# Chapter 14

# Graph Enumeration B: Lec 18

**14.3 Burnside's Lemma**

**14.4 Cycle-Index Polynomial of a Permutation Group**

# 14.3 BURNSIDE'S LEMMA

There is a mathematical principle whose application permits quick calculation of the number of orbits under the action of a permutation group. It is commonly called ***Burnside's lemma***, even though it was previously published by Frobenius, and yet earlier by Cauchy.

- We will illustrate the definitions and results in this section using the basic actions of $\mathcal{A}ut(G)$ on $V_G$ and $E_G$.
- However, we will apply Burnside's lemma (in §14.5) to the *induced* actions of $\mathcal{A}ut(G)$ on the sets $Col_k(V_G)$ and $Col_k(E_G)$.

# Stabilizer of an Object

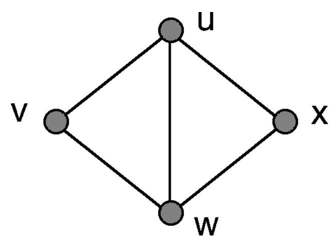
DEF: Let $\mathcal{P} = [P : Y]$ be a permutation group, and let $y \in Y$. The ***stabilizer*** of $y$ is the subgroup

$$\mathcal{S}tab\,(y) \;=\; \{\pi \in P \mid \pi(y) = y\}$$

Thus, the stabilizer of an object $y$ is simply the subgroup comprising all the permutations whose disjoint-cycle form contains the 1-cycle $(y)$.

**Example 14.3.1:** The analysis of $K_4{-}K_2$ and $\mathcal{A}ut(K_4{-}K_2)$ continues.

| Symmetry | $\pi \in \mathcal{A}ut_V(G)$ |
|---|---|
| identity | $\epsilon = (u)(v)(w)(x)$ |
| refl. thru vert. axis | $\pi_1 = (u)(w)(v\ x)$ |
| refl. thru horiz. axis | $\pi_2 = (v)(x)(u\ w)$ |
| 180° rotation | $\pi_3 = (u\ w)(v\ x)$ |

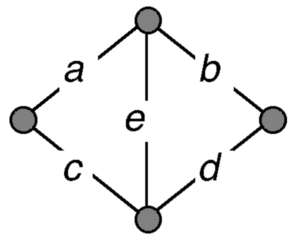



In $\mathcal{A}ut_V(K_4{-}K_2)$, the stabilizers are as follows:

$$\mathcal{S}tab\,(u) = \{\epsilon,\ \pi_1\} \qquad \mathcal{S}tab\,(w) = \{\epsilon,\ \pi_1\}$$
$$\mathcal{S}tab\,(v) = \{\epsilon,\ \pi_2\} \qquad \mathcal{S}tab\,(x) = \{\epsilon,\ \pi_2\}$$

**Example 14.3.2:** That stabilizers of different objects need not have the same number of permutations is illustrated by $\mathcal{A}ut_E(K_4{-}K_2)$.

| Symmetry | $\pi \in \mathcal{A}ut_E(G)$ |
|---|---|
| identity | $\epsilon = (a)(b)(c)(d)(e)$ |
| refl. thru vert. axis | $\pi_1 = (e)(a\ b)(c\ d)$ |
| refl. thru horiz. axis | $\pi_2 = (e)(a\ c)(b\ d)$ |
| 180° rotation | $\pi_3 = (e)(a\ d)(b\ c)$ |




In $\mathcal{A}ut_E(K_4{-}K_2)$, the stabilizers are as follows:

$$\begin{aligned} \mathcal{S}tab\,(a) &= \mathcal{S}tab\,(b) = \mathcal{S}tab\,(c) = \mathcal{S}tab\,(d) = \{\epsilon\} \\ \mathcal{S}tab\,(e) &= \{\epsilon,\ \pi_1,\ \pi_2,\ \pi_3\} \end{aligned}$$

