WEAK REPRESENTATIONS OF RELATION ALGEBRAS AND RELATIONAL BASES

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Abstract. It is known that for all finite $n \geq 5$, there are relation algebras with n-dimensional relational bases but no weak representations. We prove that conversely, there are finite weakly representable relation algebras with no n-dimensional relational bases. In symbols: neither of the classes RA_n and wRRA contains the other.

The main result of this paper concerns the relationship between two classes each of which contains the class of representable relation algebras (RRA). Representable relation algebras are isomorphic to genuine fields of binary relations with natural set-theoretically defined operators (union, set complement, identity, converse and composition). It is known that RRA is badly behaved in a number of ways. RRA cannot be defined by finitely many axioms, nor by any set of equations using only finitely many variables, nor by any set of first-order sentences containing only finitely many non-canonical sentences. It is undecidable whether a finite relation algebra is representable or not. The equational theory of RRA is undecidable. See, e.g., [16, 9, 7, 5, 18]. Consequently, researchers have defined and investigated other classes of relation algebras, not identical to RRA but with some common features.

One of these classes is the class of *weakly representable relation algebras* (wRRA), which have representations rather like classical representations, but the Boolean sum and negation operators are not required to be interpreted as set union and complement respectively [8, page 459]. Clearly $\mathsf{wRRA} \supseteq \mathsf{RRA}$ and it turns out that the inclusion is proper.

Another kind of representation which allows us to generalise RRA is called a *relativised representation.* The definition is like the definition of a classical representation, but all operators are relativised to the unit element in the representation, which is simply a reflexive and symmetric binary relation containing all the others relations. The class of relation algebras with relativised representations turns out to be very well behaved—all relation algebras have relativised representations, the class of relation-type algebras with relativised representations is finitely axiomatisable [12, theorem 5.20], and all finite relation algebras have finite relativised representations [5, theorem 19.13]. There are many variants of this definition which impose different restrictions on the type of unit that is allowed. A set of points in the domain of a relativised representation is called a *clique* if every pair of points from the set belongs to the unit of the representation. A special kind of relativised representation, called an *n-square representation,* has the following property: for any clique C with fewer than n

points, and any pair of points from the clique, belonging to the representation of a composition of two elements from the algebra, C is contained in some clique that contains a point witnessing the composition. The class of relation algebras with n-square representations is called RA_n (though the standard definition of this class is given in terms of *bases*: see below). Although $\mathsf{RA}_\omega = \mathsf{RRA}$ (see [13, theorem 6] or [15, theorem 418]), RA_n is quite well-behaved for finite n. Every finite relation algebra in RA_n has a finite n-square representation [5, theorem 19.18] and a finite n-dimensional relational basis [15, theorems 411, 325], and it follows that the problem of determining whether a finite relation algebra belongs to RA_n is decidable. The classes $\mathsf{RA}_n : n < \omega$ form a sequence of better and better approximations to RRA in the sense that $\mathsf{RA} = \mathsf{RA}_4 \supset \mathsf{RA}_5 \supset \mathsf{RA}_6 \supset \cdots$ (these containments are strict [14]—indeed, for $n \geq 4$, RA_{n+1} is not finitely axiomatisable over RA_n [5, theorem 17.18]) and

$$(1) \qquad \bigcap_{4 \leq n < \omega} \mathsf{RA}_n = \mathsf{RA}_\omega = \mathsf{RRA}.$$

These two kinds of generalisation of classical representability have very different definitions and each class has its strengths and weaknesses. For quite some time it was not known whether wRRA contained RA_n (any n with $5 \leq n < \omega$) nor whether RA_n contained wRRA. The Jónsson axioms (4), (5), (6) below are all valid over both classes (see remark 0.12 below for more about this) so they have something in common. Jónsson [8, problem 3] asked whether the inclusion $\mathsf{wRRA} \supseteq \mathsf{RRA}$ is strict, and Andréka [1] proved that it is, but the algebras she constructs turn out to belong to RA_5 and RA_6. Of course, each of Andréka's algebras must fail to belong to RA_n for some $n < \omega$ by (1), since they have no classical representations, and therefore $\mathsf{wRRA} \not\subseteq \mathsf{RA}_n$, for some $n < \omega$, but we do not know the least value of n such that one or her weakly representable algebras is not in RA_n, and this least value is certainly greater than six, so her construction does not tell us exactly how these classes relate to each other.

Maddux suggested that the relationship between RA_n and wRRA should be clarified, and in particular he asked about the relationship between RA_5 and wRRA. This problem was stated as [5, problem 18.26]. In [5, corollary 18.25] it was shown that $\mathsf{RA}_5 \not\subseteq \mathsf{wRRA}$ (and indeed that $\mathsf{RA}_n \not\subseteq \mathsf{wRRA}$ for every finite $n \geq 3$). In this paper, we prove that there are weakly representable relation algebras not belonging to RA_5 (hence not belonging to RA_n for $5 \leq n < \omega$). So neither wRRA nor RA_5 contains the other.

We will say a bit more about these classes, after we have given the formal definitions.

DEFINITION 0.1 (Tarski). A *relation algebra* $\mathcal{A} = (A, 0, 1, +, \cdot, -, 1', \breve{\ }, ;)$ is a Boolean algebra $(A, 0, 1, +, \cdot, -)$ together with a constant $1' \in A$, a unary function $\breve{\ }$, and a binary function $;$, such that

- $(A, ; , 1')$ is a monoid,
- $(a;b) \cdot c = 0 \iff (\breve{a};c) \cdot b = 0 \iff a \cdot (c;\breve{b}) = 0$, for all $a, b, c \in A$ (the so-called 'Peircean law').

The class of all relation algebras is denoted RA.

DEFINITION 0.2 (Jónsson). Let $\mathcal{A}$ be a relation algebra. A *weak representation* h over the base set X is a 1–1 map $h : \mathcal{A} \to \wp(X \times X)$ such that for all elements a, b of $\mathcal{A}$, the following hold:

$$h(0) = \emptyset$$
$$h(a \cdot b) = h(a) \cap h(b)$$
$$h(1') = \{(x, x) : x \in X\}$$
$$h(\breve{a}) = \{(x, y) : (y, x) \in h(a)\}$$
$$h(a;b) = \{(x, y) : \exists z \, ((x, z) \in h(a) \wedge (z, y) \in h(b))\}$$

The class wRRA is defined to be the class of relation algebras with weak representations.

Until recently, not much was known about wRRA. The class is clearly closed under subalgebras and direct products, and by [17] we now know that this class is also closed under homomorphic images and therefore forms an equational variety. But it is not finitely axiomatisable [3, 4, 6], and there is no algorithm for determining whether a finite relation algebra has a weak representation [5]. Indeed, we have:

THEOREM 0.3 ([5, theorem 18.23]). *If K is any class of relation algebras such that* RRA $\subseteq K \subseteq$ wRRA *then the problem of determining whether an arbitrary finite relation algebra belongs to K is undecidable.*

So in one sense, RRA and wRRA are very close: restricted to finite algebras, RRA and (the complement of) wRRA are recursively inseparable.

We now come to the classes RA_n. To define them, we need to consider *atoms* and *networks*. Let $\mathcal{A}$ be a relation algebra. We can define a Boolean ordering by $x \leq y \iff x + y = y$. An *atom* of $\mathcal{A}$ is a minimal non-zero element in this ordering. We write $At(\mathcal{A})$ for the set of atoms of $\mathcal{A}$. $\mathcal{A}$ is *atomic* if every non-zero element is above some atom. Because $(x \mapsto \breve{x})$ is an automorphism of the Boolean reduct of $\mathcal{A}$, the converse of an atom is an atom.

DEFINITION 0.4 (Lyndon). Let $\mathcal{A}$ be an atomic relation algebra. The *atom structure* $\alpha(\mathcal{A}) = (At(\mathcal{A}), Id, \breve{}, C)$ of $\mathcal{A}$ is a four-tuple where Id is the set of atoms below the identity of $\mathcal{A}$, $\breve{}$ is the restriction of the converse operator of $\mathcal{A}$ to atoms, and C is the set of triples of atoms (a, b, c) such that $a;b \geq c$.

The *Peircean transforms* of a triple of atoms (a, b, c) are

(2) $\qquad (a, b, c), (b, \breve{c}, \breve{a}), (\breve{c}, a, \breve{b}), (\breve{b}, \breve{a}, \breve{c}), (\breve{a}, c, b), (c, \breve{b}, a)$

It follows from the relation algebra axioms that if $(a, b, c) \in C$, then all Peircean transforms of (a, b, c) also belong to C.

