosphere through A with axis AA' cuts BB' and CC' at B'' and C'', its intersections with the planes formed by the parallels produce a horospherical triangle with sides $B''C'' = p$, $C''A = q$, $B''A = r$ and opposite angles $\Pi(\alpha')$, $\Pi(\alpha)$, $\pi/2$.

Consequently (TP 34),

$$p = r \sin \Pi(\alpha) \text{ and } q = r \cos \Pi(\alpha).$$

Along BB', break the connection of the three principal planes, turning them out from one another so that they lie in a single plane. In this plane, the arcs p, q, r unite into an arc of a single horocycle, which passes through A and has axis AA'.



Thus, the following lie on one side of AA': arcs p and q; side b of the rectilinear triangle, which is perpendicular to AA' at A; axis CC', which emanates from the endpoint of b, then passes through C″, the join of p and q, and is parallel to AA'; and the axis BB', which emanates from the endpoint of a, then passes through B″, the endpoint of arc p, and is parallel to AA'. On the other side of AA' lie the following: side c, which is perpendicular to AA' at point A, and axis BB', which emanates from the endpoint of c, then passes through B″, the endpoint of arc r, and is parallel to AA'.

Moreover, we see (by TP 33) that

$$t = pe^{f(b)} = r \sin \Pi(\alpha) e^{f(b)}.$$

If we were to erect the perpendicular to triangle ΔABC 's plane at B, instead of A, then the lines c and r would remain the same, while the arcs q and t would change to t and q, the straight lines a and b would change to b and a, and the angle $\Pi(\alpha)$ would change to $\Pi(\beta)$. From this it follows that

$$q = r \sin \Pi(\beta) e^{f(a)}.$$

Thus, by substituting the value that we previously obtained for q, we find that

$$\cos \Pi(\alpha) = \sin \Pi(\beta) e^{f(a)}.$$

If we change α and β into b′ and c, then

$$\sin \Pi(b) = \sin \Pi(c) e^{f(a)}.$$

Multiplying by $e^{f(b)}$ yields

$$\sin \Pi(b) e^{f(b)} = \sin \Pi(c) e^{f(c)}.$$

Consequently, it follows that

$$\sin \Pi(a) e^{f(a)} = \sin \Pi(b) e^{f(b)}.$$

Because the lengths a and b are independent of one another and, moreover, $f(b) = 0$ and $\Pi(b) = \pi/2$ when $b = 0$, it follows that for every a,

$$e^{-f(a)} = \sin \Pi(a).$$

Therefore,

$$\sin \Pi(c) = \sin \Pi(a) \sin \Pi(b)$$
$$\sin \Pi(\beta) = \cos \Pi(\alpha) \sin \Pi(a).$$

Moreover, by transforming the letters, these equations become

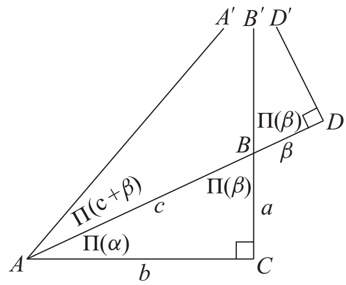
$$\sin \Pi(\alpha) = \cos \Pi(\beta) \sin \Pi(b)$$
$$\cos \Pi(b) = \cos \Pi(c) \cos \Pi(\alpha)$$
$$\cos \Pi(a) = \cos \Pi(c) \cos \Pi(\beta).$$

In the spherical right triangle, if the sides Π(c), Π(β), Π(a) and opposite angles Π(b), Π(α')are renamed a, b, c, A, B, respectively, then the preceding equations will assume forms that are known as established theorems of the ordinary spherical trigonometry of right triangles. Namely,

$$\sin(a) = \sin(c) \sin(A)$$
$$\sin(b) = \sin(c) \sin(B)$$
$$\cos(B) = \cos(b) \sin(A)$$
$$\cos(A) = \cos(a) \sin(B)$$
$$\cos(c) = \cos(a) \cos(b).$$

From these equations, we may derive those for all spherical triangles in general. Consequently, the formulae of spherical trigonometry do not depend upon whether or not the sum of the three angles in a rectilinear triangle is equal to two right angles.

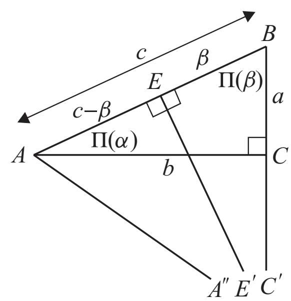
Proposition 36



We now return to the rectilinear right triangle ΔABC with sides a, b, c, and opposite angles $\Pi(\alpha), \Pi(\beta), \pi/2$. Extend this triangle's hypotenuse beyond B to a point D at which $BD = \beta$, and erect a perpendicular DD' from BD. By construction, DD' is parallel to BB', the extension of side a beyond B. Finally, draw AA' parallel to DD'; it will be parallel to CB' as well (TP 25).