# Fixed-Point Set of a Permutation

DEF: Let $\mathcal{P} = [P : Y]$ be a permutation group and $\pi \in P$. The ***fixed-point set*** of the permutation $\pi$ is the subset

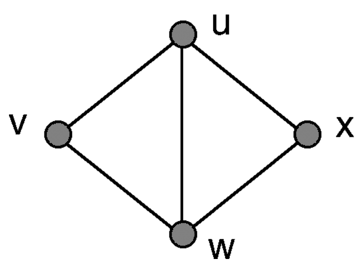
$$Fix\,(\pi) \;=\; \{y \in Y \mid \pi(y) = y\}$$

Thus, $Fix\,(\pi)$ consists of of the objects of $Y$ appearing as 1-cycles in the disjoint-cycle form of $\pi$.

TERMINOLOGY: For a given automorphism $\pi$ on a graph, the ***fixed-vertex set*** and ***fixed-edge set*** are the fixed-point sets of $\pi_V$ and $\pi_E$, respectively.

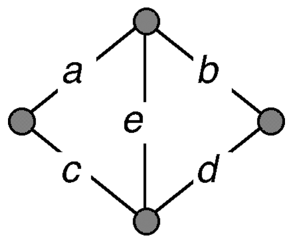
**Example 14.3.3:** The vertex-perms in $\mathcal{A}ut_V(K_4-K_2)$ have the following fixed-point sets

$$\begin{aligned} Fix\,\big((u)(v)(w)(x)\big) &= \{u, v, w, x\} \\ Fix\,\big((u)(w)(v\ x)\big) &= \{u, w\} \\ Fix\,\big((v)(x)(u\ w)\big) &= \{v, x\} \\ Fix\,\big((u\ w)(v\ x)\big) &= \emptyset \end{aligned}$$



**Example 14.3.4:** The edge-permutations in $\mathcal{A}ut_E(K_4 - K_2)$ have the following fixed-point sets

$$
\begin{aligned}
Fix\big((a)(b)(c)(d)(e)\big) &= \{a, b, c, d, e\} \\
Fix\big((e)(a\ b)(c\ d)\big) &= \{e\} \\
Fix\big((e)(a\ c)(b\ d)\big) &= \{e\} \\
Fix\big((e)(a\ d)(b\ c)\big) &= \{e\}
\end{aligned}
$$



The next three lemmas are needed for the proof of Burnside's lemma.

## Stabilizers and Fixed-Point Sets

**Lemma 14.3.1.** *Let $\mathcal{P} = [P : Y]$ be a permutation group. Then*

$$\sum_{y \in Y} |\mathcal{S}tab\,(y)| \quad = \quad \sum_{\pi \in P} |Fix\,(\pi)|$$

**Pf:** Consider a matrix whose rows are indexed by the objects of the set $Y$ and whose columns are indexed by the permutations in $P$, and whose entry in row $y$ and column $\pi$ is 1 if $\pi(y) = y$, but 0 otherwise. Then for each $y$, the summand on the left side of the equation is the sum of row $y$ (i.e., the number of 1's), and the summand on the right side is the sum of column $\pi$. The equation simply asserts that the sum of the row sums of the matrix equals the sum of the column sums. $\diamondsuit$

**Example 14.3.5:** In $\mathcal{A}ut_V(K_4 - K_2)$, the sum of the stabilizer sizes (from Example 14.3.1) is

$$\sum_{y \in \mathcal{A}ut_V(K_4 - K_2)} |\mathcal{S}tab\,(y)| \quad = \quad 2+2+2+2 \quad = \quad 8$$

and the sum of the sizes of the fixed-vertex sets (from Example 14.3.3) is

$$\sum_{\pi \in P} |Fix\,(\pi)| \quad = \quad 4+2+2+0 \quad = \quad 8$$

**Example 14.3.6:** In $\mathcal{A}ut_E(K_4 - K_2)$, the sum of the stabilizer sizes (from Example 14.3.2) is

$$\sum_{y \in \mathcal{A}ut_E(K_4 - K_2)} |\mathcal{S}tab\,(y)| \quad = \quad 1+1+1+1+4 \quad = \quad 8;$$

