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(Mostly Real) Quantifier Elimination

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Overview

Quantifier Elimination and Decision

Example (Tarski Algebra = real numbers with arithmetic and ordering)

$$\mathbb{R} \models \underbrace{\forall x \exists y (x^2 + xy + b > 0 \land x + ay^2 + b \leq 0)}_{\varphi} \longleftrightarrow \underbrace{a < 0 \land b > 0}_{\varphi'}$$

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Formally: Given 1st-order theory Θ, find **algorithm** with input φ and output φ' quantifier-free such that

$$\Theta \models \varphi \longleftrightarrow \varphi',$$

or prove that no such algorithm exists.

Important aspects: theoretical complexity, practical performance

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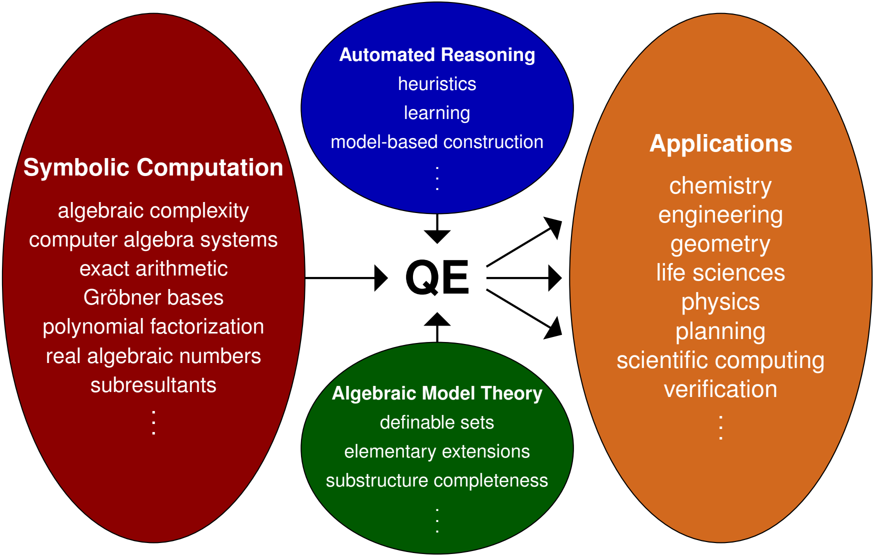
Important Special Cases

- all variables in φ are quantified $\rightsquigarrow$ **decision problem**
- only existential quantifiers $\rightsquigarrow$ **satisfiability problem**

Quantifier Elimination-relevant Research Topics



Definitions

Syntax and Semantics

Language (= Signature): $L = (0, 1, +, -, \cdot, <, \leq, \neq, >, \geq)$

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Important convention in algebraic model theory

There is always"$=$" which is formally not in the language.

Semantics of "$=$" is Leibniz's (second-order) definition of **equality**

$$x = y :\Longleftrightarrow \forall p\big(p(x) \longleftrightarrow p(y)\big)$$

in contrast to its first-order theory.

For convenience, define $L_{=} := L \cup \{=\}$.

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Remark

There is no multiplicative inverse or division in L.

We do not want to deal with partial functions.

Terms and Atomic Formulas

Terms

are w.l.o.g. polynomials with integer coefficients in a recursive representation

$$t \in (\dots(((\mathbb{Z}[x_n])[x_{n-1}])\dots)[x_2])[x_1]$$

Representation is unique and isomorphic to "distributive" $\mathbb{Z}[x_1, \dots, x_n]$.

Example

$f = x_1 + (x_2 + x_3), \quad f^2 = x_1^2 + (2x_2 + 2x_3)x_1 + (x_2^2 + 2x_3x_2 + x_3^2)$

We can efficiently **reorder** such polynomials, i.e., change the main variable.

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Atomic formulas (atoms) are of the form $f \; R \; 0$, where

- $R \in L_= = \{\leq, <, \neq, \geq, >, =\}$ as discussed
- f a recursive polynomial in some variables $x_1, \dots, x_n$ as above
- $L_=$ is closed under negation: For $R \in L_=$ there is $\bar{R} \in L_=$ such that

$$\mathbb{R} \models \neg(f \; R \; 0) \longleftrightarrow f \; \bar{R} \; 0.$$

Quantifier-free Formulas and First-order Formulas

First-order formulas are obtained from atomic formulas using operators

$$\text{true}, \quad \text{false}, \quad \wedge, \quad \vee, \quad \exists x, \quad \forall x, \qquad \text{where } x \text{ is a variable}$$

Further Boolean Operators

- $\longrightarrow$ and $\longleftrightarrow$ can be expressed without introducing quantifiers:

$$\alpha \longrightarrow \beta \quad \rightsquigarrow \quad \neg\alpha \vee \beta, \qquad \alpha \longleftrightarrow \beta \quad \rightsquigarrow \quad \alpha \longrightarrow \beta \wedge \beta \longrightarrow \alpha.$$

- Eliminate $\neg$ using de Morgan's law and closure property of L w.r.t. negation, e.g.:

$$\neg(x = 0 \wedge y > 0) \quad \rightsquigarrow \quad x \neq 0 \vee y \leq 0.$$

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Practical reson for restricting to $\wedge$ and $\vee$: **Simplification**

Quantifier-free formulas are first-order formulas not containing $\exists x$ or $\forall x$.

Convention: the only formulas containing true, false are true, false themselves.

Prenex Formulas

We assume w.l.o.g. that all first-order formulas are in a **prenex normal form**

$$Q_n x_n \ldots Q_1 x_1 (\psi)$$

with $Q_1, \ldots, Q_n \in \{\exists, \forall\}$ and ψ quantifier-free.

Fact

(i) For every first-order formula $\tilde{\varphi}$ there is an equivalent prenex formula

$$\varphi = Q_n x_n \ldots Q_1 (\psi).$$

(ii) φ can be efficiently computed from $\tilde{\varphi}$ such that the number of alternations in the sequence $Q_n, \ldots, Q_1$ is minimized.

Virtual Substitution

Eliminate from the Inside to the Outside

Given $\varphi = \mathrm{Q}_n x_n \dots \mathrm{Q}_1 x_1(\psi)$

- ψ is quantifier-free
- the variables of ψ are a subset of **quantified (bound) variables** $X = \{x_1, \dots, x_n\}$ and **(free) parameters** $U = \{u_1, \dots, u_m\}$, where

$$X \cap U = \emptyset.$$

We are going to eliminate $\mathrm{Q}_1 x_1$.