From this, we have $\measuredangle A'AC = \Pi(b)$ and $\measuredangle A'AD = \Pi(c+\beta)$, from which it follows that

$$\Pi(b) = \Pi(\alpha) + \Pi(c+\beta).$$



Now let E be the point on ray BA for which $BE = \beta$. Erect the perpendicular EE' to AB, and draw AA'' parallel to it. Line BC', the extension of side a beyond C, will be a third parallel.

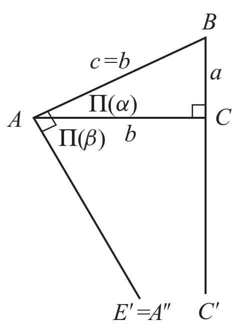
If $\beta < c$, as in the figure, we see that $\measuredangle CAA'' = \Pi(b)$ and $\measuredangle EAA'' = \Pi(c-\beta)$, from which it follows that

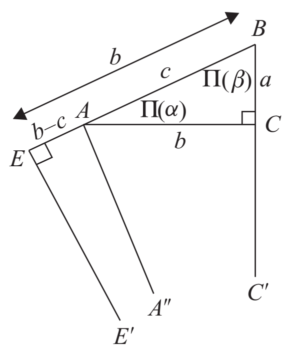
$$\Pi(c-\beta) = \Pi(\alpha) + \Pi(b).$$

In fact, this last equation remains valid even when $\beta = c$, or $\beta > c$.

If $\beta = c$ (see figure at left), the perpendicular AA' erected upon AB is parallel to BC, and hence to CC', from which it follows that $\Pi(\alpha) + \Pi(b) = \pi/2$. Moreover, $\Pi(c-\beta) = \pi/2$ (TP 23).

If $\beta > c$ (see figure at right), E falls beyond point A. In this case, we have $\measuredangle EAA'' = \Pi(c-\beta)$, from which it follows that





$$\Pi(\alpha) + \Pi(b) = \pi - \Pi(\beta - c) = \Pi(c-\beta) \quad \text{(TP 23).}$$

Combining the two equations yields

$$2\Pi(b) = \Pi(c-\beta) + \Pi(c+\beta)$$
$$2\Pi(\alpha) = \Pi(c-\beta) - \Pi(c+\beta),$$

from which follows

$$\frac{\cos\Pi(b)}{\cos\Pi(\alpha)} = \frac{\cos[\frac{1}{2}\Pi(c-\beta) + \frac{1}{2}\Pi(c+\beta)]}{\cos[\frac{1}{2}\Pi(c-\beta) - \frac{1}{2}\Pi(c+\beta)]}.$$

Using the substitution

$$\frac{\cos \Pi(b)}{\cos \Pi(\alpha)} = \cos \Pi(c) \text{ (from TP 35)}$$

yields

$$\tan^2\left(\frac{\Pi(c)}{2}\right) = \tan\left(\frac{\Pi(c-\beta)}{2}\right)\tan\left(\frac{\Pi(c+\beta)}{2}\right).$$

Because the angle $\Pi(\beta)$ at B may have any value between 0 and $\pi/2$, β itself can be any number between 0 and ∞. By considering the cases in which $\beta = c, 2c, 3c$, *etc*., we may deduce that for all positive values of r,[4]

$$\tan^r\left(\frac{\Pi(c)}{2}\right) = \tan\left(\frac{\Pi(rc)}{2}\right).$$

If we view r as the ratio of two values x and c, and assume that $\cot(\Pi(c)/2) = e^c$, we find that for all values of x, whether positive or negative,

$$\tan\left(\frac{\Pi(x)}{2}\right) = e^{-x},$$

where e is an indeterminate constant, which is larger than 1, since $\Pi(x) = 0$ when $x = \infty$.

Since the unit with which we measure lengths may be chosen at will, we may choose it so that e is the base of the natural logarithm.

Proposition 37

Of the five equations above (TP 35), the following two

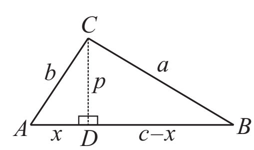
$$\sin \Pi(c) = \sin \Pi(a) \sin \Pi(b)$$
$$\sin \Pi(\alpha) = \cos \Pi(\beta) \sin \Pi(b)$$

suffice to generate the other three: we can obtain one of the others by applying the second equation to side a rather than side b; we then deduce another by combining the equations already established. There will be no ambiguities of algebraic sign, since all angles here are acute.