and the sum of the sizes of the fixed-edge sets (from Example 14.3.4) is

$$\sum_{\pi \in P} |Fix\,(\pi)| \quad = \quad 5+1+1+1 \quad = \quad 8$$

# Relationship Between Stabilizers and Orbits

**Lemma 14.3.2.** *Let $\mathcal{P} = [P : Y]$ be a permutation group and $y \in Y$. Then*

$$|\mathcal{S}tab\,(y)| \;=\; \frac{|P|}{|orbit(y)|}$$

**Pf:** Suppose that

$$orbit(y) \;=\; \{y = y_1,\, y_2,\, \ldots\, y_n\}$$

and that, for $j = 1, \ldots, n$, $P_j$ is the subset of permutations of $P$ that maps object $y$ to object $y_j$. Then the subsets

$$P_1,\, P_2,\, \ldots,\, P_n$$

partition the permutation group $P$, and $P_1 = \mathcal{S}tab\,(y)$.

For $j = 1, \ldots, n$, let $\pi_j$ be any permutation such that

$$\pi_j(y) \;=\; y_j$$

Then the rule $\pi \mapsto \pi \circ \pi_j$ (composition with $\pi_j$) is a bijection from $P_1$ to $P_j$, which implies that

$$|P_j| \;=\; |P_1| \;=\; |\mathcal{S}tab\,(y)|$$

for $j = 1, \ldots, n$.

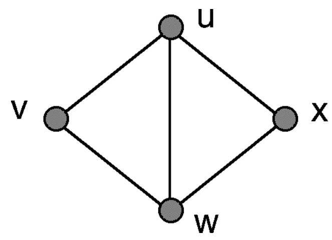
Since each of the $n$ partition cells $P_1, P_2, \ldots, P_n$ of group $P$ has cardinality $|\mathcal{S}tab\,(y)|$, it follows that

$$n \cdot |\mathcal{S}tab\,(y)| \;=\; |P|$$

But $n = |orbit(y)|$, which completes the proof. $\diamondsuit$

**Example 14.3.7:** Analysis of $\mathcal{A}ut_V(K_4 - K_2)$ continues.

| Symmetry | $\pi \in \mathcal{A}ut_V(G)$ |
|---|---|
| identity | $\epsilon = (u)(v)(w)(x)$ |
| refl. thru vert. axis | $\pi_1 = (u)(w)(v\ x)$ |
| refl. thru horiz. axis | $\pi_2 = (v)(x)(u\ w)$ |
| 180° rotation | $\pi_3 = (u\ w)(v\ x)$ |




The orbits of $\mathcal{A}ut_V(K_4 - K_2)$ are $\{u, w\}$ and $\{v, x\}$, both of cardinality 2. All the stabilizers are of cardinality 2, as determined in Example 14.3.1. The cardinality of the group $\mathcal{A}ut_V(K_4 - K_2)$ is 4. Thus, for each vertex, the equation

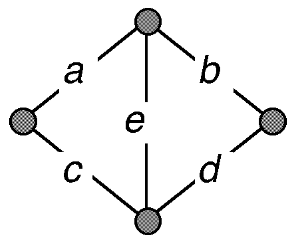
$$|\mathcal{S}tab\,(y)| \;=\; \frac{|P|}{|orbit(y)|}$$

takes the form

$$2 \;=\; \frac{4}{2}$$

**Example 14.3.8:** Analysis of $\mathcal{A}ut_E(K_4 - K_2)$ continues.

| Symmetry | $\pi \in \mathcal{A}ut_E(G)$ |
|---|---|
| identity | $\epsilon = (a)(b)(c)(d)(e)$ |
| refl. thru vert. axis | $\pi_1 = (e)(a\ b)(c\ d)$ |
| refl. thru horiz. axis | $\pi_2 = (e)(a\ c)(b\ d)$ |
| 180° rotation | $\pi_3 = (e)(a\ d)(b\ c)$ |