Conversely, given a four-tuple $\alpha = (A, Id, \breve{}, C)$, where $Id \subseteq A$, $\breve{} : A \to A$ and $C \subseteq A \times A \times A$, we define the *complex algebra* $\mathfrak{Cm}(\alpha)$ to be the algebra $(\wp(A), \emptyset, A, \cup, \cap, \setminus, Id, \breve{}, ;)$ where $\breve{S} = \{\breve{s} : s \in S\}$ for any $S \subseteq A$, and $S;T = \{u \in A : \exists s \in S, \ t \in T, \ (s, t, u) \in C\}$ for any $S, T \subseteq A$.

For a finite relation algebra $\mathcal{A}$ we have $\mathfrak{Cm}(\alpha(\mathcal{A})) \cong \mathcal{A}$ (see [10, §4]), so we can define $\mathcal{A}$, up to isomorphism, by defining its atom structure.

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The following definition differs from the definition of a relation algebra network [5, definition 7.1], where we only require that $N_2(x,y);N_2(y,z) \cdot N_2(x,z) \neq 0$ and the definition of network inclusion is also weaker, but the stricter definition here can be used to find sufficient conditions for weak representability.

DEFINITION 0.5. Let $\mathcal{A}$ be a relation algebra. A *network* over $\mathcal{A}$ is a pair (N_1, N_2), where N_1 is a set (of 'nodes') and $N_2 : N_1 \times N_1 \to \mathcal{A}$ is a function such that

$$N_2(x,x) \leq 1\text{'}$$
$$N_2(y,x) = N_2(x,y)^{\smile}$$
$$N_2(x,y);N_2(y,z) \geq N_2(x,z)$$

for $x,y,z \in N_1$. The network is said to be *strict* if $N_2(x,y) \leq 1\text{'} \Rightarrow x = y$.

Let $(M_1, M_2), (N_1, N_2)$ be networks. We write $(M_1, M_2) \leq (N_1, N_2)$ if $M_1 \subseteq N_1$ and $N_2{\restriction}_{M_1 \times M_1} = M_2$. If $(N_1^0, N_2^0) \leq (N_1^1, N_2^1) \leq \cdots \leq (N_1^n, N_2^n) \leq \cdots$ is a countable sequence of nested networks then we define the *limit* $\bigcup_{n<\omega}(N_1^n, N_2^n) = (N_1, N_2)$ by $N_1 = \bigcup_{n<\omega} N_1^n$ and for any $x,y \in N_1$ we let $N_2(x,y) = N_2^n(x,y)$ for any $n < \omega$ such that $x,y \in N_1^n$ (since the networks are nested this is well-defined).

Now suppose that $\mathcal{A}$ is atomic. Let $N = (N_1, N_2)$ be a network over $\mathcal{A}$. N is said to be *atomic* if $N_2(x,y) \in At(\mathcal{A})$ for every $x,y \in N_1$. (Sometimes these networks have been called *basic matrices*.) An *atomic refinement* of N is a pair (N_1, N_2^a) such that $N_2^a(x,y) \in At(\mathcal{A})$ and $N_2^a(y,x)^{\smile} = N_2^a(x,y) \leq N_2(x,y)$ for every $x,y \in N_1$; it may or may not be a network.

In this paper we drop the subscripts and use N to refer to the network (N_1, N_2), the set of nodes N_1 and the labelling N_2, distinguishing these uses by context.

DEFINITION 0.6 (Maddux, [15, §6.21, §6.24]). Let $\mathcal{A}$ be an atomic relation algebra and $4 \leq n < \omega$.[1] An *n-dimensional relational basis for* $\mathcal{A}$ is a set B of atomic networks over $\mathcal{A}$ with nodes $\{0, 1, \ldots, n-1\}$ and such that

- for all $a \in At(\mathcal{A})$ there is $N \in B$ with $N(0,1) = a$, and
- for all $N \in B$, $i,j,k < n$ and $a,b \in At(\mathcal{A})$, if $k \notin \{i,j\}$ and $a;b \geq N(i,j)$ then there is $M \in B$ with $M(x,y) = N(x,y)$ for every $x,y \in \{0, \ldots, n-1\} \setminus \{k\}$ (we write $M \equiv_k N$ for this), $M(i,k) = a$, and $M(k,j) = b$.

The class RA_n is defined to be the closure under subalgebras of the class of atomic relation algebras with n-dimensional relational bases.

This definition turns out to be equivalent to the sketched definition we gave in the introduction (see [5, theorem 13.46] for a more precise formulation and proof of this). RA_n has at least two positive features: it is a canonical variety (by [13, theorems 8,9] or [15, theorems 414, 420]), and there is an algorithm that determines whether a finite relation algebra belongs to this class or not (by [15, theorems 411, 325]). This latter property is also a consequence of the following easy lemma.

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[1]RA_2 and RA_3 can be defined as well [15, §6.24], but we are not concerned with these cases here.

LEMMA 0.7. *Let $4 \leq n < \omega$ and let $\mathcal{A} \subseteq \mathcal{B}$ be atomic relation algebras. If $\mathcal{A}$ is finite and $\mathcal{B}$ has an n-dimensional relational basis then so does $\mathcal{A}$.*

Hence, a finite relation algebra $\mathcal{A}$ belongs to RA_n if and only if $\mathcal{A}$ has an n-dimensional relational basis (no need to worry about the case where $\mathcal{A}$ has no basis but an extension of $\mathcal{A}$ does). From this, a non-deterministic algorithm to test whether $\mathcal{A} \in \mathsf{RA}_n$ picks an arbitrary set B of maps $f : \{0, \ldots, n-1\}^2 \to At(\mathcal{A})$ (there are only finitely many such maps, in fact $O(|At(\mathcal{A})|^{n^2})$), checks to see if each map in B is a network, and then checks whether B forms a relational basis.

The fact that we can tell by an algorithm whether a finite relation algebra is in RA_n distinguishes these classes from wRRA. Since $\mathsf{RA}_n \supseteq \mathsf{RRA}$, it follows from theorem 0.3 above that RA_n cannot be contained in wRRA, for any n with $5 \leq n < \omega$.[2] Indeed, the undecidability proof for theorem 0.3 yields an example of a finite relation algebra with a five-dimensional relational basis but no weak representation. If τ is a non-tiling instance of the *deterministic tiling problem* with the property that for each $t \in \tau$ there are four tiles in τ that match t on the left, right, top and bottom respectively, then it can be checked that the relation algebra $RA(\tau)$ of [5, section 18.3] is not weakly representable but does have a five-dimensional relational basis. So we have:

PROPOSITION 0.8 ([5, corollary 18.25]). *For all $n \geq 5$, there are finite relation algebras in $\mathsf{RA}_n \setminus \mathsf{wRRA}$.*

The remaining problem, then, is to show that the other inclusion fails—$\mathsf{wRRA} \not\subseteq \mathsf{RA}_n$, for $n \geq 5$. A fairly direct proof that $\mathsf{wRRA} \not\subseteq \mathsf{RA}_n$ for $n \geq 6$ can be obtained using the *rainbow algebras* of [5, chapter 16] (see remark 0.21 below), but it seems impossible to use that construction to find an algebra in $\mathsf{wRRA} \setminus \mathsf{RA}_5$. This is the problem we tackle in the remainder of this paper.

We will need to show that the algebra we construct is weakly representable. To that end, we now define a game to establish weak representability.

DEFINITION 0.9. Let $\mathcal{A}$ be a relation algebra. The two player game $\mathcal{G}(\mathcal{A})$ has ω rounds. A play of the game consists of a countable sequence of strict networks

$$(3) \qquad\qquad N_0 \leq N_1 \leq N_2 \leq \ldots$$

In the initial round, $\forall$ picks non-zero $a_0 \in \mathcal{A}$ and $\exists$ has to play a strict network N_0 containing nodes x_0, y_0, say, such that $N_0(x_0, y_0) = a_0$. In round $k > 0$ of the play, N_{k-1} has been defined in the previous round, $\forall$ picks two nodes $m, n \in N_{k-1}$ and two elements $Y, Z \in \mathcal{A}$ such that $Y; Z \geq N_{k-1}(m, n)$. We denote this move as (m, n, Y, Z). $\exists$ is required to respond with a strict network $N_k \geq N_{k-1}$ containing a node p such that $N_k(m, p) \leq Y$ and $N_k(p, n) \leq Z$. If, in some round, she fails to provide a suitable extension network then she loses the play. Otherwise, she wins.

The main difference between this game and the classical representation games of [5, chapter 7] is that the labels of edges of networks are not refined, and indeed in each round when $\exists$ extends the previous network she is not allowed to refine

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[2]It also holds for $n = 4$, but that was known from the beginning [8].

the labels on edges. This is essentially because a weak representation is not required to respect $+$ or $-$.