The rest is iteration with some optimizations to discuss later on.

We may assume that $\mathrm{Q}_1 = \exists$, because $\forall x_1 \varphi \longleftrightarrow \neg\exists x_1 \neg\varphi$.

Elimination of One Existential Quantifier

Given $\varphi = \exists x_1(\psi)$

- The variables in ψ are among x_1 and $V_1 := (X \setminus \{x_1\}) \cup U$.
- All variables from V_1 will play the same role now, say, $V_1 = \{v_1, \dots, v_k\}$.

If x_1 does not occur in ψ, then we are done.

Key Idea

- **Intuitively**, $\exists x$ is like a big disjunction over all real **numbers**.
- Could there be a finite E set of **terms** t such that

$$\mathbb{R} \models \exists x_1(\psi) \longleftrightarrow \bigvee_{t \in E} \psi[x_1/t] \quad ?$$

Modulo a couple of technical problems, there is essentially such a set.

Thought Experiment

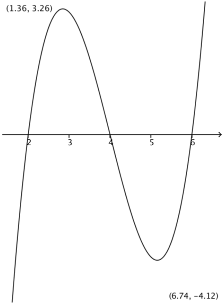
Given $\varphi = \exists x_1(\psi)$

Temporarily and only in our minds (not in any algorithm) fix

$$(v_1, \dots, v_k) := (a_1, \dots, a_l) \in \mathbb{R}^k$$

such that ψ becomes univariate in x_1.

Left hand sides of atomic formulas in ψ become univariate polynomials $f \in \mathbb{R}[x_1]$.



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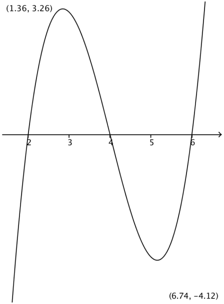

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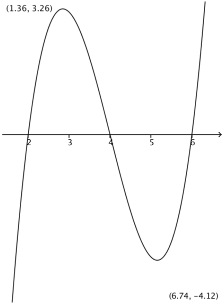

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- if $b_i \in \mathbb{R}$, then $f(b_i) = 0$
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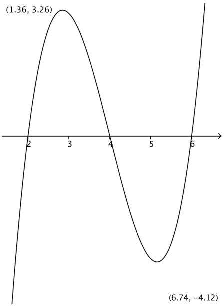

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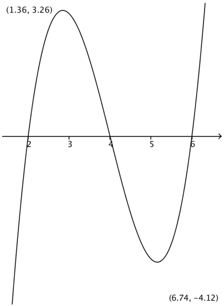

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- if $b_i \in \mathbb{R}$, then $f(b_i) = 0$
- Set of satisfying values for x_1 in ψ has the same form. $\wedge$ is cut and $\vee$ is intersection of satisfying sets.
- Idea: E = all b_2 or $b_2 - \varepsilon$ and ∞.

Elimination Sets

Given $\varphi = \exists x_1(\psi)$

Supersets of the zeros of the left hand side terms

$$f \in (\dots(((\mathbb{Z}[v_1])[v_2])\dots)[v_k])[x_1]$$

can be computed **symbolically** and **uniformly**.

Example

$f = a(v_1 \dots, v_k)x_1^2 + b(v_1, \dots, v_k)x_1 + c(v_1, \dots, v_k)$ yields candidate solutions

$$\underbrace{(-b \pm \sqrt{b^2 - 4ac})/2a}_{t} \text{ for } \underbrace{a \neq 0 \wedge b^2 - 4ac \geq 0}_{\gamma}, \qquad \underbrace{-c/b}_{t} \text{ for } \underbrace{a = 0 \wedge b \neq 0}_{\gamma}.$$

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An **elimination set** E for x_1 and ψ is a finite set of pairs (γ, t) such that

$$\mathbb{R} \models \exists x_1(\psi) \longleftrightarrow \bigvee_{(\gamma,t)\in E} \gamma \land \psi[x_1 /\!/ t].$$

Virtual Substitution

Given $\varphi = \exists x_1(\psi)$ and E such that $\mathbb{R} \models \exists x_1(\psi) \longleftrightarrow \bigvee_{(\gamma,t)\in E} \gamma \wedge \psi[x_1 /\!/ t]$.

Remaining Problem

t contain $/$, $\sqrt{\cdot}$, ∞, ε, $\dots$, which are not in our language L.

Solution: Virtual Substitution

$$[x /\!/ t] : \text{atomic formulas} \rightarrow \text{quantifier-free formulas}$$

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And beyond degree 2?

- Method generalizes to arbitrary degrees (in principle long known).
- first implementation will be available this year (PhD thesis by M. Košta).
- For higher degrees, t will be way more abstract.

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Important

In practice, good simplification of quantifier-free (intermediate) results is crucial!

Virtual Substitution by Example

Conventions: $f \in \mathbb{Z}[\mathbf{y}][x]$, f_i, g_i, $g_i^* \in \mathbb{Z}[\mathbf{y}]$

Quotients

$$(f_1 x + f_0 \leq 0)\left[x /\!/ \tfrac{g_1}{g_2}\right] \equiv f_1 \frac{g_1}{g_2} + f_0 \leq 0 \equiv f_1 g_1 g_2 + f_0 g_2^2 \leq 0$$

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Formal solutions of quadratic equations

$$\left(f = 0\right)\left[x /\!/ \tfrac{g_1 + g_2\sqrt{g_3}}{g_4}\right] \equiv \frac{g_1^* + g_2^*\sqrt{g_3}}{g_4^*} = 0$$

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Infinity

$$(f_2 x^2 + f_1 x + f_0 < 0)[x /\!/ \infty] \equiv f_2 < 0 \vee (f_2 = 0 \wedge f_1 < 0) \vee (f_2 = 0 \wedge f_1 = 0 \wedge f_0 < 0)$$

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Positive infinitesimals

$$(3x^2 + 6x - 3 > 0)[x /\!/ t - \varepsilon] \equiv 3t^2 + 6t - 3 > 0 \vee (3t^2 + 6t - 3 = 0 \wedge 6t + 6 \le 0)$$

Elimination of Several Existential Quantifiers by Block

Back to the bigger picture

$$\ldots\forall^*\exists^*\forall^*\exists^*\exists x_1(\psi) \quad \rightsquigarrow \quad \ldots\forall^*\exists^*\forall^*\exists^* \bigvee_{(\gamma,t)\in E} \gamma \wedge \psi[x_1 /\!/ t]$$

Disjunction $\bigvee$ is compatible with existential quantifiers $\exists^*$:

