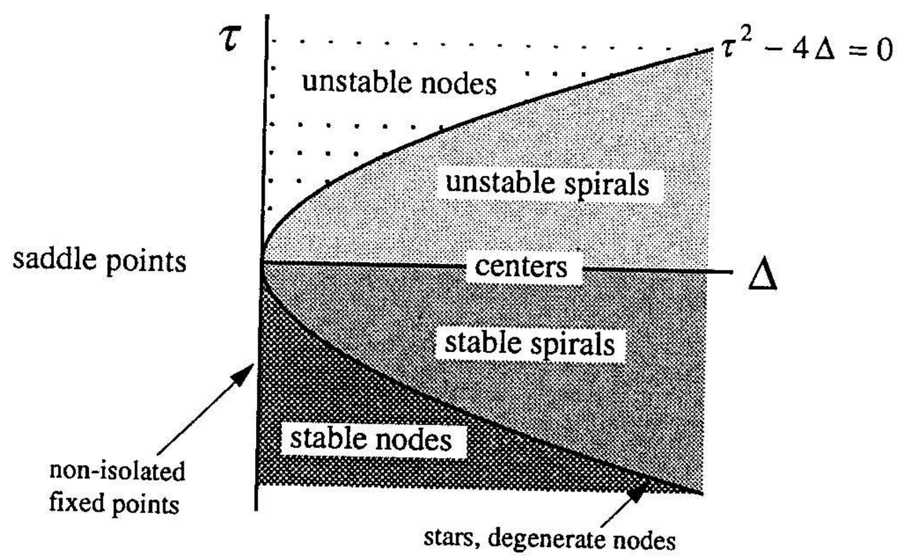
Fixed Points and Linearization

- Suppose (x^*, y^*) is fixed point, **linearized system** is

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}_{(x^*,y^*)} \begin{pmatrix} u \\ v \end{pmatrix}$$

  where $u = x - x^*, v = y - y^*$. **Jacobian matrix**.

- If fixed point for linearized system is not one of borderline cases, linearized system give a qualitatively correct picture near (x^*, y^*).
- Borderline cases can be altered by small nonlinear terms.

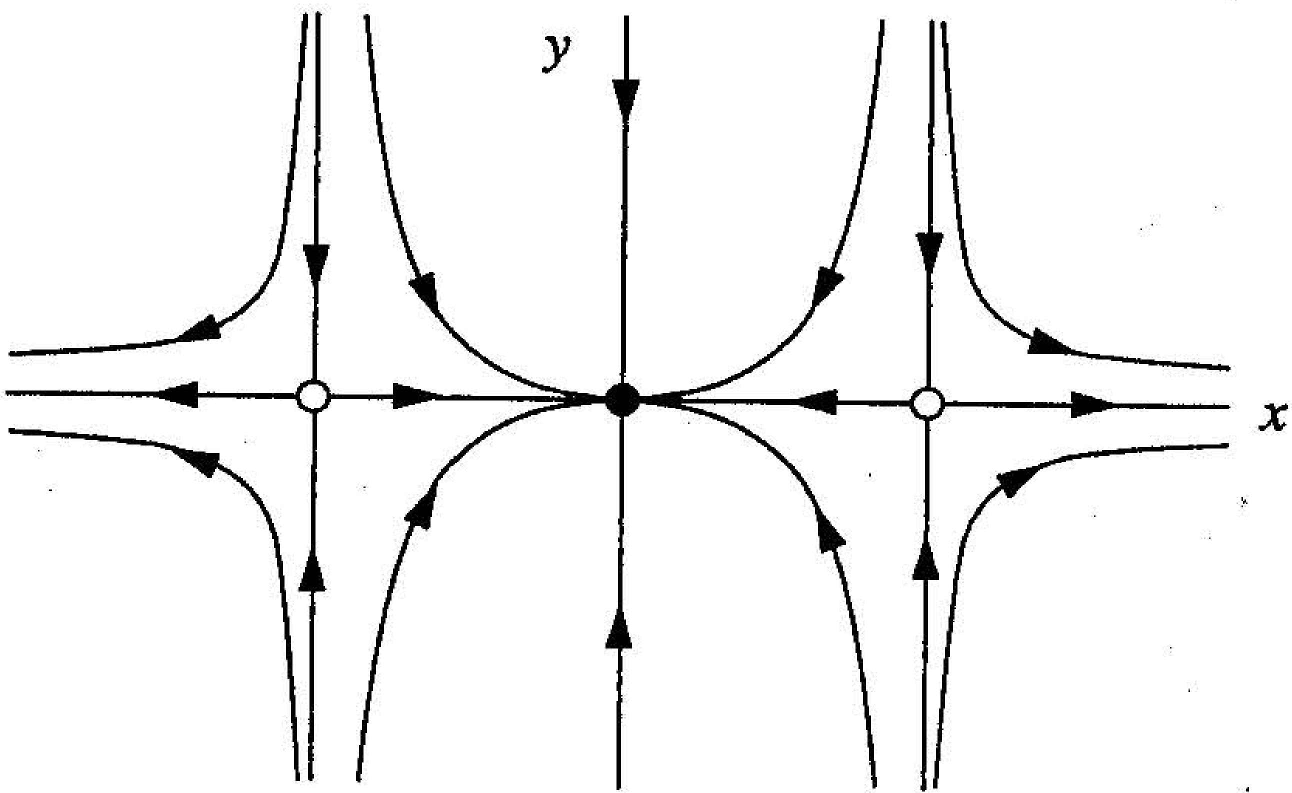


Example 6.3.1

- Consider

$$\dot{x} = -x + x^3, \qquad \dot{y} = -2y.$$

- 3 fixed points: $(0,0)$ (stable **node**), $(1,0), (-1,0)$ (**saddles**) by analyzing **linearized system**.
- Not borderline cases. Fixed points for **nonlinear system** is similar to **linearized system**.



