Nonstandard Models

- First-order theories (without finite models) have unintended, non-isomorphic, models. Some are of a different cardinality than the intended model. Others are merely of a different order-type. Either way, the fact that nothing that we can say (in a first-order language) pins down the subject of our mathematical theories (even up to isomorphism) raises questions about the determinacy of basic terms, like 'uncountable' and 'finite'.

Completeness Theorem

- Let us fix on some standard (classical) first-order proof relation, $\vdash$, and let $\models$ be the standard Tarkian consequence relation (for first-order languages). Then we have:
  - Soundness Theorem: If Γ is a set of sentences, and δ is a sentence, then $(\Gamma \vdash \delta) \rightarrow (\Gamma \models \delta)$.
  - Completeness Theorem: If Γ is a set of sentences, and δ is a sentence, then $(\Gamma \models \delta) \rightarrow (\Gamma \vdash \delta)$.
- The proof of Soundness is straightforward. One verifies that each instance of the axiom schemas is valid (i.e., true in all models), and that the inference rule(s) preserve validity.
  - *Note*: Proof of Soundness also amounts to a proof of the consistency of the proof system. If it were inconsistent, then, by Soundness, we would have $\varnothing \models \delta$ and $\varnothing \models {\sim}\delta$. But, by the definition of $\models$, ${\sim}\delta$ is true in all models if/f δ is true in none.
- The proof of Completeness is more subtle. The first point is that Completeness means that if Con(Γ & ${\sim}\delta$) then $\exists M \models (\Gamma \,\&\, {\sim}\delta)$. $(\Gamma \vdash \delta)$ means that ~Con(Γ & ${\sim}\delta$) and $(\Gamma \models \delta)$ means that there is no model of (Γ & ${\sim}\delta$). So, if Con(Γ & ${\sim}\delta$), then there *is* a model of (Γ & ${\sim}\delta$). Thus, to prove Completeness it suffices to prove: $\mathbf{Con(\Sigma)} \rightarrow \exists \boldsymbol{M} \models \boldsymbol{\delta}$**, for** $\boldsymbol{\delta} \in \boldsymbol{\Sigma}$.
  - *Note*: The proof of Completeness will actually show more than this. It will show that if Σ is consistent, then Σ has a countable model. This will generate a puzzle.
- Let us begin with a consistent theory, Σ. Add countably-many constants to the language, $\mathbf{c}_1, \mathbf{c}_2, \ldots \mathbf{c}_n$, and amend the formal system's properties correspondingly. The constants are called witnesses. Clearly, Σ in the supplemented language, call it Σ_+, remains consistent.

- *Note*: For simplicity, we assume that the language has only one binary predicate, *P*. But the technique described generalizes to any predicates of arbitrary arity.

- Next, let us enumerate the formulas in the enriched language with some free variable, x, $\varphi_1(x), \varphi_2(x)...\varphi_n(x)...$, and define Φ_n to be the formula, $[(\exists x)\varphi_n(x) \rightarrow \varphi(\mathbf{c}_{n*})]$. The constant $\mathbf{c}_{n*}$ is the *first* constant from our enumeration that fails to occur inside any *prior* φ or Φ.

- We now want to add each Φ_n to our theory, Σ_+. The idea is to make sure that *whenever our theory proves that there is an x such that φ, it proves this of some **c***. So, we define:

    - $\Sigma^0 = \Sigma_+$
    - $\Sigma^{n+1} = \Sigma^n \cup \{\Phi_n\}$ (i.e., $\Sigma^n \cup \{[(\exists x)\varphi_n(x) \rightarrow \varphi_n(\mathbf{c}_{n*})]\}$)
    - $\Sigma^\infty = \cup\Sigma^n$

- It is clear that each Σ^n is consistent since each new witness acts like a free variable. But any proof is finite, contained in some Σ^n. So, if each Σ^n is consistent, then Σ^∞ must be too.