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Effect

- more local substitution of test points With the elimination of the next quantifiers
- even improves upper bound on asymptotic worst-case complexity

Complexity of Virtual Substitution

Upper bound on asymptotic worst-case complexity

doubly exponential in the input word length (and thus optimal)

More precisely

doubly exponential	in # quantifier alternations
singly exponential	in # quantifiers **thanks to elimination by block**
polynomial	in # parameters (= unquantified variables)
polynomial	in # atomic formulas

particularly good for

low degrees and many parameters

For comparision: Cylindrical Algberaic Decomposition (CAD)

[Collins 1973, Hong, Brown, . . .] doubly exponential in the number of **all** variables

For comparison: Asymptotically fast procedures

[Renegar, Basu–Pollack–Roy, Grigoriev, . . .] no practical relevance (so far)

Variants of Quantifier Elimination

Extended Quantifier Elimination

Generalize $\exists x \varphi \longleftrightarrow \bigvee_{(\gamma,t)\in E} \gamma \wedge \varphi[t/\!/x]$ to $\exists x\varphi \rightsquigarrow \begin{bmatrix} \vdots & \vdots \\ \gamma \wedge \varphi[t/\!/x] & x = t \\ \vdots & \vdots \end{bmatrix}$

Simple example revisited

$$\varphi \equiv \exists x(ax^2 + bx + c = 0) \rightsquigarrow \begin{bmatrix} a \neq 0 \wedge b^2 - 4ac \geq 0 & x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a} \\ a = 0 \wedge b \neq 0 & x = -\dfrac{c}{b} \\ a = 0 \wedge b = 0 \wedge c = 0 & x = \infty_1 \end{bmatrix}$$

Semantics (for fixed parameters)

Whenever some left hand side condition holds, then $\exists x\varphi$ holds and the corresponding right hand side term is **one** sample solution.

[M. Kosta, T.S., A. Dolzmann, J. Symb. Comput. 2016]

For fixed choices of parameters, standard values can be efficiently computed for all ∞_i and ε_i in a post-processing step.

Generic Quantifier Elimination

Collect negated equations from the γ in a global theory Θ:

$$E = \{\dots, (s \neq 0 \wedge \gamma', t), \dots\} \quad \rightsquigarrow \quad \Theta = \{\dots, s \neq 0, \dots\},\ E = \{\dots, (\gamma', t), \dots\}$$

Semantics

φ' is correct for all choices of parameters satisfying Θ:

$$\bigwedge \Theta \longrightarrow (\varphi' \longleftrightarrow \varphi).$$

Important observation

exception set has a lower dimension than the parameter space

Simple example revisited

$$\varphi \equiv \exists x(ax^2 + bx + c = 0) \quad \rightsquigarrow \quad \Theta = \{a \neq 0\}, \quad \varphi' \equiv b^2 - 4ac \geq 0$$

Software

Redlog and Reduce

Everything discussed here is available in our computer logic system Redlog:

`http://www.redlog.eu`

- interactive system, QE and decision for many domains, normal forms, simplification, construction and decomposition of large formulas, ...
- interfaces to Qepcad B, Gurobi, Mathematica, Z3, ...
- more than 300 citations of applications in the literature:

  geometry, verification, chemistry, life sciences, physics and engineering, scientific computation, geometry and planning, ...
- Redlog development since 1992 as part of the CAS Reduce [Hearn, 1968]
- Reduce/Redlog open-source (free-BSD) on Sourceforge since 12/2008

  `http://reduce-algebra.sourceforge.net`
- 48,318 downloads since 12/2008 (7,496 in 2014), 500+ SVN commits per year

Further Theories in Redlog

Integers (AAECC 2007, CASC 2007, CASC 2009)

- Presburger Arithmetic
- weak quantifier elimination for the full linear theory
- weak quantifier elimination also for higher degrees (special cases)

Mixed Real-Integer (Weispfenning at ISSAC 1999)

- experimental

Complex Numbers (using Comprehensive Gröbner Bases)

- language of rings only

Differential Algebras (CASC 2004)

- language of rings with unary differential operator
- computation in differentially closed field (A. Robinson, Blum)

Further Theories in Redlog

Padic Numbers (JSC 2000, ISSAC 1999, CASC 2001)

- linear formulas over p-adic fields for p prime
- optionally uniform in p
- used e.g. for solving parametric systems of congruences over the integers

Terms (CASC 2002)

- Malcev-type term algebras (with functions instead of relations)

Queues (C. Straßer at RWCA 2006)

- two-sided queues over the other theories (2-sorted)
- Implemented at present for queues of reals

Propositional Formulas (CASC 2003, ISSAC 2010)

- generalization of SAT solving
- quantified propositional calculus, i.e., parametric QSAT (aka QBF) solving

Some Other Software

- Qepcad B (Hong and Brown)
  is the reference implementation for cylindrical algebraic decomposition (CAD).
- The computer algebra system Mathematica
  has real QE: essentially CAD + virtual substitution for preprocessing.
- The computer algebra system Maple
  has been used in recent research on CAD (Davenport et al.)
- The computer algebra system Risa/Asir (originally by Fujitsu)
  has QE by virtual substitution (TS, 1996)
- Some prototypes in Japan
  based on comprehensive Gröbner bases (Sato et al.)
  or Sturm–Habicht sequences (Anai et al. in Matlab)
- Specialized implementations of CAD in SMT solvers (z3)
- Specialized implementations of virtual substitutions for SMT (SMT-RAT)

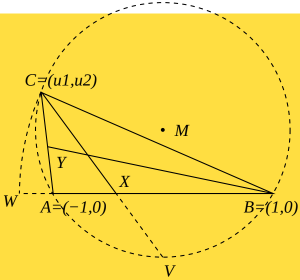
Applications in Geometry and Verification

Variant of the Steiner–Lehmus-Theorem

[J. Autom. Reasoning 1998 – Joint work with A. Dolzmann, V. Weispfenning]