The orbits of $\mathcal{A}ut_E(K_4 - K_2)$ are $\{e\}$ and $\{a, b, c, d\}$. For each of the edges $a$, $b$, $c$, and $d$, the equation

$$|\mathcal{S}tab\,(y)| \;=\; \frac{|P|}{|orbit(y)|}$$

becomes

$$1 \;=\; \frac{4}{4}$$

For edge $e$, it becomes

$$4 \;=\; \frac{4}{1}$$

**Remark**: Although the next lemma is expressed in terms of the orbits of permutation groups, it is really a fact about set partitions.

**Lemma 14.3.3.** *Let $\mathcal{P} = [P : Y]$ be a perm gp w. $n$ orbits.*

Then
$$\sum_{y\in Y} \frac{1}{|orbit(y)|} \quad = \quad n$$

**Pf:** Let $Y_1, \ldots, Y_n$ be the orbits, so
$$Y \;=\; Y_1 \cup \cdots \cup Y_n$$

oIt follows that
$$\sum_{y\in Y} \frac{1}{|orbit(y)|} = \sum_{j=1}^{n} \sum_{y\in Y_j} \frac{1}{|orbit(y)|} = \sum_{j=1}^{n} \sum_{y\in Y_j} \frac{1}{|Y_j|}$$
$$= \sum_{j=1}^{n} \frac{1}{|Y_j|} \sum_{y\in Y_j} 1 \quad = \quad \sum_{j=1}^{n} \frac{1}{|Y_j|}|Y_j|$$
$$= \sum_{j=1}^{n} 1 \quad = \quad n \qquad \diamondsuit$$

**Example 14.3.9:** The left side of the equation in Lemma 14.3.3 is the sum of the reciprocals of the sizes of the cells in a partition. Thus, the sum of all the reciprocals must equal the number of cells, as on the right side of the equation in Lemma 14.3.3.