PROPOSITION 0.10. *Let $\mathcal{A}$ be a countable relation algebra. If $\exists$ has a winning strategy in the game $\mathcal{G}(\mathcal{A})$ then $\mathcal{A}$ is weakly representable.*

To prove the proposition, suppose $\exists$ has a winning strategy in the game. For each $a \in \mathcal{A}$ let $N_0 \le N_1 \le \ldots$ be a play of the game in which $\forall$ plays a initially and schedules all possible moves into the subsequent play of the game, *i.e.*, if $i < \omega$, $m, n \in N_i$ and $U; V \ge N_i(m, n)$ then eventually $\forall$ plays (m, n, U, V)—and $\exists$ uses her winning strategy. Since the networks are finite and $\mathcal{A}$ is countable, this scheduling can be done. Let N^a be the limit of this play. By renaming nodes we can arrange that the nodes of N^a and N^b are disjoint, when $a \ne b$. Let X be the disjoint union of the nodes of the N^as, as a ranges over $\mathcal{A}$. Define a map $h : \mathcal{A} \to \wp(X \times X)$ by $h(b) = \{(x, y) : \exists a \in \mathcal{A}(x, y \in N^a \land N^a(x, y) \le b)\}$. It is easy to check that h is a weak representation.

PROBLEM 0.11. Let $\mathcal{A}$ be a relation algebra and define a *weak complete representation* h of X to be a weak representation such that for any subset S of $\mathcal{A}$ whose infimum $\inf(S)$ exists in $\mathcal{A}$ we have $h(\inf(S)) = \bigcap\{h(s) : s \in S\}$.

Is it the case that every weakly representable relation algebra has a weak complete representation? If not, is the class of relation algebras with weak complete representations an elementary class?

If $\mathcal{A}$ has a weak complete representation, must $\exists$ have a winning strategy in $\mathcal{G}(\mathcal{A})$?

REMARK 0.12. We have some comments on this game, the original axioms for wRRA given by Jónsson (which include (4), (5), (6) below), and the Lyndon conditions. For those readers who have come across some of this material we hope that the following remarks will shed light on how we constructed a finite relation algebra in wRRA $\setminus$ RA$_5$. For other readers it might be more convenient to skip this remark.

In [8, theorem 1], Jónsson gave a recursive axiomatisation of wRRA, namely the axioms of RA plus the following conditions on an algebra $\mathcal{A}$: suppose $1 \le n \in \omega$, $a_0, a_1, \cdots, a_{2n}$ are in $\mathcal{A}$, and $k \ge \phi_k, \psi_k \in \omega$ whenever $1 \le k \le n$. Let

(Γ_1) $\mathsf{N}_{0,1}^1 = a_0$, $\mathsf{N}_{1,0}^1 = \breve{a}_0$, $\mathsf{N}_{0,0}^1 = \mathsf{N}_{1,1}^1 = 1$',

and, for $1 \le k \le n$ and $0 \le i, j \le k$, let

(Γ_2) $\mathsf{N}_{i,j}^{k+1} = \prod_{p \le k}(\mathsf{N}_{i,p}^k ; \mathsf{N}_{p,j}^k)$,

(Γ_3) $\mathsf{N}_{i,k+1}^{k+1} = (\mathsf{N}_{i,\phi_k}^{k+1} ; a_{2k-1}) \cdot (\mathsf{N}_{i,\psi_k}^{k+1} ; (a_{2k})^{\smile})$,

(Γ_4) $\mathsf{N}_{k+1,j}^{k+1} = (\mathsf{N}_{j,k+1}^{k+1})^{\smile}$,

(Γ_5) $\mathsf{N}_{k+1,k+1}^{k+1} = 1$'.

Then

$$\bigwedge_{k=1}^{n} \left(\mathsf{N}_{\phi_k,\psi_k}^k \le a_{2k-1} ; a_{2k}\right) \Longrightarrow a_0 \le \mathsf{N}_{0,1}^{n+1}.$$

To give specific examples of Jónsson's conditions we will use the notational conventions introduced by Lyndon [10]. Variables are indexed by pairs of digits, sometimes with primes, subject to the tacit assumption that if x_{ij} is a

variable then x_{ji} is the term $(x_{ij})^{\smallsmile}$. More complicated terms are determined by these abbreviations: $x_{ij}^k = x_{ik};x_{kj}$, $x_{ij}^{kl} = x_{ik};x_{kj} \cdot x_{il};x_{lj}$, and $x_{ij}^{klm} = x_{ik};x_{kj} \cdot x_{il};x_{lj} \cdot x_{im};x_{mj}$.

For $1 \leq n \leq 3$, all of the Jónsson conditions hold in RA. For $n = 4$, every Jónsson condition either holds in RA or is equivalent to (4), (5), or (6).

(4) $$x_{23}^{01} \leq x_{23}^4 \Longrightarrow x_{01}^{23} \leq x_{04}^{23};x_{41}^{23}$$

(5) $$x_{01} \cdot x_{01}^{234} \leq x_{02};(x_{24}^{01} \cdot x_{23}^{01};x_{34}^{01});x_{41}$$

(6) $$x_{01};(x_{02} \cdot x_{02}^3);(x_{21} \cdot x_{21}^4) \leq x_{03};(x_{34}^2 \cdot x_{30};x_{04}^{12} \cdot x_{31}^{02};x_{14});x_{41}$$

The formulas (4), (5), and (6) were called (J), (L), and (M), respectively, in [15, page 30]. (5) was implicitly used by Lyndon [10] and is explicitly mentioned by Chin-Tarski [2]. The independence of (4), (5), and (6) is proved in [15, page 454], where it is observed that there are many relation algebras with four or five atoms in which (5) and (6) hold while (4) fails, many others in which (4) and (5) hold while (6) fails, but only two (the algebras 1857_{3013} and 1858_{3013}) in which (4) and (6) hold while (5) fails. (The two algebras described in [15, page 454] are isomorphic to 1857_{3013} and 1858_{3013} but were incorrectly identified there as algebras 2793_{3013} and 2795_{3013}.)

Here are five more examples of formulas equivalent to Jónsson conditions: (7) and (11) are equivalent to conditions with $n = 5$, while (8), (9), and (10) are equivalent to Jónsson conditions with $n = 6$.

(7) $$x_{23}^{01} \leq x_{23}^4 \Longrightarrow x_{01} \cdot x_{01}^3 \cdot (x_{02} \cdot x_{02}^{3'});x_{21}$$
$$\leq x_{03'};(x_{3'1}^{02} \cdot (x_{3'4}^2 \cdot x_{3'0};(x_{04}^3 \cdot x_{01};x_{14}^{23}));x_{41}^{23})$$

(8) $$x_{04'}^{13} \leq x_{04'}^{1'} \Longrightarrow x_{01} \cdot (x_{02} \cdot x_{02}^4);x_{21} \cdot x_{03};(x_{31} \cdot x_{31}^{4'})$$
$$\leq x_{01'};(x_{1'2}^0 \cdot x_{1'3}^{04'};(x_{32}^0 \cdot (x_{34}^0 \cdot x_{31};x_{14}^{02});x_{42})$$
$$\cdot x_{1'4'};(x_{4'2}^1 \cdot x_{4'3};((x_{31};x_{14}^{02} \cdot x_{34}^0);x_{42} \cdot x_{32}^0)));x_{21}$$

(9) $$x_{01} \cdot x_{02};(x_{21} \cdot x_{21}^3) \cdot (x_{04} \cdot x_{04}^{1'});x_{41} \cdot x_{03'};(x_{3'1} \cdot x_{3'1}^{2'})$$
$$\leq x_{02};(x_{21'}^0 \cdot (x_{24}^0 \cdot x_{23};(x_{34}^1 \cdot x_{30}^{12};x_{04}));x_{41'}$$
$$\cdot x_{23'}^{01};(x_{3'1'}^0 \cdot (x_{3'4}^0 \cdot x_{3'2'};(x_{2'4}^1 \cdot x_{2'0}^{13'};x_{04}));x_{41'}));x_{1'1}^4$$

(10) $$x_{01} \cdot (x_{02} \cdot x_{02}^{33'});x_{21} \cdot (x_{04} \cdot x_{04}^{1'});x_{41}$$
$$\leq x_{03};(x_{32} \cdot ((x_{31'}^0 \cdot (x_{34}^0 \cdot x_{31}^{02};x_{14});x_{41'});(x_{1'2}^0$$
$$\cdot x_{1'4};(x_{42}^1 \cdot (x_{43'}^0 \cdot x_{41};x_{13'}^{02});x_{3'2})))) ;x_{21}$$

