Practice Final

The following problem set has 20 sub-problems. This is likely too long for two hours, but see how much you can do – most of the problems have simple answers. In the exam, you may use any written material you want (textbook, notes, review notes, assignments and solutions, etc), but not a friend or the internet.

1: About rates Consider the iteration

$$x_{k+1} = x_k + \cos(x_k).$$

1. What are the fixed points?
2. Classify each fixed point as attractive or repulsive.
3. For the attractive fixed points, what is the rate of convergence?

2: Rewrite for accuracy Consider $f(x) \equiv \sqrt{1+x} - \sqrt{1-x}$ when $x \ll 1$. Write a routine to compute f without catastrophic cancellation.

3: Q-less QR MATLAB's sparse QR solver computes a "Q-less" QR decomposition, i.e. a sparse R is computed explicitly but not the dense Q factor. Given a sparse triangular R and a sparse A, describe:

1. How would one efficiently solve a least squares problem involving A?
2. Solves with Q-less QR iteration are less well-behaved than those involving standard QR, and so a typical implementation will do a step of iterative refinement. Write a MATLAB fragment to describe iterative refinement in this setting.

4: Funky fill Consider an SPD matrix with the nonzero pattern

$$A = \begin{bmatrix} \times & & & \times \\ & \times & \times & \times \\ & \times & \times & \\ \times & \times & & \times \end{bmatrix}.$$

What is the nonzero structure of the Cholesky factor?

5: Frobenius Write a line of MATLAB to minimize $\|XA - B\|_F^2$ where $X \in \mathbb{R}^{n \times m}$ is unknown and $A \in \mathbb{R}^{m \times p}$ and $B \in \mathbb{R}^{n \times p}$ are given, with $m < p$.

6: Residual Suppose $r = Ax - b$ is the residual in a least squares problem. Given only b and the Q factor in an economy QR decomposition of A, what is $\|r\|$?

7: Constrained LS Write a routine to minimize $\|Ax - b\|^2$ subject to $\sum_{j=1}^{n} x_j = 1$.

8: Transfer trouble Suppose $A = QHQ^T$ is upper Hessenberg. Argue that the Hessenberg form provides a way of computing the transfer function $h(s) = c^T(A - sI)^{-1}b$ in $O(n^2)$ time for arbitrary s. Give MATLAB code; you may assume backslash with a Hessenberg matrix requires $O(n^2)$ time (because it does!) – explain why.

9: Diagonal decisions Consider the block 2×2 matrix

$$M = \begin{bmatrix} A & B \\ B & D \end{bmatrix}$$

where $A, B, D \in \mathbb{R}^{n \times n}$ are orthogonal. Describe an efficient algorithm for computing all the eigenvalues of M.

10: Jacobi jumble What is the rate of convergence of Jacobi on a diagonal matrix?

11: Killer Krylov Suppose $\hat{A} = A + uv^T$. Given A^{-1} as a preconditioner, how many steps of a Krylov subspace method are required to solve a system with $\hat{A}$, and why?

12: Line search Does the Armijo condition for a line search guarantee that $\|x_{k+1} - x_*\| < \|x_k - x_*\|$? Why or why not?

13: Simple solver Suppose $(A + \eta \operatorname{diag}(x))x = b$. Under what conditions does the fixed point iteration

$$Ax_{k+1} = b - \eta \operatorname{diag}(x_k)x_k$$

converge? Give a bound on the rate of convergence.

14: Differential deal Suppose H is positive semi-definite, and consider the trust region step

$$(H + \lambda I)p = -\nabla\phi.$$

show that at $\lambda = 0$, $dp/d\lambda = H^{-2}\nabla\phi$.

15: Modified Gauss-Newton Consider the Gauss-Newton-like iteration

$$p_k = M^{-1}J(x_k)^T r(x_k)$$

where M is a fixed positive definite matrix that we hope approximates the matrix $J^T J$ at the minimizer of $\phi(x) = \|r(x)\|^2/2$.

1. Write a short MATLAB code to implement the iteration efficiently. You may take $O(mn^2)$ setup time, but you should only require $O(mn)$ time per step.
2. Show that p_k is a descent direction (so the iteration will converge with line search).
3. Give conditions on M and r such that the iteration is guaranteed to be locally convergent without line search. You may assume the Jacobian satisfies a Lipschitz condition $\|J(x) - J(y)\| \leq \gamma\|x - y\|$.