- *Lindenbaum's Lemma*: Every consistent theory has a *complete* and *consistent* extension in the same language. We proceed as in the propositional case. Let $\varphi_1, \varphi_2...\varphi_n...$ be an enumeration of all of the sentences in the language of consistent theory, Σ^∞. Then define:

    - $\Sigma_0 = \Sigma^\infty$
    - $\Sigma_{n+1} = \Sigma_n \cup \{\varphi_n\}$, if this is consistent, and let $\Sigma_{n+1} = \Sigma_n$ if not.
    - $\Sigma^* = \cup\Sigma_n$ $(n \in \mathbb{N})$

  Then Σ^* is a complete consistent extension of Σ^∞ (in Σ^∞'s language) with the features:

    - $\Sigma^* \vdash \delta$ or $\Sigma^* \vdash \sim\delta$ (because Σ^* is complete)
    - $\Sigma^* \vdash \sim\delta$ if/f $\Sigma^* \nvdash \delta$ (because Σ^* is also consistent)
    - $\Sigma^* \vdash (\delta \,\&\, \alpha)$ if/f $\Sigma^* \vdash \delta$ and $\Sigma^* \vdash \alpha$
    - $\Sigma^* \vdash (\exists x)\varphi_n(x)$ if/f $\Sigma^* \vdash \varphi_n(\mathbf{c}_{n*})$

- We can now define a model, $M = \langle U, R\rangle$, of Σ^* by conflating names and their referents.

    - $U = \{\mathbf{c}_1, \mathbf{c}_2,...\mathbf{c}_n,...\}$

        - *Note*: Technically, we identify the elements of U with <u>equivalence classes</u>. $\mathbf{c}_j \sim \mathbf{c}_k \longleftrightarrow T \vdash \mathbf{c}_j = \mathbf{c}_k$. Thus, $[\mathbf{c}_j]$ is the class of $\mathbf{c}_k$s, $[\mathbf{c}_j] = \{\mathbf{c}_k : T \vdash \mathbf{c}_j = \mathbf{c}_k\}$.

- R = {<$\mathbf{c}_j$, $\mathbf{c}_k$>} such that $\Sigma^* \vdash P(\mathbf{c}_j, \mathbf{c}_k)$

- As $\Sigma \subseteq \Sigma^*$, $M \models \delta$, for $\delta \in \Sigma$, this completes the proof.

Löwenheim–Skolem Theorem

- So, if Σ is a (syntactically) consistent set of sentences, then it has a countable model. This might strike you as perplexing. What if the Σ is (first-order) *ZF*, which proves Cantor's Theorem, along with the existence of (a set-theoretic surrogate of) ℕ and *P*(ℕ)?

    - *Cantor's Theorem*: For any set, *X*, there is no one-to-one correspondence between *X* and *P*(*X*).

    - *Proof*: Suppose, for *reductio*, that $f: X \rightarrow P(X)$ is such a correspondence. Define a subset of *X*, $Y \in P(X)$, as follows. $Y = \{x \in X : x \notin f(x)\}$. Now suppose that there is an *x* with $f(x) = Y$. Then, if $x \in Y$, $x \notin f(x) = Y$, which is a contradiction. On the other hand, if $x \notin Y = f(x)$, then $x \in Y$, which is also a contradiction. Therefore, the assumption that *f* is a one-to-one correspondence between *X* and *P*(*X*) is false.

- The standard view is that *Cantor's Theorem* shows that cardinality of the powerset of *X* is strictly greater than that of *X*. Hence, the cardinality of the powerset of ℕ is uncountable.

    - *Skolem*: "By virtue of the [*Zermelo*] axioms we can prove the existence of higher [uncountable] cardinalities….How can it be, then, that the entire domain…can already be enumerated by means of the finite positive integers [1922, 295]?"