(11) $$x_{01} \cdot x_{02};(x_{21} \cdot (x_{23} \cdot (x_{24} \cdot x_{24}^{3'});x_{41}^3))$$
$$\leq x_{03'}^2;(x_{3'1}^2 \cdot (x_{3'0}^2 \cdot x_{3'4};(x_{40}^2 \cdot x_{43};x_{30}^{12}));x_{01}$$
$$\cdot x_{3'4};(x_{41}^3 \cdot (x_{40}^2 \cdot x_{43};x_{30}^{12});x_{01}))$$

28_{37}	a	r	ř
a	1	$a+r+ř$	$a+r$
r	$a+ř$	ar	1
ř	$a+r+ř$	$1'+r+ř$	$a+ř$

38_{65}	a	b	c
a	$1'+a+b$	$a+b+c$	$b+c$
b	$a+b+c$	$1'+a+b$	$a+c$
c	$b+c$	$a+c$	1

44_{65}	a	b	c
a	1	$a+b+c$	$a+b$
b	$a+b+c$	$1'+a$	$a+c$
c	$a+b$	$a+c$	$1'+b$

45_{65}	a	b	c
a	$1'+b+c$	$a+b+c$	$a+b$
b	$a+b+c$	$1'+a+b$	$a+c$
c	$a+b$	$a+c$	$1'+b$

FIGURE 1. Four relation algebras in $\mathsf{RA} \setminus \mathsf{wRRA} \setminus \mathsf{RA}_5$

Formulas (7)–(11) are all valid in wRRA, and each of them is independent of (4), (5), and (6). To show this we use the four integral relation algebras $28_{37}, 38_{65}, 44_{65}$, and 45_{65}. 28_{37} has atoms 1', r, and ř, while $38_{65}, 44_{65}$, and 45_{65} have symmetric atoms 1', $a = ă$, $b = b̆$, and $c = č$. The multiplication tables for the diversity atoms of these algebras are shown in Figure 1. Algebras $28_{37}, 38_{65}, 44_{65}, 45_{65}$ all satisfy (4), (5), (6) (see [15]), but (7) fails in 28_{37} when $x_{01} = r$, $x_{02} = a$, $x_{21} = r$, $x_{03'} = ř$, $x_{3'2} = ř$, $x_{03} = ř$, $x_{31} = a$, and $x_{24} = r$; (8) fails in 38_{65} when $x_{01} = a$, $x_{02} = b$, $x_{21} = c$, $x_{04} = a$, $x_{42} = a$, $x_{03} = c$, $x_{31} = c$, $x_{34'} = b$, $x_{14'} = a$, $x_{01'} = b$, $x_{1'4'} = b$; (9) fails in 44_{65} when $x_{01} = a$, $x_{02} = c$, $x_{21} = a$, $x_{23} = c$, $x_{31} = b$, $x_{04} = b$, $x_{01'} = c$, $x_{1'4} = c$, $x_{41} = a$, $x_{03'} = c$, $x_{3'1} = b$, $x_{3'2'} = c$, $x_{2'1} = c$; (10) fails in 45_{65} when $x_{01} = a$, $x_{02} = b$, $x_{03} = c$, $x_{32} = a$, $x_{03'} = a$, $x_{3'2} = c$, $x_{21} = a$, $x_{04} = a$, $x_{01'} = b$, $x_{1'4} = c$, and $x_{41} = c$; (11) fails in 45_{65} when $x_{01} = a$, $x_{02} = c$, $x_{21} = b$, $x_{23} = a$, $x_{24} = a$, $x_{23'} = b$, $x_{3'4} = c$, $x_{43} = c$, $x_{31} = a$. Therefore (7)–(11) are independent of (4)–(6) (relative to RA). From [15] we know that $28_{37}, 38_{65}, 44_{65}, 45_{65}$ have no 5-dimensional relational bases and hence are not in RA_5. One of (7)–(11) fails in each of these algebras, so they are also not weakly representable.

By [15, theorem 341], (4)–(6) are valid in RA_5. By similar but more complicated proofs one can also show that (7)–(11) are valid in RA_5. (The subscripts and superscripts in (7)–(11) are remnants of those proofs.) Thus the Jónsson conditions for $1 \le n \le 4$ and some conditions with $n = 5, 6$ do not distinguish wRRA from RA_5. On the other hand, we know by Proposition 0.8 that some Jónsson condition must fail in RA_5.

PROBLEM 0.13. Find the smallest n for which some Jónsson condition fails in RA_5.

Let us say that a Jónsson condition is *equational* if it is equivalent to an equation of the form $\sigma = \tau$, where σ and τ are terms built up from variables using the operations ·, ;, and ˘. Examples include the conditions equivalent to (5), (6), (9), (10), and (11), but not necessarily those equivalent to (4), (7), and (8).

PROBLEM 0.14. Is there an equational Jónsson condition that fails in RA_5?

On the basis of limited evidence, including the validity of (4)–(11) in RA_5 and several other such examples, Maddux suggested that perhaps every equational Jónsson condition is valid in RA_5.

The smallest Jónsson condition that is not valid in RA_5 may be long, but by applying methods of Lyndon [11] to (4) we can obtain a reasonably simple formula that is valid in RA_5 but fails in wRRA, namely,

$$(12) \qquad x_{23}^{01} \leq x_{23}^{4} + x_{23}^{4'} \Longrightarrow x_{01}^{23} \leq x_{04}^{23}; x_{41}^{23} + x_{04'}^{23}; x_{4'1}^{23}$$

It is easy to show $\mathsf{RA}_5 \models (12)$. However, while wRRA $\models$ (4) we have on the other hand wRRA $\not\models$ (12). Our algebra shows this. The argument that our algebra has no 5-dimensional basis actually also shows that it fails to satisfy (12) when $x_{01} = \mathsf{v}$, $x_{02} = \mathsf{g}$, $x_{21} = \mathsf{y}$, $x_{03} = \mathsf{g}'$, $x_{31} = \mathsf{y}'$, $x_{24} = \alpha_1$, $x_{43} = \beta_1$, $x_{24'} = \alpha_0$, and $x_{4'3} = \beta_0$. We know (12) holds in RA_5, so our algebra cannot be in RA_5.

Now let $\mathcal{A}$ be a finite relation algebra. In [10] Lyndon defines a set of formulas (now called the *Lyndon conditions*) that define representability for finite relation algebras. These Lyndon conditions are closely related to formulas that state that $\exists$ has a winning strategy in a two player game rather like the one given in definition 0.9 but where $\forall$ is only allowed to choose atoms for his moves and $\exists$ has to play labellings throughout, *i.e.*, all edge labels have to be atoms. One of these Lyndon conditions has some similarity to axiom (4) above, but here all variables a_{ij} range over atoms of the algebra. The Lyndon condition states: for all atoms $a_{ij} : i, j < 5$,

$$(13) \qquad a_{01} \leq a_{01}^{23} \to \exists a_{23}[a_{23} \leq a_{23}^{01} \wedge \forall a_{24}, a_{43}(a_{23}^{4} \geq a_{23} \to a_{01} \leq a_{04}^{23}; a_{41}^{23})]$$

This Lyndon condition is valid in RA_5. We will construct a finite relation algebra where this Lyndon condition fails, though Jónsson's axiom (4) will hold: in fact the whole construction is designed to achieve these properties. Essentially, the Lyndon condition fails because $\forall$ can win the atomic game by playing the sequence of moves illustrated in figure 3. For the atomic game $\exists$ is required to select an atom to label the edge $(2, 3)$ (corresponding to '$\exists a_{23}$' in the formula, above), and whichever label she chooses, $\forall$ has a move to win the game in the next round ($\forall$'s move will vary depending on the atom $\exists$ uses to label $(2, 3)$, see the proof of lemma 0.17). But if $\forall$ tries this in the weak representation game $\mathcal{G}(\mathcal{A})$ it will not work. $\exists$ will use a non-atomic label for the edge $(2, 3)$ and nobody can force her to choose an atom below this label. This means that the atomic moves $\forall$ had ready for an atomic label on the edge $(2, 3)$ are not legal moves, because none of them 'cover' the label on this edge, as required by the rules of the game. This indicates that (4) need not fail. The tricky part, as usual with these representation games, is proving that $\exists$ has a winning strategy for all possible plays of the game, not just the one shown in figure 3. A consequence of this proof is that (4) does hold for the relation algebra we construct.