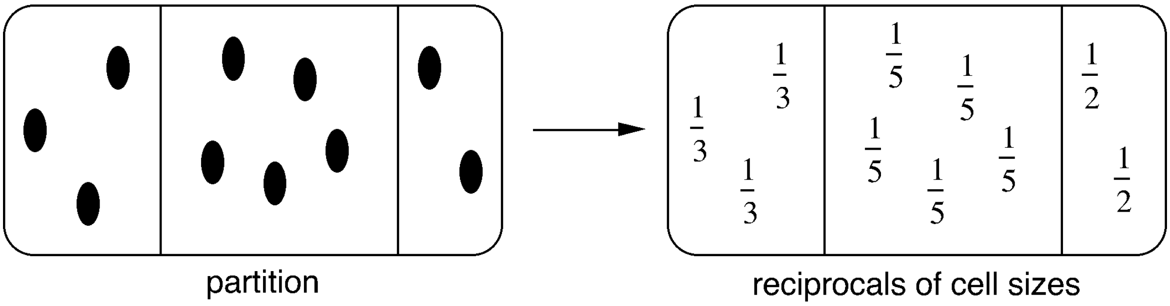



**Fig 14.3.1 Reciprocals of cell sizes of a partition.**

# Proof of Burnside's Lemma

**Theorem 14.3.4. (Burnside's lemma)** *Let $\mathcal{P} = [P : Y]$ be a permutation group with $n$ orbits. Then*

$$n = \frac{1}{|P|} \sum_{\pi \in P} |Fix(\pi)|$$

**Pf:** Lemmas 14.3.1, 14.3.2, and 14.3.3 establish the following chain of equalities, which proves Burnside's lemma.

$$\begin{aligned}
\frac{1}{|P|} \sum_{\pi \in P} |Fix(\pi)| &= \frac{1}{|P|} \sum_{y \in Y} |\mathcal{S}tab(y)| && \text{(Lemma 14.3.1)} \\
&= \frac{1}{|P|} \sum_{y \in Y} \frac{|P|}{|orbit(y)|} && \text{(Lemma 14.3.2)} \\
&= \frac{1}{|P|} |P| \sum_{y \in Y} \frac{1}{|orbit(y)|} \\
&= \sum_{y \in Y} \frac{1}{|orbit(y)|} = n \quad \text{(Lemma 14.3.3)} \quad \diamondsuit
\end{aligned}$$

# Direct Application of Burnside's Lemma

The most powerful applications of Burnside's lemma are not to counting vertex orbits or edge orbits, but rather, to counting induced equivalence classes, for which purpose they require the use of auxiliary results and techniques, which are the focus of the rest of the chapter.

Nonetheless, a direct orbit-counting application of Burnside's lemma to a permutation group $\mathcal{P} = [P : Y]$ would proceed as follows:

1. the values of $|Fix\,(\pi)|$ are added over all $\pi \in P$;
2. the resulting sum is divided by $|P|$.

The following two examples apply the direct method to counting vertex-orbits and edge-orbits.

**Example 14.3.10:** $\mathcal{A}ut_V(K_4{-}K_2)$ has four automorphisms. The sum of the sizes of their fixed-point sets, previously calculated in Example 14.3.5, is

$$\sum_{\pi\in P} |Fix\,(\pi)| \quad = \quad 4+2+2+0 \quad = \quad 8$$

The vertex orbits are $\{u, w\}$ and $\{v, x\}$. Since

$$|P| \;=\; |\mathcal{A}ut_V(K_4 - K_2)| \;=\; 4$$

Burnside's lemma implies correctly that the number of vertex orbits in $K_4 - K_2$ is

$$2 \;=\; \frac{1}{|P|}\sum_{\pi\in P} |Fix\,(\pi)| \;=\; \frac{1}{4}\cdot 8$$

**Example 14.3.11:** $\mathcal{A}ut_E(K_4 - K_2)$ has four automorphisms. The sum of the sizes of their fixed-point sets, previously calculated in Example 14.3.6, is

$$\sum_{\pi \in P} |Fix(\pi)| \quad = \quad 5+1+1+1 \quad = \quad 8$$

The edge orbits are $\{a, b, c, d\}$ and $\{e\}$. Since

$$|P| \;=\; |\mathcal{A}ut_V(K_4 - K_2)| \;=\; 4$$

Burnside's lemma implies correctly that the number of edge orbits is

$$2 \;=\; \frac{1}{|P|} \sum_{\pi \in P} |Fix(\pi)| \;=\; \frac{1}{4} \cdot 8$$

# 14.4 CYCLE-INDEX OF A PERM GROUP

Substituting the value $k$ into the *cycle-index polynomial* yields the number of equivalence classes of $(\leq k)$-colorings. This section examines some examples of applications of this substitution principle and then proves its correctness.

## Cycle-Structure Monomial of a Permutation

TERMINOLOGY: The ***cycle structure of a permutation*** is the number of cycles of each length in its disjoint-cycle form.

TERMINOLOGY: A ***monomial*** is a polynomial with only one term.

DEF: Let $\mathcal{P} = [P : Y]$ be a permutation group on a set $Y$ of $n$ objects, and let $\pi \in P$. The ***cycle-structure monomial*** of $\pi$ is the $n$-variable monomial

$$\zeta(\pi) \quad = \quad \prod_{k=1}^{n} z_k^{r_k} \quad = \quad z_1^{r_1} z_2^{r_2} \cdots z_n^{r_n}$$

where $z_k$ is a formal variable and $r_k$ is the number of $k$-cycles in the disjoint-cycle form of $\pi$.