    - *Note*: The standard view is not beyond question. Dummett maintains that "The argument does not show that [*P*(ℕ)] form[s] a non-denumerable totality unless we assume…that [*P*(ℕ)] form[s] a determinate totality comprising all that we shall ever recognize as a real number: the alternative is to regard the concept of real number as an indefinitely extensible one [1993, 27]." This interpretation, however, is objectionably psychologistic. The claim appears to be that we form a 'definite conception' of subsets of *X* by pairing them with members of *X*. Then we use that pairing – i.e., that conception – to literally create a new subset of *X*.

- The incongruence of the *Completeness Theorem* and *Cantor's Theorem* can be extended.

- *Löwenheim-Skolem Theorem*: If a set of sentences, Σ (in a countable language) has a model (where '=' means identity), it has a model of *any infinite cardinality*.

- The 'downwards' implication of this can be refined using Mostowski Collapse:

  - *Transitive Submodel Theorem*: If *N* is a transitive model of infinite cardinality κ, and λ < κ, then there is at transitive submodel of *N*, *M*, such that the cardinality of dom(*M*) is λ, and, for any δ, *N* |= δ if/f *M* |= δ.

    - A model, *M*, is *transitive* when x ∈ dom(*M*) → x ⊆ dom(*M*), and a *submodel* of *N* when dom(*M*) ⊆ dom(*N*) and *M* and *N* agree on their interpretation of constants, predicates, and function symbols.

      - *Note*: The assumption that there is a *transitive* model of Σ is stronger than the assumption that there is just a model of Σ!

- Skolem's 'paradox' relies on the assumption of (classical) first-order logic. If we help ourselves to second-order *ZF* and its standard (non-Henkin) semantics, then the discrepancies above evaporate. But if the theorems are taken to preclude explaining the determinacy of 'uncountable', then nothing is gained by resorting to second-order logic. The problem just becomes to explain the determinacy of the second-order quantifiers.

Compactness Theorem

- Theories may have 'untended' models of the intended cardinality as well. (Indeed, *ZF* is not κ-categorical, it is not the case that all models of *ZF* of cardinality κ are isomorphic.)

  - Compactness Theorem: If every finite subset of Σ has a model, then Σ does too.

  - Proof: If ~*Con*(Σ), then there is a formula, δ, such that Σ ⊢ (δ & ~δ), i.e., some *n* ∈ ℕ, and a sequence of formulae, $\theta_1, \theta_2, \ldots \theta_n$, such that θ_n is (δ & ~δ), and each θ_i is a logical axiom, an element of Σ, or such that θ_i follows from previous θs by a rule of inference. As this list is finite, the number of formulas in it that are also members of Σ is. So, ~*Con*(Σ) → ~*Con*(Σ_{Fin}), for some finite subset of Σ, Σ_{Fin}. Equivalently, **if Con(Σ_{Fin}), for *every* finite subset of Σ, Σ_{Fin}, then *Con*(Σ)**. Now suppose that every finite subset of Σ, Σ_{Fin}, has a model. Then, by Soundness, every Σ_{Fin} is consistent. As this implies that *Con*(Σ), ∃*M* |= Σ, by Completeness.

- *Illustration*: Consider the following theory in language $\mathcal{L}$= {<}.

- (i) $(\forall x)\sim(x < x)$
- (ii) $(\forall x)(\forall y)(\sim[(x < y) \& (y < x)])$
- (iii) $(\forall x)(\forall y)(\forall z)([(x < y) \& (y < x)] \rightarrow (x < z))$
- (iv) $(\forall x)(\forall y)((x < y) \vee (y < x) \vee (x = z))$
- (v) $(\exists x)(\forall y)(\sim(y < x))$
- (vi) $(\forall x)(\exists y)[(x < y) \& (\forall z)(\sim(x < z) \& (z < y))]$
- (vii) $(\forall x)[(\exists y)(y < x) \rightarrow (\exists z)[(z < x) \& (\forall w)\sim[(w < x) \& (z < w)]])$

- It turns out that sentences, (i) – (vii), completely axiomatize the structure, $\mathcal{N} = (\mathbb{N}, <)$. For any sentence, δ, and any model of the sentences, (i) – (vii), M, $\mathcal{N} \models \delta$ just in case $M \models \delta$. It follows that the sentences, (i) – (vii), *prove* δ just in case $\mathcal{N}$ itself is a model of δ.