This illustrates the difference between the classes wRRA and RA_5. A game to test whether a finite relation algebra belongs to RA_5 can be obtained by taking the atomic game we just mentioned but restricting all networks to at most five nodes. This RA_5 game can be more demanding for $\exists$ than the game for wRRA because she is forced to choose atoms as labels of edges, leaving a wider choice of possible moves to her opponent $\forall$. On the other hand, the game for wRRA can

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be more demanding than this RA_n game (any $n < \omega$) because $\exists$ may be required to label edges in a network with more than n nodes.

We will now construct our algebra, a kind of 'rainbow algebra' (cf. [5, chapter 16]), and show it is a weakly representable relation algebra but is not in RA_5.

DEFINITION 0.15. We define a finite algebra $\mathcal{A}_0$, which we will see in lemma 0.16 to be a relation algebra, by defining its atom structure. The set of atoms At is

$$(14) \qquad \{1', \mathsf{g}, \mathsf{g}', \mathsf{y}, \mathsf{y}', \mathsf{b}, \mathsf{\breve{b}}, \mathsf{w}_i, \mathsf{s}, \mathsf{v}, \alpha_i, \beta_i, \mathsf{r}_i, \mathsf{\breve{r}}_i : i < 2\}.$$

There are 19 atoms here. It may help to think of g, g' as green, y, y' as yellow, $\mathsf{b}, \mathsf{\breve{b}}$ as black, w_i as white, s as silver, v as violet, and $\mathsf{r}_i, \mathsf{\breve{r}}_i$ as red. The only atom below the identity is $1'$ itself. The atoms $1', \mathsf{v}, \mathsf{g}, \mathsf{g}', \mathsf{y}, \mathsf{y}', \mathsf{w}_i, \mathsf{s}, \alpha_i, \beta_i$ are self-converse and the other atoms form converse pairs: $(\mathsf{b}, \mathsf{\breve{b}})$ and $(\mathsf{r}_i, \mathsf{\breve{r}}_i)$ for $i < 2$. Let

$$\mathsf{A} = \mathsf{g} + \mathsf{g}' + \mathsf{y} + \mathsf{y}' + \sum_{i<2}(\alpha_i + \beta_i)$$

$$\mathsf{E} = \mathsf{b} + \mathsf{\breve{b}} + \mathsf{s} + \sum_{i<2}(\mathsf{w}_i + \mathsf{r}_i + \mathsf{\breve{r}}_i)$$

$$\Gamma = \mathsf{g} + \mathsf{g}' + \mathsf{y} + \mathsf{y}'$$

$$\mathsf{R} = \mathsf{r}_0 + \mathsf{r}_1$$

Note that $\Gamma \leq \mathsf{A}$ and $1 = 1' + \mathsf{E} + \mathsf{A} + \mathsf{v}$. Lower case variables will be used for atoms and upper case variables stand for elements of $\mathcal{A}_0$. For example, if we write $a \leq \mathsf{A}$ it will be implicit that a is an atom.

Forbidden triples. (*i.e.*, atoms (a, b, c) such that $a; b \cdot c = 0$): Peircean transforms of

  I. $(1', x, y) : x \neq y$,
 II. $(a_1, a_2, a_3) : a_1, a_2, a_3 \leq \mathsf{A}$,
III. $(\mathsf{g}, \mathsf{g}', \mathsf{v}), (\mathsf{g}, \mathsf{g}', \mathsf{b}), (\mathsf{g}, \mathsf{g}', \mathsf{\breve{b}})$, and $(\mathsf{y}, \mathsf{y}', \mathsf{s}), (\mathsf{y}, \mathsf{y}', \mathsf{w}_i)$ for each $i < 2$,
 IV. $(\alpha_i, \beta_i, \mathsf{r}_i), (\alpha_i, \beta_i, \mathsf{\breve{r}}_i) : i < 2$,
  V. a) $(\gamma, \alpha_i, \mathsf{v}), (\gamma, \alpha_i, \mathsf{w}_i), (\gamma, \alpha_i, \mathsf{b}), (\gamma, \alpha_i, \mathsf{s})$: $\gamma \leq \mathsf{g} + \mathsf{y}, i < 2$,

  b) $(\gamma', \beta_i, \mathsf{w}_{1-i}), (\gamma', \beta_i, \mathsf{\breve{b}}), (\gamma', \beta_i, \mathsf{r}_j)$: $\gamma' \leq \mathsf{g}' + \mathsf{y}', i, j < 2$,
 VI. $(\mathsf{r}_i, \mathsf{v}, \mathsf{r}_j) : i, j < 2$.

Rules II–VI are illustrated in figure 2. The set of forbidden triples is F. Let $C = At^3 \setminus F$ (the consistent triples). This defines the atom structure At. We define $\mathcal{A}_0$ to be the complex algebra over At.

LEMMA 0.16. $\mathcal{A}_0$ *is a relation algebra.*

PROOF. By [5, lemma 3.24], the following three properties prove that $\mathcal{A}$ is a relation algebra: (i) For all $x, y \in At$ we have $x = y \iff (1', x, y) \in C$, (ii) C is closed under Peircean transforms and (iii) for all $b, c, d, e \in At$, $\exists a[(b, c, a), (d, e, a) \in C \leftrightarrow \exists f((b, f, d), (f, e, c) \in C)]$. The first two properties are true by definition of C. The third property is easily seen to be true if $1' \in \{b, c, d, e\}$. For example, if $b = 1'$ then the only possibility for a is c, the only possibility for f is d, and both sides reduce to $(d, e, c) \in C$. If $1' \notin \{b, c, d, e\}$ then both sides of the equivalence are true. This is because plainly, each rule can
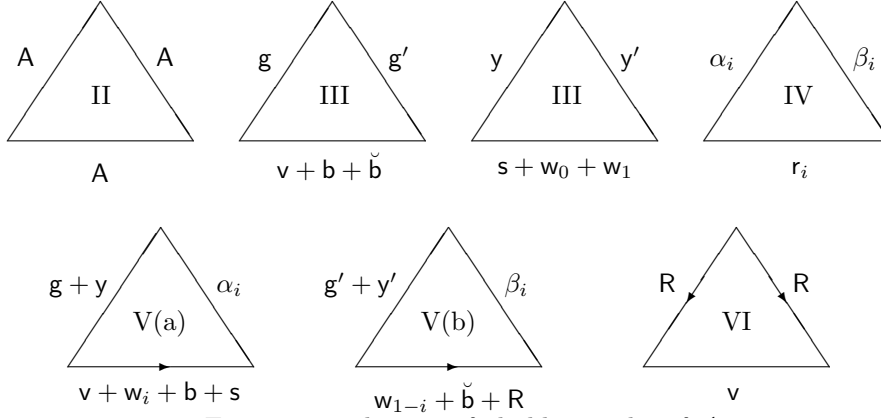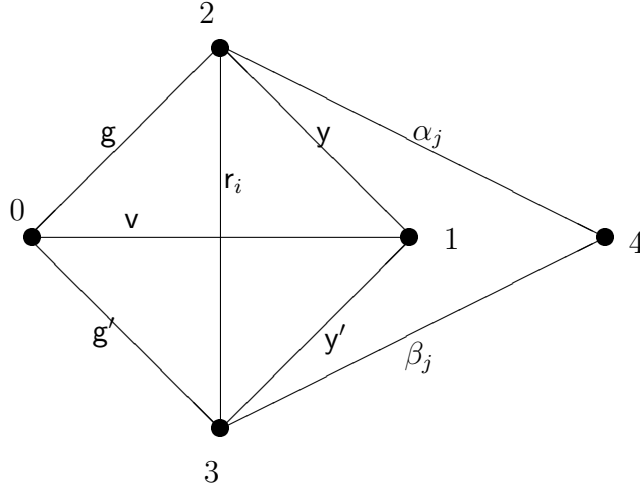
FIGURE 2. the main forbidden triples of $\mathcal{A}_0$

FIGURE 3. Impossible to complete a labelling by giving a consistent label to $(4,0), (4,1)$. The edge $(2,3)$ has no arrow, as both r_i and $\check{r}_i$ are possible.

now eliminate *at most four of the nine atoms beneath* E. Rule I is not applicable because $b, c, d, e \neq 1$' and $\mathsf{E} \cdot 1$' $= 0$. Rule V(b) can eliminate four atoms beneath E (namely, $\mathsf{w}_{1-i}, \check{\mathsf{b}}, \mathsf{r}_0, \mathsf{r}_1$), and each of the other rules can eliminate at most three. So for any b, c, d, e, the two conditions $(b, c, a), (d, e, a) \in C$ on the left of the equivalence can eliminate at most eight of the nine atoms $a \leq \mathsf{E}$, and so there is always a possible choice of $a \leq \mathsf{E}$ making the left-hand side true. Similarly, there is always a value of $f \leq \mathsf{E}$ making the right-hand side true too.   ⊣

