An Intuitionistic Completeness Theorem for Classical First-Order Logic

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October 28, 2019

Abstract

Gödel proved the *completeness* of classical First-Order Logic (FOL) in 1929. His proof is quite hard to understand, and several alternative accounts are now available, including the very readable proof in Smullyan's excellent small textbook, *First-Order Logic* [4]. An interesting question about these proofs is whether they are constructive. Here we show that Smullyan's account is intuitionistically provable. The intuitionistic part is the use of *Brouwer's Fan Theorem.*

We note for historical interest that Church [2] proved in 1936 that FOL *validity* is *undecidable.*

1 Smullyan's Completeness Theorem for Classical First-Order Logic

Smullyan's 158 page book *First-Order Logic* [4] uses a proof system called *Analytic Tableaux.* It is very easy to understand, and the proof rules are simple. The motivating idea is that we attempt to falsify the proposition, say P. If our approach is sufficiently general and yet we fail to falsify P, then we know that P is valid. Moreover, the record of our failed attempt to falsify the formula can be seen as a proof of the formula.

Smullyan has only four rules for building FOL proofs. They are presented on page 53 in a particularly concise format. Here are the two propositional rules, A, B followed by the two quantifier rules, C, D.

A: $\alpha/\alpha_1, \alpha_2$ **B:** $\beta/\beta_1|\beta_2$ **C:** $\gamma/\gamma(a)$ a is *any* parameter **D:** $\delta/\delta(a)$ a is a *new* parameter.

Typically Smullyan presents the rules using *signed formulas*. Those are formulas preceded by either T or F, for True and False. Here are the quantifier rules in terms of *signed formulas*:

C $T(\forall x.A(x))\backslash TA(a)$ for a a parameter.

C $F(\exists x.A(x))\backslash FA(a)$ for a a parameter

D $T(\exists x)A(x)\backslash TA(a)$ for a a *new* parameter.

D $F(\forall x.A(x))\backslash FA(a)$ for a a *new* parameter.

On page 57 Smullyan defines an *Hintikka set* over a domain of discourse D. He proves the easy theorem that every Hintikka set S over non-empty D is first-order satisfiable in that domain. This is an easy lemma, so we simply cite Smullyan's proof on page 58 of his book. It is interesting that for FOL, unlike for the propositional calculus, the construction of an Hintikka set might generate an unbounded path. It is important to guarantee that this path will be a Hintikka set itself. On page 69 Smullyan shows how to guarantee this. It is a simple matter. If we encounter an α node we add α_1 and α_2 to the path. If we encounter a β node, we branch. If we encounter a δ node we, take the first parameter a not already in the tree and add $\delta(a)$. The delicate case is when we encounter a γ node. In this case, we find the first parameter a such that $\gamma(a)$ does not occur in the tree and extend every path θ to $(\theta, \gamma(a), \gamma)$. We are adding $\gamma(a)$ and then repeating the formula γ.

Definition: We call a systematic tableaux *finished* if it cannot be extended by the procedure we just defined or it is unbounded. By Smullyan's *Theorem 1* on page 60 in the unbounded case, every open branch is a Hintikka sequence, hence is first-order satisfiable.

We can now prove the completeness theorem as in Smullyan. He relies on *König*'s Lemma page 32, namely: Every finitely generated tree T with infinitely many points must contain at least one infinite branch. He cites this theorem on page 61 to claim that a closed infinite tableaux is impossible. His proof of this lemma on page 32 is not constructive. That means that his completeness theorem is not constructive.

However, as we will see later in the course, using Brouwer's intuitionistic mathematics, we can cite his famous *Fan Theorem* from 1924. This is a constructive result. It is discussed in Crystal Cheung's Cornell Master's Thesis [1]. Her excellent thesis is available on the course web page for Lecture 2 and on the PRL project web page as well. This means that it is plausible that *our version of Smullyan's proof is intuitionistically valid*. This is not a very widely known fact. Indeed there are published claims that it is not possible.

What is more widely known is that FOL completeness is not constructive in the more narrow sense of the term. I have not yet written the observation about an intuitionistic proof for my colleagues to check, so it is best for now to take this as a *plausible conjecture* with some evidence, but as yet no formally verified proof in intuitionistic type theory. We will soon be able to check this idea using Nuprl because *Nuprl implements Brouwer's intuitionistic mathematics.*

Completeness Theorem for FOL: If X is a valid FOL formula, then the systematic tableaux for X will close after finitely many steps. This *closed tableaux* is a formal *first-order classical proof* of X.

There is evidence that the above theorem is intuitionistically valid. A much harder result is a proof that intuitionistic FOL, iFOL, is complete. New semantic methods are needed for the result obtained at Cornell, that iFOL is complete with respect to uniform evidence semantics [3].

References

[1] Crystal Cheung. Brouwer's Fan Theorem: An Overview. Master's thesis, Cornell University, Ithaca, NY, 2015.

[2] Alonzo Church. An unsolvable problem of elementary number theory. *American Journal of Math*, 58:345–363, 1936.

[3] Robert Constable and Mark Bickford. Intuitionistic Completeness of First-Order Logic. *Annals of Pure and Applied Logic*, 165(1):164–198, January 2014.

[4] R. M. Smullyan. *First–Order Logic*. Springer-Verlag, New York, 1968.