The longer bisector goes to the shorter side

$$
\begin{aligned}
h_1 &\equiv u_2 \geq 0 \wedge x_1 \geq 0 \\
h_2 &\equiv r^2 = 1 + x_1^2 = u_1^2 + (u_2 - x_1)^2 \\
h_3 &\equiv x_2 \leq 0 \wedge r^2 = (x_2 - x_1)^2 \\
h_4 &\equiv u_1 x_2 + u_2 x_3 - x_2 x_3 = 0 \\
h_5 &\equiv x_4 \leq 1 \wedge (x_4 - 1)^2 = (u_1 - 1)^2 + u_2^2 \\
h_7 &\equiv (-1 - u_1)^2 + u_2^2 < 2^2 \\
h_6 &\equiv (x_4 - x_5)^2 + x_6^2 = (u_1 - x_5)^2 + (u_2 - x_6)^2 \wedge u_1 x_6 - u_2 x_5 - u_2 + x_6 = 0 \\
g &\equiv (u_1 - x_3)^2 + u_2^2 < (x_5 - 1)^2 + x_6^2
\end{aligned}
$$



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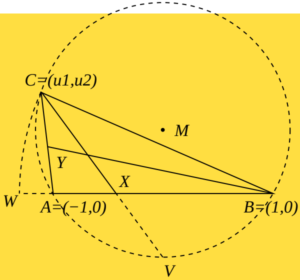

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- $\varphi \equiv \forall x_6 \forall x_5 \forall x_4 \forall x_3 \forall x_2 \forall x_1 \forall r \left(\bigwedge_{i=1}^{7} h_i \longrightarrow g \right)$

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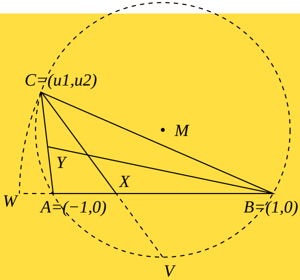

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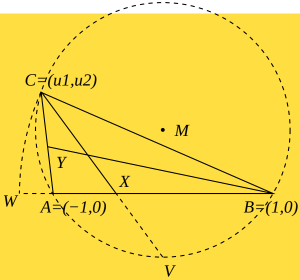

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- CAD (0.9 s): $\forall u_1 \forall u_2 (\bigwedge \Theta \longrightarrow \varphi')$ ✓

Collision Avoidance with Adaptive Cruise Control

[ISSAC 2011 – Joint Work with A. Tiwari @SRI]

System dynamics

$$\begin{aligned}
\dot{v}_f &= a_f \in [-5, 2] && \textit{velocity and accelleration of leading car} \\
\dot{v} &= a \in [-5, 2] && \textit{velocity and accelleration of rear car} \\
\dot{\mathrm{gap}} &= v_f - v && \\
\dot{a} &= -3a - 3(v - v_f) + (\mathrm{gap} - (v + 10)) && \textit{control law for rear car}
\end{aligned}$$

Initial states and safe states

$$\begin{aligned}
\mathrm{Init} &\equiv \mathrm{gap} = 10 \wedge a = 0 \wedge v_f = c_1 \wedge v = c_2 \\
\mathrm{Safe} &\equiv \mathrm{gap} > 0
\end{aligned}$$

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Certificate-based approach to find a set Inv such that

1. Init $\subseteq$ Inv
2. Inv $\subseteq$ Safe
3. System dynamics cannot cause the system to leave Inv.

Collision Avoidance with Adaptive Cruise Control

Linear ansatz

$$\begin{aligned} \mathrm{Inv} &\equiv p \geq 0 \quad \text{where} \quad p := c_3 v + c_4 v_f + c_5 a + \mathrm{gap} + c_6 \\ \mathrm{Inv}' &\equiv -5 \leq a \leq 2 \ \wedge\ -5 \leq a_f \leq 2 \ \wedge\ v \geq 0 \ \wedge\ v_f \geq 0 \end{aligned}$$

Certificate as a formula

$\exists c_3 \exists c_4 \exists c_5 \exists c_6 \forall v \forall v_f \forall \mathrm{gap} \forall a \forall a_f (\varphi_1 \wedge \varphi_2 \wedge \varphi_3)$

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After 1 minute of computation:

- 584 disjuncts, 33365 atomic formulas, depth 13, some still containing $\exists c_5$
- first 33 disjuncts automatically simplify to $c_2^2 - 30c_2 - 75 \leq 0$ for $c_1 > 0$, $c_2 > 0$.
- $\Rightarrow$ no collision for $c_2 = v \leq 32$

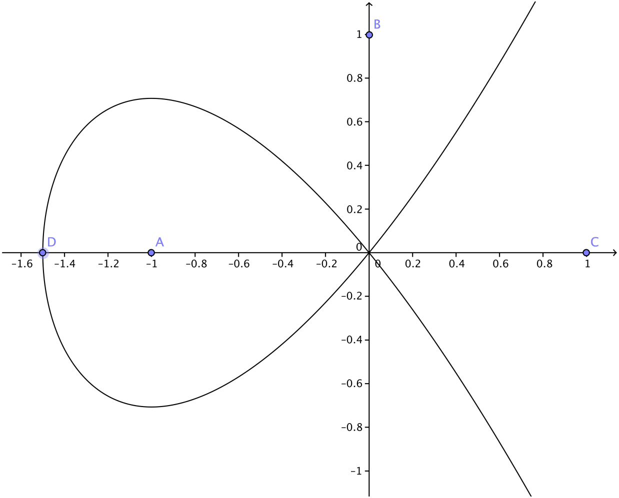
Cylindrical Algebraic Decomposition (CAD)

From Sign Invariant Regions to CAD Cells

$\varphi(f_1, f_2)$ is a Boolean combination of constraints with left hand sides f_1, f_2 and right hand sides 0.

$$f_1(x, y) = 2y^2 - 2x^3 - 3x^2$$

$f_1(A) = -1 < 0$
$f_1(B) = 2 > 0$
$f_1(C) = -5 < 0$
$f_1(D) = 0$



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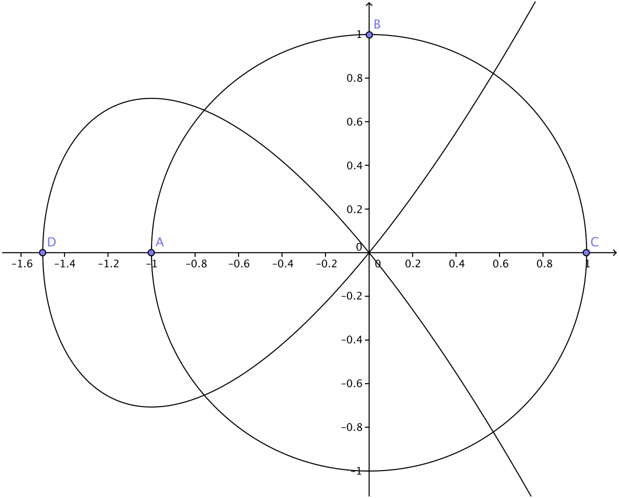
$$f_2(x, y) = y^2 + x^2 - 1$$