LEMMA 0.17. $\mathcal{A}_0$ *does not have a five-dimensional relational basis.*

Suppose B is a five-dimensional relational basis for $\mathcal{A}_0$, for contradiction. See figure 3. Then there is $N_0 \in B$ with $N_0(0, 1) = \mathsf{v}$. Since $(\mathsf{g}, \mathsf{y}, \mathsf{v})$ is not forbidden
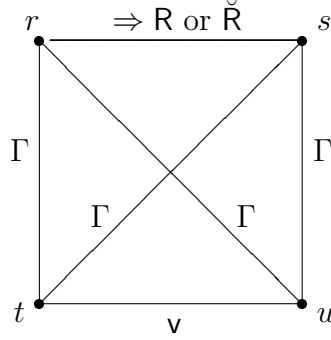
FIGURE 4. Critical edge (r, s) of N

we have $\mathsf{g};\mathsf{y} \geq \mathsf{v}$ so there must be $N_1 \in B$ with $N_1 \equiv_2 N_0$ and $N_1(0, 2) = \mathsf{g}$, $N_1(2, 1) = \mathsf{y}$. Similarly, since $(\mathsf{g}', \mathsf{y}', \mathsf{v})$ is not forbidden, there must be $N_2 \in B$ with $N_2 \equiv_3 N_1$, $N_2(0, 3) = \mathsf{g}'$, and $N_2(3, 1) = \mathsf{y}'$. Note that

$$N_2(2, 3) \cdot (1' + \mathsf{A} + \mathsf{w}_0 + \mathsf{w}_1 + \mathsf{b} + \check{\mathsf{b}} + \mathsf{s} + \mathsf{v}) = 0,$$

by forbidden triples I, II and III, therefore $N_2(2, 3) = \mathsf{r}_i$ or $N_2(2, 3) = \check{\mathsf{r}}_i$, for some $i < 2$. Say, $N_2(2, 3) = \rho_i \leq \mathsf{r}_i + \check{\mathsf{r}}_i$.

Write j for $1 - i$. Since $(\alpha_j, \beta_j, \rho_i)$ is not forbidden by IV, or indeed by any forbidden triple, there is $N_3 \in B$ with $N_3 \equiv_4 N_2$ and $N_3(2, 4) = \alpha_j$, $N_3(4, 3) = \beta_j$. Then $N_3(0, 4) \leq \mathsf{g};\alpha_j \cdot \mathsf{g}';\beta_j = \check{\mathsf{R}}$ (by I, II and V) and similarly, $N_3(1, 4) \leq \mathsf{y};\alpha_j \cdot \mathsf{y}';\beta_j = \check{\mathsf{R}}$. But then $N_3(0, 4) = N_3(0, 1); N_3(1, 4) \cdot N_3(0, 4) \leq \mathsf{v}; \check{\mathsf{R}} \cdot \check{\mathsf{R}} = 0$, by VI. This is a contradiction and proves that a relational basis does not exist. $\dashv$

THEOREM 0.18. $\mathcal{A}_0$ *is weakly representable.*

PROOF. We show that $\exists$ has a winning strategy in the weak representation game $\mathcal{G}(\mathcal{A}_0)$ (see definition 0.9). Suppose that $\forall$ selects $a_0 \in \mathcal{A} \setminus \{0\}$ in his initial move. $\exists$ responds with a strict atomic network $N_0 = (\{x_0, y_0\}, l)$, where $l(x_0, y_0)$ is an arbitrary atom $\leq a_0$, and $x_0 = y_0$ iff $l(x_0, y_0) = 1'$. This determines N_0.

Suppose at some point in a play of the game that the current network is N. An *edge of N* is an ordered pair of nodes of N. An edge (r, s) of N is said to be *critical (in N)* if $r \neq s$ and there are nodes t, u of N such that $N(r, t), N(r, u), N(s, t), N(s, u) \leq \Gamma$ and $N(t, u) = \mathsf{v}$. We will assume inductively that for every $r, s \in N$:

I1) If (r, s) is non-critical then $N(r, s) \in At(\mathcal{A})$.
I2) If (r, s) is critical then $N(r, s) \in \{\mathsf{R}, \check{\mathsf{R}}\}$. See figure 4.
I3) Every atomic refinement of N is a strict network.

Clearly, these assumptions hold for the initial network N_0. Assume that they hold for N. Let $\forall$'s move in the current round of the game be (m, n, Y, Z), where $m, n \in N$ and $Y; Z \geq N(m, n)$. Nodes $m, n \in N$ and elements $Y, Z \in \mathcal{A}_0$ are fixed for the duration of the proof of this theorem. $\exists$'s job is to find a network $N^+ \geq N$ satisfying the three inductive conditions and containing a node p with $N^+(m, p) \leq Y$ and $N^+(p, n) \leq Z$. This is not so hard. We will now explain how she can do it.

First of all, if N already contains a node p with $N(m,p) \leq Y$ and $N(p,n) \leq Z$, then $\exists$ may and does respond to $\forall$'s move by playing $N^+ = N$. This obviously keeps all the inductive conditions, and it completes the round of the game in this case. So assume from now on that there is no such p in N.

Claim. There are atoms $b \leq Y$, $c \leq Z$ such that $b;c \geq N(m,n)$.

Proof of claim. If (m,n) is non-critical then $N(m,n)$ is already an atom of $\mathcal{A}_0$, so such atoms b, c exist by additivity of ';' in $\mathcal{A}_0$. If (m,n) is critical, then $N(m,n)$ is R or $\check{\mathsf{R}}$. Suppose that $N(m,n) = \mathsf{R}$ (the case where $N(m,n) = \check{\mathsf{R}}$ is similar). Suppose for contradiction that no suitable b, c exist. Since $Y;Z \geq \mathsf{R} = \mathsf{r}_0 + \mathsf{r}_1 \geq \mathsf{r}_0$, there are atoms $b_0 \leq Y$, $c_0 \leq Z$ with $b_0;c_0 \geq \mathsf{r}_0$, and therefore by hypothesis, $b_0;c_0 \not\geq \mathsf{r}_1$. Similarly, there are atoms $b_1 \leq Y$, $c_1 \leq Z$ with $b_1;c_1 \geq \mathsf{r}_1$ but $b_1;c_1 \not\geq \mathsf{r}_0$. Referring back to the list of forbidden triples, we see that only rules I and IV distinguish between r_0 and r_1, so there are rather few possibilities for these atoms. They are tabulated in figure 5. Entries in the body of the table

$b_0, c_0 \ \backslash \ b_1, c_1$	1', r_1	r_1, 1'	α_0, β_0
1', r_0	Id-move	$\mathsf{r}_1, \mathsf{r}_0$	α_0, r_0
r_0, 1'	$\mathsf{r}_0, \mathsf{r}_1$	Id-move	r_0, β_0
α_1, β_1	α_1, r_1	r_1, β_1	α_0, β_1

FIGURE 5. Atoms b_0, c_0, b_1, c_1

are of two kinds. Seven of them consist of a pair of atoms, b, c say. It can be checked that in each case, $b \in \{b_0, b_1\}$ and so $b \leq Y$, $c \in \{c_0, c_1\}$ and so $c \leq Z$, and $b;c \geq \mathsf{R}$, contrary to our hypothesis that no such atoms exist. So we are in fact in one of the two remaining cases 'Id-move': either $b_0 = b_1 = 1$' and $c_i = \mathsf{r}_i$ for each i, or the other way round, swapping b, c. But then, N already contains a node p with $N(m,p) \leq Y$ and $N(p,n) \leq Z$, namely $p = m$ (or $p = n$ when b, c are swapped). This contradicts our assumption above that there is no such p in N, and proves the claim.

$\exists$ chooses atoms b, c as in the claim. If $b = 1$' or $c = 1$', then again she has the trivial response of $N^+ = N$ to $\forall$'s move, contrary to assumption. So this is not the case. $\exists$ now begins the construction of N^+, by creating a new node $p \notin N$. The nodes of N^+ will be those of N together with p. She defines $N^+(m,p) = b$ and $N^+(p,n) = c$. Further, she lets $N^+(p,m) = \check{b}$ and $N^+(n,p) = \check{c}$. This is well defined in the case where $m = n$: for then, $b;c \geq N(m,m) = 1$', so by rule I, $b = \check{c}$. She sets $N^+(p,p) = 1$' of course.

It remains for her to define $N^+(p,q)$ and $N^+(q,p)$ for each $q \in N \setminus \{m,n\}$. She will define them to be mutually converse, so it is enough to specify either one. Her strategy is as follows:

S1) If $N(q,m), N(q,n), b, c \leq \Gamma$ and $N(m,n) = \mathsf{v}$, then define $N^+(q,p) = \mathsf{R}$.