Example 6.3.2

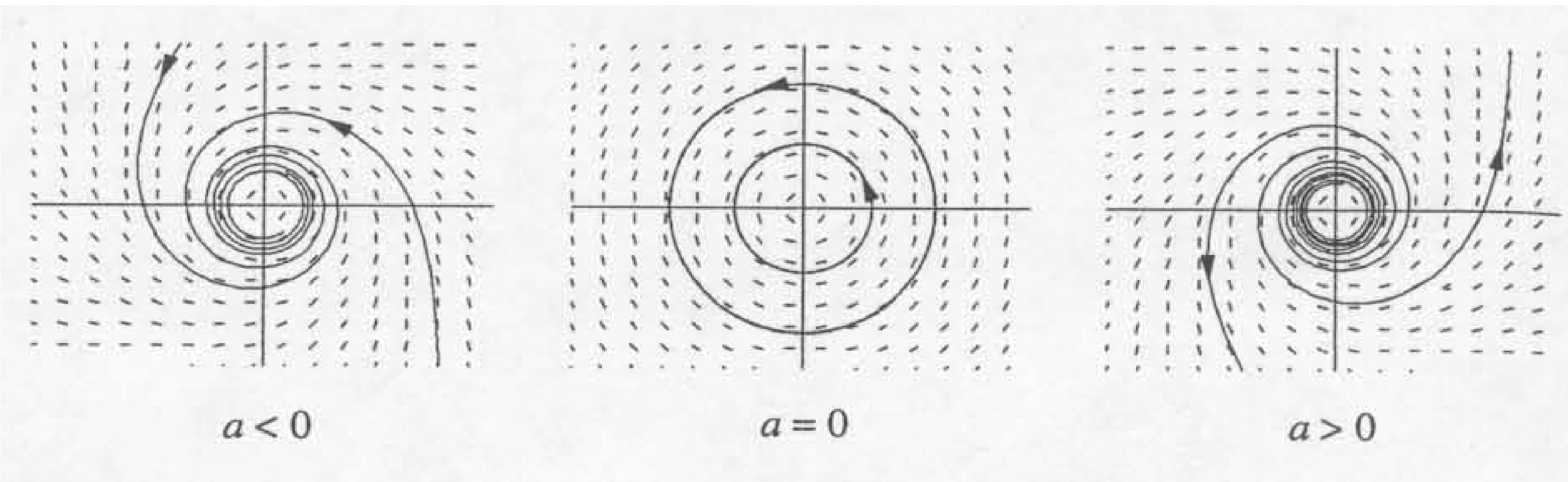
- Consider

$$\dot{x} = -y + ax(x^2 + y^2), \qquad \dot{y} = x + ay(x^2 + y^2).$$

- **Linearized system** predicts: $(0, 0)$ is a **center** for all a.
- In polar coordinates

$$\dot{r} = ar^3, \qquad \dot{\theta} = 1.$$

- In fact, $(0, 0)$ is a **spiral** (stable if $a < 0$, unstable if $a > 0$).
- Stars and degenerate nodes can be altered by small nonlinearities, but their stability doesn't change.



Hyperbolic Fixed Points and Structural Stability

- If only interested in stability
  - **Robust Cases**:
    Repellers (sources): $\mathrm{Re}(\lambda_1)$, $\mathrm{Re}(\lambda_2) > 0$.
    Attractors (sinks): $\mathrm{Re}(\lambda_1)$, $\mathrm{Re}(\lambda_2) < 0$.
    Saddles: $\lambda_1 > 0, \lambda_2 < 0$.
  - **Marginal Cases**:
    Centers: both eigenvalues are pure imaginary.
    Higher-order and **Non-isolated fixed points**: at least one eigenvalues is zero.
- If $\mathrm{Re}(\lambda) \neq 0$ for both eigenvalues, fixed point is **hyperbolic**.
- **Hartman-Grobman Theorem**: Local phase portrait near a hyperbolic fixed point is topologically equivalent to phase portrait of its linearized system.
- A phase portrait is **structurally stable** if its topology cannot be changed by an arbitrarily small perturbation to the vector field.

Lotka-Volterra Model of Competition

- Consider Rabbit (x) vs Sheep (y)

$$\dot{x} = x(3 - x - 2y), \qquad \dot{y} = y(2 - y - x).$$

- Find **fixed points**: $(0,0), (0,2), (3,0), (1,1)$.
- Compute Jacobian matrix and **classify** fixed points: $(0,0)$ (unstable node), $(0,2), (3,0)$ (stable nodes), $(1,1)$ (saddle).
- Draw **phase portrait**. **Basic of attraction** for fixed points.
- Principle of competitive exclusion.