**Example 14.4.1:** The permutation

$$\pi = (1\ 7\ 9\ 3)(2\ 4\ 8\ 6)(5)$$

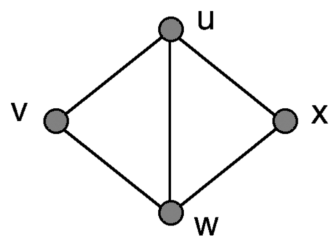
has cycle-structure monomial

$$\zeta(\pi) = z_1 z_4^2$$

because $\pi$ has one 1-cycle and two 4-cycles.

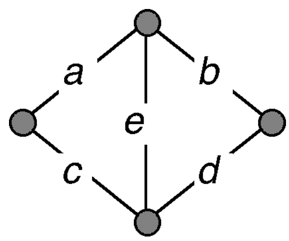
**Example 14.4.2:** The cycle structure of all the vertex-perms in $\mathcal{A}ut_V(K_4{-}K_2)$ is given in the following table.

| Symmetry | $\pi \in \mathcal{A}ut_V(G)$ | $\zeta(\pi)$ |
|---|---|---|
| identity | $(u)(v)(w)(x)$ | $z_1^4$ |
| refl. thru vert. axis | $(u)(w)(v\ x)$ | $z_1^2 z_2$ |
| refl. thru horiz. axis | $(v)(x)(u\ w)$ | $z_1^2 z_2$ |
| 180° rotation | $(u\ w)(v\ x)$ | $z_2^2$ |




**Example 14.4.3:** The cycle structure of all the edge-perms in $\mathcal{A}ut_E(K_4{-}K_2)$ is given in the following table.

| Symmetry | $\pi \in \mathcal{A}ut_E(G)$ | $\zeta(\pi)$ |
|---|---|---|
| identity | $(a)(b)(c)(d)(e)$ | $z_1^5$ |
| refl. thru vert. axis | $(e)(a\ b)(c\ d)$ | $z_1 z_2^2$ |
| refl. thru horiz. axis | $(e)(a\ c)(b\ d)$ | $z_1 z_2^2$ |
| 180° rotation | $(e)(a\ d)(b\ c)$ | $z_1 z_2^2$ |



# Cycle-Index Polynomial

The last two examples used the actions of $\mathcal{A}ut(G)$ on $V_G$ and $E_G$, but the remaining examples use the *induced* actions of $\mathcal{A}ut(G)$ on the sets $Col_k(V_G)$ and $Col_k(E_G)$.

DEF: Let $\mathcal{P} = [P : Y]$ be a permutation group on a set of $n$ objects. Then the ***cycle-index polynomial*** of $\mathcal{P}$ is the polynomial

$$\mathcal{Z}_{\mathcal{P}}(z_1, \ldots, z_n) \;=\; \frac{1}{|P|} \sum_{\pi \in P} \zeta(\pi)$$

where $\zeta(\pi)$ is the cycle-structure monomial of the perm $\pi$.

**Example 14.4.4:** From Example 14.4.2, it follows that the cycle-index polynomial of the perm group $\mathcal{A}ut_V(K_4{-}K_2)$ is

$$\mathcal{Z}_{\mathcal{A}ut_V(K_4\text{-}K_2)}(z_1, z_2) \;=\; \frac{1}{4}(z_1^4 + 2z_1^2 z_2 + z_2^2)$$

Substituting 2, the number of colors, for both of the variables $z_1$ and $z_2$ yields the number

$$\mathcal{Z}_{\mathcal{A}ut_V(K_4\text{-}K_2)}(2, 2) \;=\; \frac{1}{4}(2^4 + 2 \cdot 2^2 \cdot 2 + 2^2) \;=\; 9$$

which was shown in Figure 14.2.5 (previous lecture) to be the number of $\mathcal{A}ut_V(K_4{-}K_2)$-orbits of vertex-($\leq$2)-colorings of the graph $K_4{-}K_2$.