S2) Otherwise, if there is $a \leq \mathsf{w}_0 + \mathsf{w}_1 + \mathsf{b} + \check{\mathsf{b}} + \mathsf{s} + \mathsf{R}$ such that for every atomic refinement N^a of N, the triples $(N^a(q,m), b, a)$ and $(N^a(q,n), \check{c}, a)$ are not forbidden, then choose any such a and set $N^+(q,p) = a$. (For definiteness,
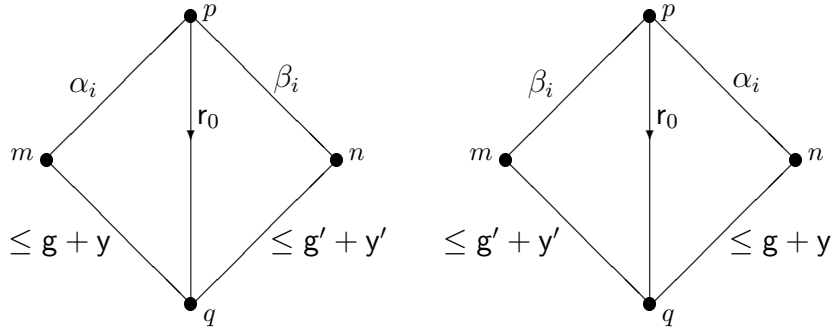
FIGURE 6. case S3 of ∃'s strategy

we could let a be the first applicable of the atoms $\mathsf{w}_0, \mathsf{w}_1, \mathsf{b}, \check{\mathsf{b}}, \mathsf{s}, \mathsf{r}_0, \mathsf{r}_1$ in that order.)

S3) Otherwise, define $N^+(p, q) = \mathsf{r}_0$. (r_1 would also do.)

This is plainly well defined and completes the definition of N^+. Note that S1–S3 only select elements beneath E. (Broadly, elements below E are most useful for ∃, and those below A are most useful for ∀.)

Before we proceed, we make two remarks.

REMARK 0.19. Let us analyse case S3 of the strategy a little. Suppose that ∃ used S3, so that $N^+(p, q) = \mathsf{r}_0$. So S2 does not apply, and therefore all seven atoms w_0, w_1, b, $\check{\mathsf{b}}$, s, r_0, and r_1 are forbidden by some rules and atomic refinements of N in the triangles $(q, m, p), (q, n, p)$.

How can this be? Atomic refinements differ from N only by having labels r_0 or r_1 instead of R (or with converses). Examination of the list of forbidden triples shows that even allowing for various different atomic refinements, the only rules that can forbid more than three of the seven atoms are rules I and V(b). Rule I is not involved, since (inductively) every atomic refinement of N is strict, we assumed $b, c \neq 1$', and none of $\mathsf{w}_0, \mathsf{w}_1, \mathsf{b}, \check{\mathsf{b}}, \mathsf{s}, \mathsf{r}_0, \mathsf{r}_1$ are 1'. Therefore, V(b) must be involved, and either b or c is β_i for some $i < 2$. Suppose first that $c = \beta_i$, and $N(q, n) \leq \mathsf{g}' + \mathsf{y}'$. Now, $\mathsf{w}_{1-i}, \check{\mathsf{b}}, \mathsf{r}_0, \mathsf{r}_1$ are forbidden on (q, p) by dint of the triangle (q, n, p). So the remaining atoms $\mathsf{w}_i, \mathsf{b}, \mathsf{s}$ are being forbidden by the other triangle, (q, m, p). The only rule that can do the job is V(a), and its 'orientation' must be the same as for V(b). Therefore, $b = \alpha_i$ (the same i as for β_i) and $N(q, m) \leq \mathsf{g} + \mathsf{y}$. We have arrived at the situation shown on the left of figure 6. The other possibility, shown on the right, is obtained when $b = \beta_i$.

REMARK 0.20. Observe that no edge of N changes its criticality when viewed as an edge of N^+. Certainly, being critical is an existential condition and cannot be lost from N to N^+. Suppose for contradiction that (r, s) is non-critical in N but critical in N^+. Then there are nodes t, u of N^+ as in figure 4. Such nodes did not exist in N, so $p \in \{t, u\}$. But this means that there are three edges of N^+ involving p and labelled with atoms $\leq \Gamma + \mathsf{v}$. At most two of them can be edges from p to m or n, so at least one of them was labeled by ∃ according to S1–S3 of her strategy. Since S1–S3 never use atoms $\leq \Gamma + \mathsf{v}$, this is impossible.

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Now let us proceed to check that N^+ satisfies the inductive conditions. We need to check that if (r, s) is non-critical then $N^+(r, s)$ is an atom, and if it is critical then $N^+(r, s)$ is R or Ř. By the inductive hypothesis and remark 0.20, we only need consider edges involving the new node, p. Since always $N^+(q, p) = N^+(p, q)^\smile$, we only need consider one orientation.

- If $p = q$ then (p, q) is non-critical and $N^+(p, q) = 1$', an atom.
- The edge (m, p) is labeled by an atom (b, by construction). Also, it is not critical in N^+. For otherwise, there would be distinct nodes $t, u \in N \setminus \{m\}$ with $N^+(p, t), N^+(p, u) \leq \Gamma$ (see figure 4). As in remark 0.20, at least one of them was labeled by $\exists$ using S1–S3 of her strategy, which is impossible since S1–S3 do not use atoms $\leq \Gamma$.
- The edge (p, n) is handled similarly.
- Now let $q \in N \setminus \{m, n\}$. The only occasion when $\exists$ uses a non-atom (Ř) to label (p, q) is in case S1, and that is exactly when (p, q) is critical in N^+. Certainly, if $\exists$ uses Ř then by S1, (p, q) is critical. Conversely, if (p, q) is critical then there are distinct $t, u \in N$ as in figure 4, with $N^+(p, t), N^+(p, u) \leq \Gamma$. Since S1–S3 never use atoms $\leq \Gamma$, we must have $\{t, u\} = \{m, n\}$. Therefore, $b, c, N(m, q), N(n, q) \leq \Gamma$ and $N(m, n) = $ v. So S1 applies and $\exists$ defines $N^+(p, q) = $ Ř.

So inductive requirements I1 and I2 hold for N^+. The main work is to check I3: that every atomic refinement of N^+ is a strict network. Let N^a be such a refinement. It is clearly strict, as (inductively) every atomic refinement of N is strict, and because $b, c \neq 1$' (by assumption), the only edge that $\exists$ labels with an $\mathcal{A}_0$-element ≥ 1' is (p, p). So we need to check that for no $x, y, z \in N^+$ is $(N^a(x, y), N^a(y, z), N^a(x, z))$ a forbidden triple. The restriction of N^a to the nodes of N is an atomic refinement of N, so the condition holds inductively if $x, y, z \in N$. It is easily seen to hold if x, y, z are not all distinct. So we can assume that $p \in \{x, y, z\}$ and $|\{x, y, z\}| = 3$. Because the labels on edges of N^a are atoms and the forbidden triples are closed under Peircean transforms, the order of x, y, z is not significant. (This is why atomic refinements are helpful.)

We now divide into cases according to whether the set $\{x, y, z\} \cap \{m, p, n\}$ has size 3, 2, or 1. If $(x, y, z) = (m, p, n)$, then $N^a(x, y) = b$ and $N^a(y, z) = c$. We know that $b; c \geq N(m, n) \geq N^a(x, z)$, and we are done.

Suppose that $x \in N \setminus \{m, n\}$, $y \in \{m, n\}$, and $z = p$. If $\exists$ used case S1 of her strategy to define $N^+(x, p) = $ R, then we have $N^a(x, y), N^a(y, p) \leq \Gamma$, and $N^a(x, p) \in \{r_0, r_1\}$. No forbidden triple involves $\Gamma, \Gamma, $ R, so we are done. If she used S2, she selected some atom $a \leq w_0 + w_1 + b + \check{b} + s + $ R precisely so that the triple $(N^a(x, y), N^a(y, p), N^a(x, p))$ is not forbidden. If she used S3, so that $N^a(x, p) = $ ř$_0$, then we are in the position of figure 6 with $x = q$, from which it is clear that neither $(N^a(q, m), N^a(m, p), $ř$_0)$ nor $(N^a(q, n), N^a(n, p), $ř$_0)$ are forbidden.