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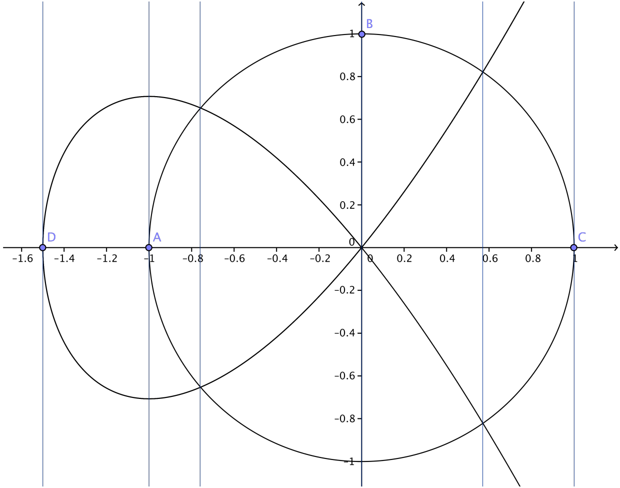

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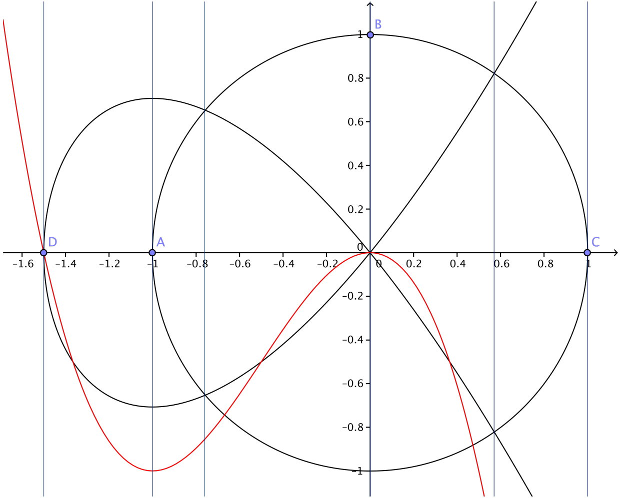
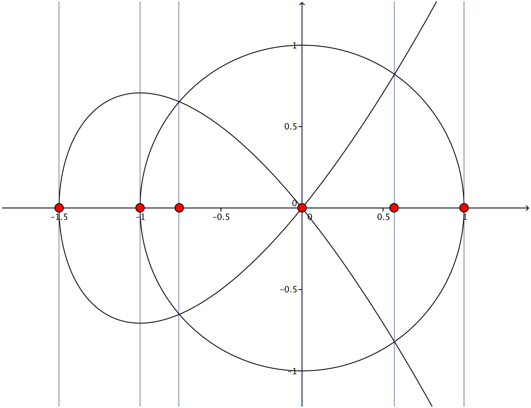
$g(x) = -2x^3 - 3x^2$
...
projection polynomials



Projection and Base Phase (1)



$\varphi(f_1, f_2)$

- **projection operator** computes **projection set**:

$$\Pi(\{f_1(x,y), f_2(x,y)\}) = \{\color{red}{g_1(x), \ldots, g_k(x)}\}$$

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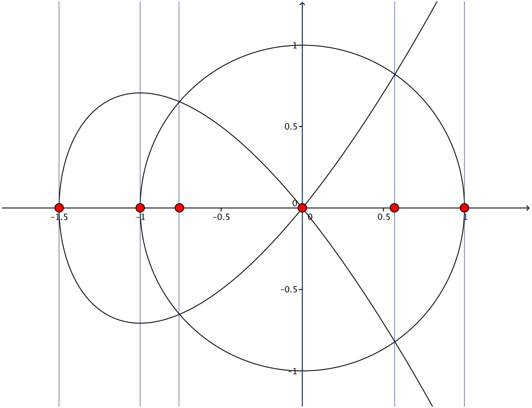

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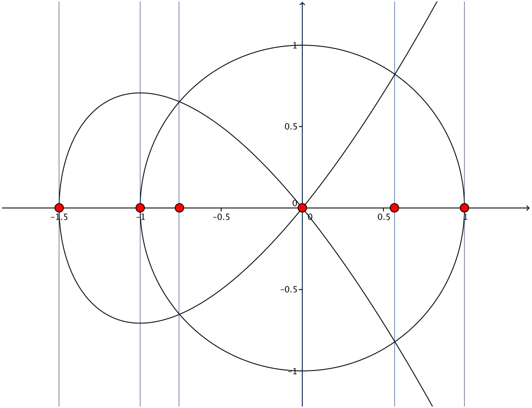

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- The zeros of the g_i are real algebraic numbers, e.g.

$$-\sqrt{2} = \left(x^2 - 2,]-10, 1[\right)$$

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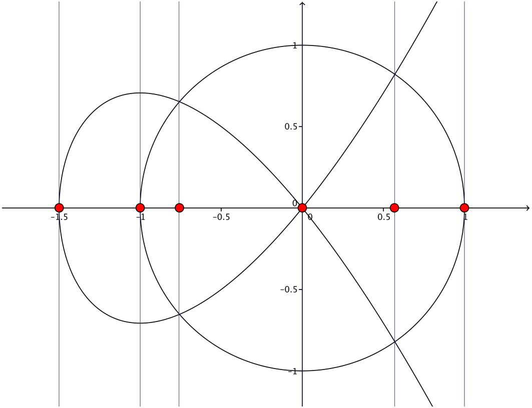

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- The zeros of the g_i are real algebraic numbers, e.g.

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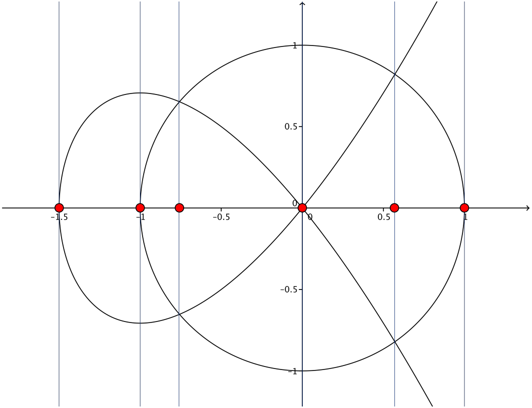
- Their computation is **univariate** computer algebra.