Similarly, we obtain the two equations:

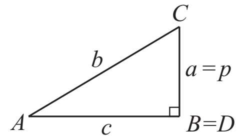
$$\textbf{(1)} \quad \tan \Pi(c) = \sin \Pi(\alpha) \tan \Pi(a)$$
$$\textbf{(2)} \quad \cos \Pi(a) = \cos \Pi(c) \cos \Pi(\beta).$$

We shall now consider an arbitrary rectilinear triangle with sides a, b, c and opposite angles A, B, C.



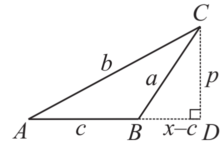
If A and B are acute angles, then the perpendicular p dropped from C will fall within the triangle and cut side c into two parts: x, on the side of A, and $c - x$, on the side of B. This produces two right triangles. Applying equation (1) to each yields

[4] Where I have r, Lobachevski uses the symbol n. Whatever one calls it, it stands for any positive *real* number. I have switched to r so as to conform with the convention of reserving n for natural numbers.



$$\tan \Pi(a) = \sin(B) \tan \Pi(p).$$
$$\tan \Pi(b) = \sin(A) \tan \Pi(p).$$

These equations hold even if one of the angles, say B, is right or obtuse. Thus, for any rectilinear triangle whatsoever, we have



(3) $\sin(A) \tan \Pi(a) = \sin(B) \tan \Pi(b).$

Applying equation (2) to a triangle with acute angles at A and B yields

$$\cos \Pi(x) = \cos(A) \cos \Pi(b)$$
$$\cos \Pi(c - x) = \cos(B) \cos \Pi(a).$$

These equations hold even when one of the angles A or B is right or obtuse.

For instance, when $B = \pi/2$, we have $x = c$; in this case, the first equation reduces to equation (2) and the second is trivially true. When $B > \pi/2$, applying equation (2) still yields the first equation; in place of the second, it yields $\cos \Pi(x - c) = \cos(\pi - B) \cos \Pi(a)$, which, however, is identical to the second, since $\cos \Pi(x - c) = - \cos \Pi(c - x)$ (TP 23), and $\cos(\pi - B) = - \cos(B)$. Finally, if A is right or obtuse, we must use $c - x$ and x , instead of x and $c - x$, so that the two equations will hold in this case also.

To eliminate x from the two equations above, we observe that

$$\begin{aligned}
\cos \Pi(c - x) &= \frac{1 - \left[\tan\left(\frac{\Pi(c-x)}{2}\right)\right]^2}{1 + \left[\tan\left(\frac{\Pi(c-x)}{2}\right)\right]^2} \\
&= \frac{1 - e^{2x-2c}}{1 + e^{2x-2c}} \\
&= \frac{1 - \left[\tan\left(\frac{\Pi(c)}{2}\right)\right]^2 \left[\cot\left(\frac{\Pi(x)}{2}\right)\right]^2}{1 + \left[\tan\left(\frac{\Pi(c)}{2}\right)\right]^2 \left[\cot\left(\frac{\Pi(x)}{2}\right)\right]^2} \\
&= \frac{\cos \Pi(c) - \cos \Pi(x)}{1 - \cos \Pi(c) \cos(x)}
\end{aligned}$$

If we substitute the expressions for $\cos \Pi(x)$ and $\cos \Pi(c - x)$ into this, it becomes from which it follows that

$$\cos \Pi(a) \cos(B) = \frac{\cos \Pi(c) - \cos \Pi(A) \cos \Pi(b)}{1 - \cos \Pi(A) \cos \Pi(b) \cos \Pi(c)},^{5}$$

[5] In Lobachevski's original, the positions of this equation and the preceding one are reversed: presumably this was a printer's error. The fact that Halsted perpetuated it in his 1891 translation leads me to suspect that Halsted, realizing that one could reach the conclusions of TP 37 by simpler arguments than Lobachevski's own, did not bother to look very closely at the details as they stand.

and finally,

$$\textbf{(4)}\quad [\sin \Pi(c)]^2 = [1 - \cos(B)\cos\Pi(c)\cos\Pi(a)]\,[1 - \cos(A)\cos\Pi(b)\cos\pi(c)]\,.$$

Similarly, we also have

$$[\sin \Pi(a)]^2 = [1 - \cos(C)\cos\Pi(a)\cos\Pi(b)]\,[1 - \cos(B)\cos\Pi(c)\cos\Pi(a)]$$
$$[\sin \Pi(b)]^2 = [1 - \cos(A)\cos\Pi(b)\cos\Pi(c)]\,[1 - \cos(C)\cos\Pi(a)\cos\Pi(b)]\,.$$

From these three equations, we find that

$$\frac{[\sin\Pi(b)]^2\,[\sin\Pi(c)]^2}{[\sin\Pi(a)]^2} = [1 - \cos(A)\cos\Pi(b)\cos\Pi(c)]^2\,.$$