The last case is when $x, y \in N \setminus \{m, n\}$ are distinct and $z = p$. Therefore, by S1–S3,

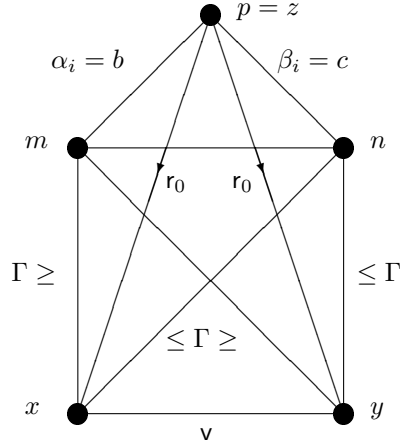$$(15) \qquad\qquad N^a(x, z), N^a(y, z) \leq \mathsf{E}.$$

FIGURE 7. last case of I3

Assume for contradiction that $(N^a(x,y), N^a(y,z), N^a(x,z))$ is a forbidden triple. The only rules that can forbid a triple with two of its entries under E are rules I and VI. Rule I is not applicable because none of $N^a(x,y)$, $N^a(y,z)$, $N^a(x,z)$ are 1'. For rule VI to be applicable, by (15) we must have $N^+(p,x), N^+(p,y) \le \mathsf{R}$, and $N^+(x,y) = \mathsf{v}$. That is, $\exists$ used case S3 of her strategy to label both $(p,x),(p,y)$. Referring to remark 0.19 and figure 6 again, we have $(b,c) \in \{(\alpha_i,\beta_i),(\beta_i,\alpha_i)\}$ for some $i < 2$; by symmetry, we can suppose $b = \alpha_i$, $c = \beta_i$. Then (see figure 6) we have $N(m,x), N(m,y) \le \mathsf{g}+\mathsf{y} \le \Gamma$ and $N(m,y), N(n,y) \le \mathsf{g}' + \mathsf{y}' \le \Gamma$. This is summarised in figure 7, from which we see that (m,n) must be a critical edge of N. Therefore, by inductive condition I2, $N(m,n) \in \{\mathsf{R}, \check{\mathsf{R}}\}$. But by rule IV, this means that $N(m,n) \not\le \alpha_i;\beta_i = b;c$, which is a contradiction.

We have verified that inductive conditions I1–I3 hold for N^+. It follows that N^+ is a network. For let $x,y,z \in N^+$. Take any atom $a \le N^+(x,z)$, and any atomic refinement N^a of N^+ with $N^a(x,z) = a$. We have proved that N^a is a network, so $a \le N^a(x,y); N^a(y,z) \le N^+(x,y); N^+(y,z)$. This holds for all such a, and so $N^+(x,z) \le N^+(x,y); N^+(y,z)$ as required.

Thus N^+ is a network extending N with $N^+(m,p) = b \le Y$ and $N^+(p,n) = c \le Z$. Since $\exists$ can always respond to $\forall$'s move with a suitable network, she is sure to win the game. Hence, by proposition 0.10, $\mathcal{A}_0$ has a weak representation. $\dashv$

REMARK 0.21. Let $\mathcal{A}$ be a rainbow algebra with the following atoms

(16) $1', \mathsf{g}_i \ (i < 4), \mathsf{w}, \mathsf{w}_{ij} \ (i < j < 4), \mathsf{y}, \mathsf{b}, \mathsf{r}, \check{\mathsf{r}}$

and let the forbidden triples of atoms be the Peircean transforms of

- $(1', x, y) : x \ne y$,
- $(\mathsf{g}_i, \mathsf{g}_j, \mathsf{g}_k)$ where $i, j, k < 4$,
- $(\mathsf{g}_i, \mathsf{g}_j, \mathsf{w})$, $(\mathsf{g}_i, \mathsf{g}_j, \mathsf{w}_{jk})$ unless $i = j$ or $i = k$,
- $(\mathsf{y}, \mathsf{y}, \mathsf{y}), (\mathsf{y}, \mathsf{y}, \mathsf{b})$,
- $(\mathsf{g}_i, \mathsf{y}, \mathsf{w}_S)$, unless $i \in S$,
- $(\mathsf{r}, \mathsf{r}, \mathsf{r})$.

Observe, by the last forbidden triple, that in any consistent triangle labelled by 'red' atoms (r or $\breve{r}$), the red atoms must form a cycle. Since there is no tournament on four nodes where every three nodes form a cycle, it follows that $\mathcal{A}$ has no six-dimensional relational basis (see [5, 16.3]). But it can be shown that $\mathcal{A}$ is weakly representable (see [5, 16.4]). In a weak representation game, whenever $\exists$ has to label a red edge, she uses the label $r + \breve{r}$. It turns out that this provides a winning strategy for $\exists$, essentially because $(r + \breve{r})\,;(r + \breve{r}) \geq r + \breve{r}$. Hence, $\mathcal{A}$ has a weak representation. However, it seems impossible to use this construction to find a weakly representable relation algebra with no five-dimensional relational basis.

PROBLEM 0.22. Let $\mathcal{A}$ be a finite relation algebra and let h be a weak representation of $\mathcal{A}$ over the base X. For simplicity, let us restrict our attention to the case where $h(1) = X \times X$. Since $\mathcal{A}$ is finite, for all $x, y \in X$ there must be a minimal element $\alpha \in \mathcal{A}$ such that $(x, y) \in h(\alpha)$. Let $(x, y)^h$ be this minimal α.

Is it necessarily the case, for all $x, y \in X$ and for all atoms $a \leq (x, y)^h$, that there is an atomic network N with set of nodes X such that $N(x, y) = a$ and for all $u, v \in X$ we have $N(u, v) \leq (u, v)^h$?

PROBLEM 0.23. Let $n \geq 5$ and $K_n = \mathsf{wRRA} \cap \mathsf{RA}_n$. Prove the following conjecture: K_n cannot be defined over wRRA using only finitely many axioms, nor can it be defined over RA_n using only finitely many axioms.

REFERENCES

[1] H ANDRÉKA, *Weakly representable but not representable relation algebras*, **Algebra Universalis**, vol. 32 (1994), pp. 31–43.

[2] L CHIN and A TARSKI, *Distributive and modular laws in the arithmetic of relation algebras*, **University of California Publications in Mathematics, New Series**, vol. 1 (1951), pp. 341–384.

[3] M HAIMAN, *Arguesian lattices which are not linear*, **Bull. Amer. Math. Soc.**, vol. 16 (1987), pp. 121–123.

[4] ———, *Arguesian lattices which are not type I*, **Algebra Universalis**, vol. 28 (1991), pp. 128–137.

[5] R HIRSCH and I HODKINSON, **Relation algebras by games**, North-Holland, Elsevier Science, Amsterdam, 2002.

[6] I HODKINSON and SZ MIKULÁS, *Non-finite axiomatizability of reducts of algebras of relations*, **Algebra Universalis**, vol. 43 (2000), pp. 127–156.

[7] I. HODKINSON and Y. VENEMA, *Canonical varieties with no canonical axiomatisation*, **Trans. Amer. Math. Soc.**, vol. 357 (2005), pp. 4579–4605.

[8] B JÓNSSON, *Representation of modular lattices and relation algebras*, **Trans. Amer. Math. Soc.**, vol. 92 (1959), pp. 449–464.

[9] ———, *The theory of binary relations*, **Algebraic logic** (H Andréka, J Monk, and I Németi, editors), Colloq. Math. Soc. J. Bolyai, vol. 54, North-Holland, Amsterdam, 1991, pp. 245–292.

[10] R LYNDON, *The representation of relational algebras*, **Annals of Mathematics**, vol. 51 (1950), no. 3, pp. 707–729.

[11] ———, *The representation of relation algebras, II*, **Annals of Mathematics**, vol. 63 (1956), no. 2, pp. 294–307.

[12] R MADDUX, *Some varieties containing relation algebras*, **Trans. Amer. Math. Soc.**, vol. 272 (1982), no. 2, pp. 501–526.

[13] ———, *A sequent calculus for relation algebras*, **Ann. Pure. Appl. Logic**, vol. 25 (1983), pp. 73–101.

[14] ———, *Relation algebras of every dimension*, **J. Symbolic Logic**, vol. 57 (1992), no. 4, pp. 1213–1229.

[15] ———, **Relation algebras**, Studies in Logic: the Foundations of Mathematics, vol. 150, Elsevier, 2006.

[16] J MONK, *On representable relation algebras*, **Michigan Mathematics Journal**, vol. 11 (1964), pp. 207–210.

[17] B PÉCSI, *Weakly representable relation algebras form a variety*, **Algebra Universalis**, (2009), To appear.

[18] A TARSKI and S GIVANT, **A formalization of set theory without variables**, Colloquium Publications, no. 41, Amer. Math. Soc., Providence, Rhode Island, 1987.

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