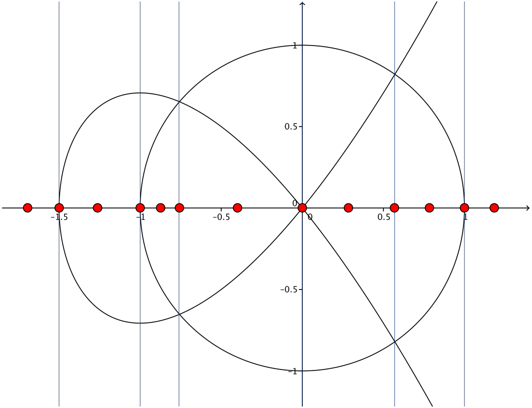
Projection and Base Phase (2)



$\varphi(f_1, f_2)$

- Add points (anywhere) between the zeros as test points for the 1-dimensional cells.

Projection and Base Phase (2)



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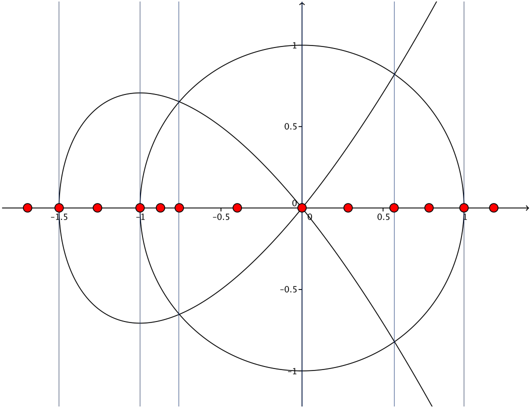

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- Add points (anywhere) between the zeros as test points for the 1-dimensional cells.
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- We want to **lift** this decomposition to $\mathbb{R}^2$.

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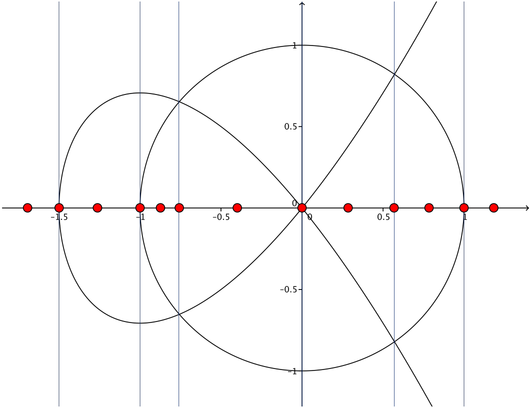

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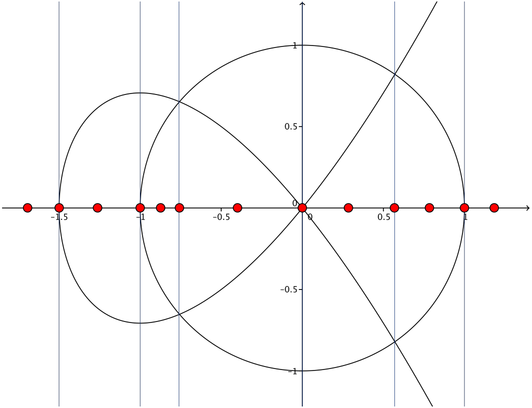
- This yields a **decomposition** of $\mathbb{R}^1$ (the x-axis).
- We want to **lift** this decomposition to $\mathbb{R}^2$.
- By the way: How many cells will there be in $\mathbb{R}^2$?

Extension Phase (Lifting)



$\varphi(f_1, f_2)$

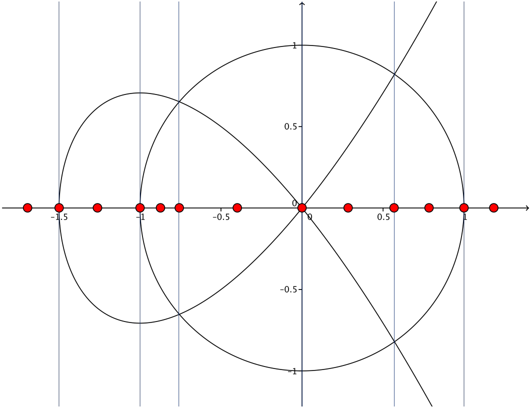
For each test point t from the base phase:

- compute **univariate**

$$f_1(t, y), \quad f_2(t, y).$$

with algebraic number coefficients.

Extension Phase (Lifting)



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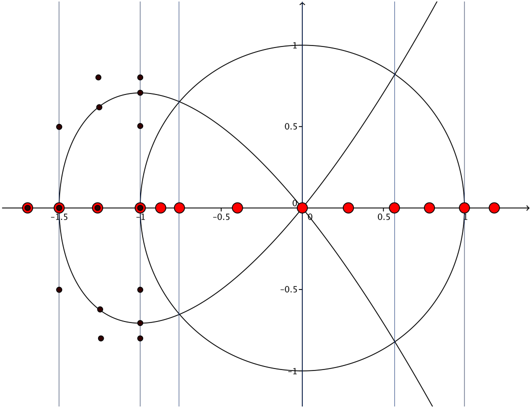

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- compute **univariate**

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with algebraic number coefficients.

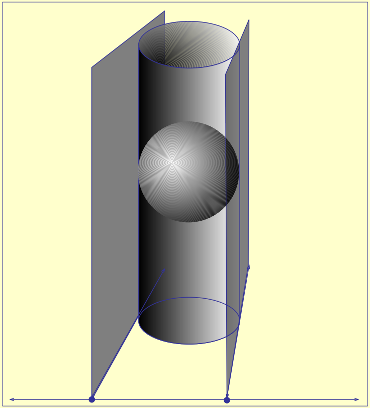
- compute zeros and points between zeros $u_1, \ldots, u_s$.
- this yields test points

$$(t, u_1), \ldots, (t, u_s) \in \mathbb{R}^2$$

for the cylinder over t.