**Example 14.4.5:** From Example 14.4.3, it follows that the cycle-index polynomial of the perm group $\mathcal{A}ut_E(K_4{-}K_2)$ is

$$\mathcal{Z}_{\mathcal{A}ut_E(K_4\text{-}K_2)}(z_1, z_2) = \frac{1}{4}(z_1^5 + 3z_1z_2^2)$$

Again, substituting 2 for both variables, we obtain

$$\mathcal{Z}_{\mathcal{A}ut_E(K_4\text{-}K_2)}(2, 2) = \frac{1}{4}(2^5 + 3 \cdot 2 \cdot 2^2) = 14$$

which was shown in Figure 14.2.6 (previous lecture) to be the number of $\mathcal{A}ut_E(K_4{-}K_2)$-orbits of edge-($\leq$2)-colorings.

That the substitutions in the last two examples yielded the number of coloring classes is not a coincidence.

# Correctness of Substituting Into the Cycle-Index Polynomial

We now show as a consequence of Burnside's lemma (Theorem 14.3.4) that

- substituting the value $k$ into the cycle-index polynomial *always* yields the number of coloring classes (orbits) of the $(\leq k)$-colorings.

We now review the terminology and the notations:

REVIEW FROM §14.2: Let $\mathcal{P} = [P : Y]$ be a permutation group acting on a set $Y$, and let $\pi \in P$.

- The mapping $\pi_{YC} : Col_k(Y) \rightarrow Col_k(Y)$ defined by

  $$\pi_{YC}(f) = f\pi \quad \text{for every coloring } f \in Col_k(Y)$$

  is a permutation on the set $Col_k(Y)$, called the ***induced permutation action*** of $\pi$ on $Col_k(Y)$.

- To distinguish between the action of a permutation $\pi$ on a set $Y$ and its induced action on the set $Col_k(Y)$ of colorings of $Y$, we let $\pi_Y$ denote its action on $Y$ and $\pi_{YC}$ its action on $Col_k(Y)$. When there is no risk of confusion, the subscripts $Y$ and $YC$ may both be omitted.

- The collection $\mathcal{P}_{\mathcal{C}} = [P : Col_k(Y)]$ of induced permutations on $Col_k(Y)$ is called the ***induced permutation group***.
- The set of orbits (coloring classes) of the induced permutation group on $Col_k(Y)$ is denoted $\{Col_k(Y)\}_{\mathcal{P}}$.
- When it is necessary to distinguish between the group that acts on the set $Y$ and the group that acts on the set $Col_k(Y)$, the respective notations $P_Y$ and $P_{YC}$ are used.

NOTATION: If $p(x_1, \ldots, x_n)$ is a multivariate polynomial, then $p(k, \ldots, k)$ denotes the result of substituting the value $k$ for every variable $x_j$.

**Lemma 14.4.1.** *Let $\mathcal{P} = [P : Y]$ be a permutation group, and let $\pi \in P$, with induced action $\pi_{Y^C}$ on $Col_k(Y)$. Then the number of ($\leq k$)-colorings of $Y$ that are fixed by $\pi_{Y^C}$ is given by*

$$|Fix\,(\pi_{Y^C})| \;=\; \zeta(\pi_Y)(k,\ldots,k)$$

**Pf:** A ($\leq k$)-coloring $c$ is fixed by the induced permutation $\pi_{Y^C}$ iff within each cycle of the permutation $\pi_Y$, all the objects are assigned the same color by $c$. Thus, there are $k$ independent choices possible for each cycle of $\pi_Y$. Therefore,

$$|Fix\,(\pi_{Y^C})| \;=\; k^n$$

where $n$ is the number of cycles in $\pi_Y$. But $k^n$ is precisely the value of $\zeta(\pi_Y)(k,\ldots,k)$. ♢

**Example 14.4.6:** Recall from Figure 14.2.2 the induced permutations from the action of $\mathcal{A}ut(C_3)$ on $Col_2(C_3)$. The following table illustrates the application of Lemma 14.4.1.