From this it follows, without ambiguity of sign, that

$$(5)\quad \cos(A)\cos\Pi(b)\cos\Pi(c) + \frac{\sin\Pi(b)\sin\Pi(c)}{\sin\Pi(a)} = 1.$$

The following expression for $\sin\Pi(c)$ follows from an alternate form of (3):

$$\sin\Pi(c) = \frac{\sin(A)}{\sin(C)}\tan\Pi(a)\cos\Pi(c).$$

If we substitute this expression into equation (5), we obtain

$$\cos\Pi(c) = \frac{\cos\Pi(a)\sin(C)}{\sin(A)\sin\Pi(b) + \cos(A)\sin(C)\cos\Pi(a)\cos\Pi(b)}.$$

If we substitute this expression for $\cos\Pi(c)$ into equation (4), we obtain

$$(6)\quad \cot(A)\sin(C)\sin\Pi(b) + \cos(C) = \frac{\cos\Pi(b)}{\cos\Pi(a)}.$$

By eliminating $\sin\Pi(b)$ with the help of equation (3), we find that

$$\frac{\cos\Pi(a)}{\cos\Pi(b)}\cos(C) = 1 - \frac{\cos(A)}{\sin(B)}\sin(C)\sin\Pi(a).$$

On the other hand, permuting the letters in equation (6) yields

$$\frac{\cos\Pi(a)}{\cos\Pi(b)} = \cot(B)\sin(C)\sin\Pi(a) + \cos(C).$$

By combining the last two equations, we obtain

$$(7)\quad \cos(A) + \cos(B)\cos(C) = \frac{\sin(B)\sin(C)}{\sin\Pi(a)}.$$

Thus, the four equations that describe how the sides a,b,c and angles A, B, C are interrelated in rectilinear triangles are [equations (3), (5), (6), (7)]:

$$(8)\quad \begin{cases} \sin(A)\tan\Pi(a) = \sin(B)\tan\Pi(b) \\ \cos(A)\cos\Pi(b)\cos\Pi(c) + \dfrac{\sin\Pi(b)\sin\Pi(c)}{\sin\Pi(a)} = 1 \\ \cot(A)\sin(C)\sin\Pi(b) + \cos(C) = \dfrac{\cos\Pi(b)}{\cos\Pi(a)} \\ \cos(A) + \cos(B)\cos(C) = \dfrac{\sin(B)\sin(C)}{\sin\Pi(a)} \end{cases}$$

When the sides a, b, c of the triangle are very small, we may content ourselves with the following approximations (TP 36):

$$\begin{aligned} \cot\Pi(a) &= a, \\ \sin\Pi(a) &= 1 - \frac{1}{2}a^2, \\ \cos\Pi(a) &= a, \end{aligned}$$

where the same approximations hold for sides b and c also.

In the scholarly journal of the University of Kazan, I have published several investigations into the measurements of curves, plane figures, surfaces, and solids, as well as the application of imaginary geometry to analysis.

In and of themselves, the equations (8) already constitute sufficient grounds for believing that the imaginary geometry might be possible. As a result, we have no means other than astronomical observations with which to judge the accuracy that follows from calculations in the ordinary geometry. Its accuracy is very far-reaching, as I have demonstrated in one of my investigations; for example, in all angles who sides we are capable of measuring, the sum of the three angles does not differ from π by so much as a hundredth of a second.

Finally, it is worth observing that the four equations (8) of plane geometry become valid formulae of spherical geometry if we substitute $a\sqrt{-1}$, $b\sqrt{-1}$, $c\sqrt{-1}$ for the sides a, b, c ; these substitutions will change

$$\begin{aligned} &\sin\Pi(a) \quad \text{to} \quad \frac{1}{\cos a}, \\ &\cos\Pi(a) \quad \text{to} \quad \sqrt{-1}\tan a. \\ &tan\Pi(a) \quad \text{to} \quad \frac{1}{\sqrt{-1}\sin a}, \end{aligned}$$

and similarly for sides b and c. Hence, these substitutions change equations (8) into the following:

$$\begin{aligned}
&\sin A \sin b = \sin B \sin a \\
&\cos a = \cos b \cos c + \sin b \sin c \cos A \\
&\cot A \sin C + \cos C \cos b = \sin b \cot a \\
&\cos A = \cos a \sin B \sin C - \cos B \cos C.
\end{aligned}$$

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About the Author

Seth Braver (B.A. San Francisco State University, M.A. University of California – Santa Cruz, Ph.D. University of Montana) was born in Atlanta, Georgia. He taught at the University of Montana (where he won several teaching awards) and St. John's College in Santa Fe (where he led classes in literature, philosophy, and ancient Greek, as well as mathematics) before joining the mathematics department of South Puget Sound Community College in Olympia, Washington, where he currently teaches.

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