Example: a CAD as a "data structure"

$P_3 = \{x_1^2 + x_2^2 + x_3^2 - 4\}$
$P_2 = \{x_2^2 + x_1^2 - 4\}$
$P_1 = \{x_1 + 2, x_1 - 2\}$



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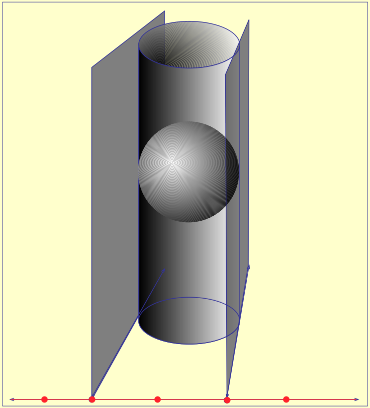

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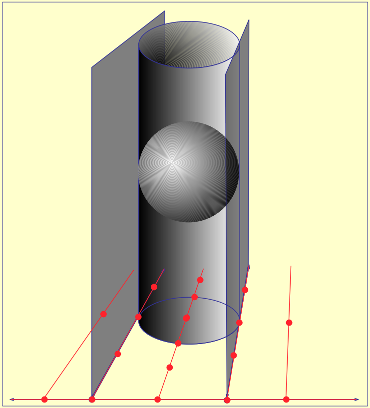

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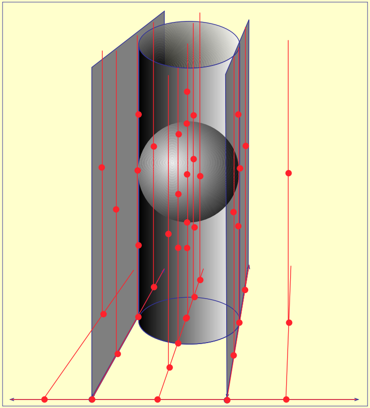
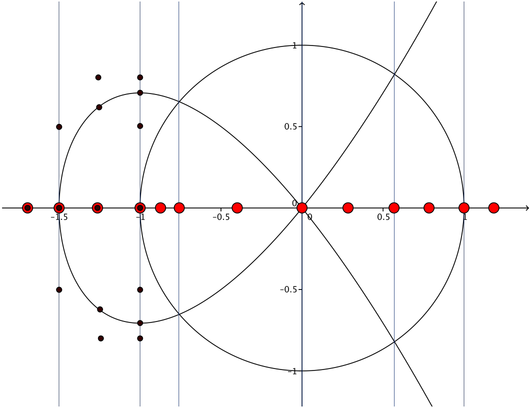


SAT-Checking



$\varphi(f_1, f_2)$

- Finitely many test points

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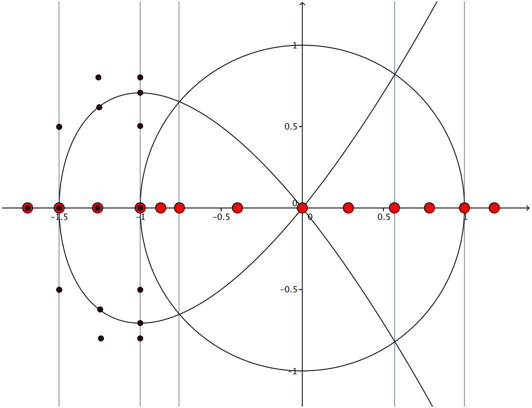

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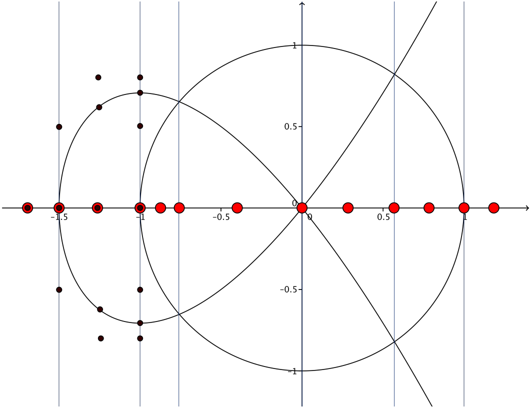
- $\mathbb{R} \models \exists\varphi(f_1, f_2)$ iff ex. $t \in T$ s.t.

$$\mathbb{R}, \ (x, y) = t \ \models \ \varphi(f_1, f_2).$$

Complete Decision Procedure



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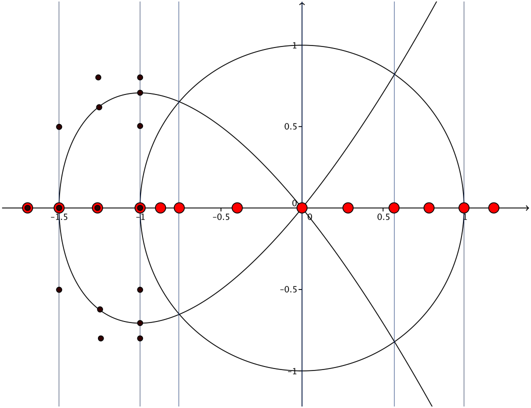

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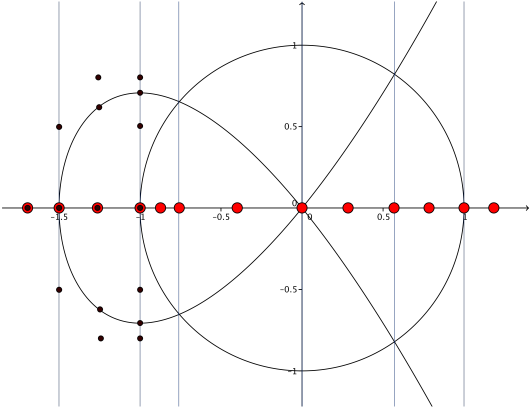

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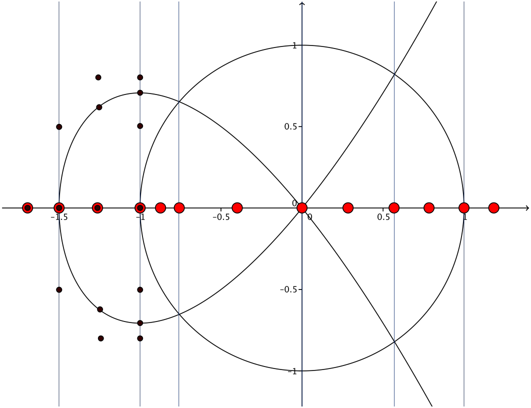

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- The innermost variable y was projected first.

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- Until now, we have not used and did not "really know" the cells – only test points.