    - *Note*: $\mathcal{N} = (\mathbb{N}, <)$ is not the structure $(\mathbb{N}, <, +, *)$, which cannot be axiomatized! But the structure $(\mathbb{N}, <, +)$ can be, as can the real and complex number fields.

- Despite being complete, the sentences (i) – (vii) are *not* categorical. That is, it is not the case that all models of (i) – (vii) are isomorphic. We can prove this using Compactness.

    - *Recall*: Categoricity is the most that we can hope for. Clearly, if $(\mathbb{N}, <)$ satisfy (i) – (vii), then so does $(\mathbb{N} - \{0\}, <)$, and so on for infinitely-many other structures.

- *Argument*: Expand the language $\mathcal{L}$ to include a constant, **c**. Call the result $\mathcal{L}^*$. Now consider the sentences (i) – (vii) *in tandem with* the following infinite set from $\mathcal{L}^*$.

    - $\Psi_1 : (\exists x_1)(x_1 < \mathbf{c})$
    - $\Psi_2 : (\exists x_1)(\exists x_2)[(x_1 < x_2) \& (x_2 < \mathbf{c})]$
    - …
    - $\Psi_n : (\exists x_1)(\exists x_2)...(\exists x_n)[(x_1 < x_2) \& (x_2 < x_3)...\& (x_n < \mathbf{c})]$
    - …

- Let us take Σ to consist of (i) – (vii) in addition to all Ψ_i above, and let $\Sigma' \subseteq \Sigma$ to be any finite subset of Σ. Then each Σ' has a model of the form $(\mathbb{N}, <, n \in \mathbb{N})$, where n is the largest $n \in \mathbb{N}$ such that $\Psi_n \in \Sigma'$. (i) – (vii) are all true in $(\mathbb{N}, <, n \in \mathbb{N})$ because they are true in $(\mathbb{N}, <)$, and Ψ_n is true in $(\mathbb{N}, <, k \in \mathbb{N})$ so long as $n \leq k$ (where, again, $0 \in \mathbb{N}$). So, by Compactness, (i) – (vii) plus all Ψ_i must have a model, M, satisfying the same sentence in $\mathcal{L}$ as $(\mathbb{N}, <)$. But M is not isomorphic to $(\mathbb{N}, <)$ because the domain of M contains an object (denoted by **c**) with *infinitely-many predecessors*, while $\mathbb{N}$ does not.

- *Note*: Although any such model must be complicated (by Tennenbaum's Theorem), a similar argument shows that there is a non-standard countable model of *PA* (in the language {<, +, *}). We will see later that such a model could satisfy *PA* + ‘there is a Gödel number of a proof of a contradiction in *PA*’. Alternatively, it might be a model of the (non-recursively enumerable) theory, True Arithmetic, i.e., every truth in the language of {ℕ, <, +, *}. So, Completeness is not sufficient for categoricity (but categoricity suffices for completeness). Only (first-order) theories with *finite* models are categorical.

  - *Details*: Any nonstandard model consists of *an initial segment that is isomorphic to the standard model*, with extra objects ‘tacked on the end’.

    - *Upshot*: Finiteness is not (first-order) definable. If it were, then we could rule out all nonstandard models by conjoining to the axioms of *PA* the sentence, ‘for all x, x has finitely-many predecessors!

      - *Clarification*: ‘Finite’ is nevertheless absolute for (standard) transitive models, *unlike* ‘countable’.