| $\pi_Y \in P_Y$ | $\pi_{Y^C} \in P_{Y^C}$ | $\zeta(\pi_Y)$ | $\lvert Fix\,(\pi_{Y^C})\rvert$ |
|---|---|---|---|
| $\epsilon_Y$ | $\epsilon_{Y^C}$ | $z_1^3$ | 8 |
| $(x\ y\ z)$ | (111)(211 121 112)<br>(122 212 221)(222) | $z_3$ | 2 |
| $(x\ z\ y)$ | (111)(211 112 121)<br>(122 221 212)(222) | $z_3$ | 2 |

**Theorem 14.4.2.** *Let $\mathcal{P} = [P : Y]$ be a permutation group. Then*

$$|\{Col_k(Y)\}_{\mathcal{P}}| = \mathcal{Z}_{\mathcal{P}}(k, \ldots, k)$$

**Pf:** Applying Burnside's lemma to the induced permutation group $\mathcal{P}_{\mathcal{C}}$ yields the equation

$$\begin{aligned} |\{Col_k(Y)\}_{\mathcal{P}}| &= \frac{1}{|P_C|} \sum_{\pi_{Y^C} \in P_C} |Fix\,(\pi_{Y^C})| \\ &= \frac{1}{|P_C|} \sum_{\pi_{Y^C} \in P_C} \zeta(\pi_Y)(k, \ldots, k) \;\; \text{(Lemma 14.4.1)} \\ &= \frac{1}{|P_Y|} \sum_{\pi_Y \in P_Y} \zeta(\pi_Y)(k, \ldots, k) \\ &= \mathcal{Z}_{\mathcal{P}}(k, \ldots, k) \end{aligned}$$

$\diamondsuit$

**Example 14.4.7:** $\mathcal{A}ut(P_5)$ has two automorphisms, the identity and the reflection, and the cycle-index polynomial for the group $\mathcal{A}ut_V(P_5)$ is given by

$$\mathcal{Z}_{\mathcal{A}ut_V(P_5)}(z_1, z_2) = \frac{1}{2}(z_1^5 + z_1 z_2^2)$$

Substituting 2 for both the variables yields

$$\mathcal{Z}_{\mathcal{A}ut_V(P_5)}(2, 2) = \frac{1}{2}(2^5 + 2 \cdot 2^2) = 20$$

which is the number of vertex-($\leq$2)-colorings calculated in Example 14.2.5.

Similarly, the cycle-index polynomial for $\mathcal{A}ut_E(P_5)$ is

$$\mathcal{Z}_{\mathcal{A}ut_E(P_5)}(z_1, z_2) \;=\; \frac{1}{2}(z_1^4 + z_2^2)$$

and, hence,

$$\mathcal{Z}_{\mathcal{A}ut_E(P_5)}(2, 2) \;=\; \frac{1}{2}(2^4 + 2^2) = 10$$

which is the number of edge-($\leq$2)-colorings calculated in Example 14.2.6.

Moreover, substituting 3 for both the variables yields

$$\mathcal{Z}_{\mathcal{A}ut_E(P_5)}(3, 3) \;=\; \frac{1}{2}(3^4 + 3^2) \;=\; 45$$

which was calculated in Example 14.2.7 to be the number of edge-($\leq$3)-colorings of $P_5$.

# 14.7 SUPPLEMENTARY EXERCISES

*In Exercises 14.7.1 through 14.7.10, do the following for the given graph:*

*b. Write the cycle index of the vertex automorphism group.*

*c. Use Burnside enumeration to count the essentially different ways to color the vertices with at most two colors.*

*g. Write the cycle index of the edge automorphism group.*

*h. Use Burnside enumeration to count the essentially different ways to color the edge with at most two colors.*

14.7.**1** The cartesian product $C_3 \times P_3$.

14.7.**2** The graph obtained by joining a 5-cycle to $K_2$.