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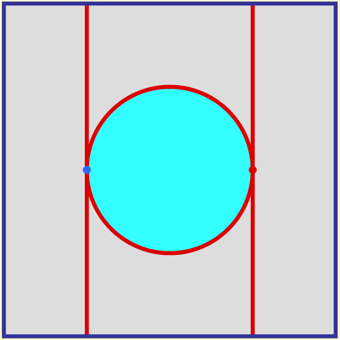
$$\mathbb{R} \models \psi \longleftrightarrow \bigvee_{c \in \mathrm{TRUECELLS}} \Delta_c.$$

Solution Formula Construction Example



$cell$	$P_{1,1}$	$P_{1,2}$	$P_{2,1}$	T/F
$1,1$	$-$	$-$	$+$	F
$2,1$	0	$-$	$+$	F
$2,2$	0	$-$	0	T
$2,3$	0	$-$	$+$	F
$3,1$	$+$	$-$	$+$	F
$3,2$	$+$	$-$	0	F
$3,3$	$+$	$-$	$-$	T
$3,4$	$+$	$-$	0	F
$3,5$	$+$	$-$	$+$	F
$4,1$	$+$	0	$+$	F
$4,2$	$+$	0	0	F
$4,3$	$+$	0	$+$	F
$5,1$	$+$	$+$	$+$	F

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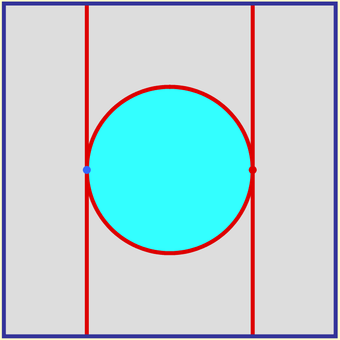

cell	$P_{1,1}$	$P_{1,2}$	$P_{2,1}$	T/F
1, 1	−	−	+	F
2, 1	0	−	+	F
2, 2	0	−	0	T
2, 3	0	−	+	F
3, 1	+	−	+	F
3, 2	+	−	0	F
3, 3	+	−	−	T
3, 4	+	−	0	F
3, 5	+	−	+	F
4, 1	+	0	+	F
4, 2	+	0	0	F
4, 3	+	0	+	F
5, 1	+	+	+	F

$$P_{2,1} < 0$$

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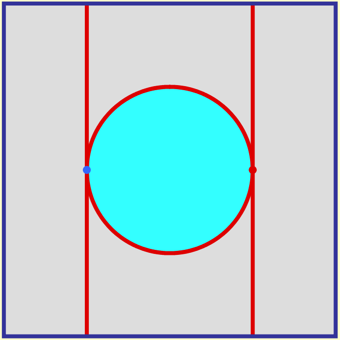

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$3,2$	$+$	$-$	0	F
$3,3$	$+$	$-$	$-$	T
$3,4$	$+$	$-$	0	F
$3,5$	$+$	$-$	$+$	F
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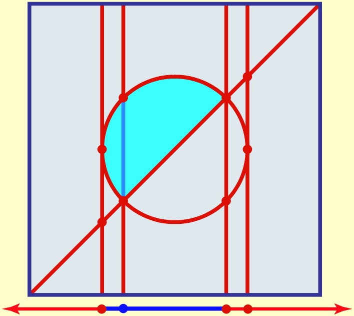
$5,1$	$+$	$+$	$+$	F

$$P_{2,1} < 0 \;\vee\; P_{1,1} = 0 \wedge P_{2,1} = 0$$

Solution Formula Construction Problem

$$\exists y[x^2 + y^2 - 1 < 0 \ \wedge \ x - y < 0]$$



cell	$x+1$	$x-1$	x^2-2	T/F
1	$-$	$-$	$+$	F
2	0	$-$	$+$	F
3	$+$	$-$	$+$	T
4	$+$	$-$	0	T
5	$+$	$-$	$-$	T
6	$+$	$-$	0	F
7	$+$	$-$	$+$	F
8	$+$	0	$+$	F
9	$+$	$+$	$+$	F

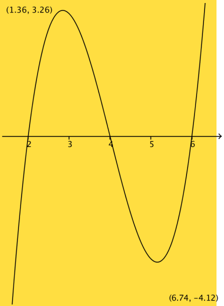
Solutions to the Solution Formula Problem (1)

Augmented Projection

- The approach of the original Collins article (1975).
- Idea: Produce sufficiently many polynomials during projection.
- Technically one adds “lots of derivatives.”

A very simple demonstration of the idea

- Consider a single polynomial $f = x^3 - 12x^2 + 44x - 48$.



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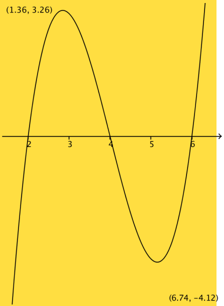

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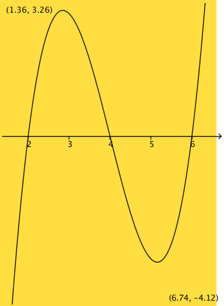

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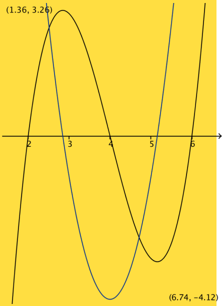

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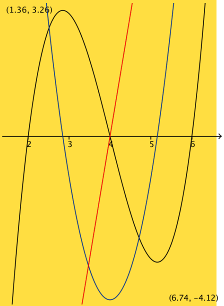

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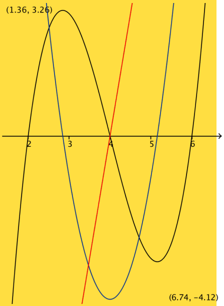

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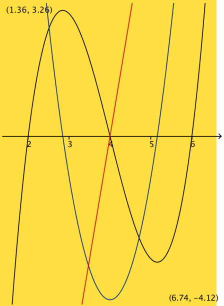

Augmented projection is considered practically infeasible.

Solutions to the Solution Formula Problem (2)

Extended Tarski Language

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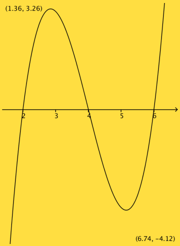

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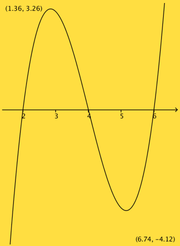

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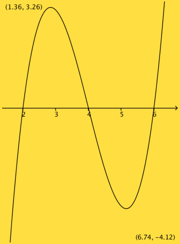

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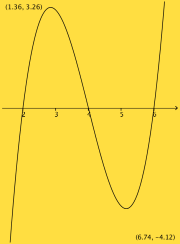

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State-of-the-art in QEPCAD and Mathematica, and used in Z3/NLSAT.

Summary

- virtual substitution for real quantifier elimination and some variants
  (extended, generic)
- software: Redlog and other
- other theories
  (integers, comples, differential, padic, terms, queues, PQSAT)
- applications in geometry, verification, ...
- cylindrical algebraic decomposition (CAD)