On applications of Orlicz spaces to Statistical Physics

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- The lecture is based on:

- L. E. Labuschagne, W. A. Majewski, : Quantum L_p and Orlicz spaces, Quantum Probability and Related Topics, vol XXIII, 176-189 (2008)

- L. E. Labuschagne, W. A. Majewski,: Maps on noncommutative Orlicz spaces, Illinois J. Math., vol. 55, Fall 2011, 1053-1081

- W. A. Majewski, L. E. Labuschagne,: On applications of Orlicz spaces to Statistical Physics, Ann. H. Poincaré, 2013; DOI 10.1007/s00023-013-0267-3

- series of joint papers with Boguslaw Zegarlinski on applications of L_p-spaces to Statistical Physics

- MOTIVATION and OUTLINE:

- *To indicate reasons why (classical as well as non-commutative) Orlicz spaces are emerging in the theory of (classical and quantum) Physics*

- "*The steady progress of Physics requires for its theoretical formulation a mathematics that gets continually more advanced. This is only natural and to be expected.*"

- P. A. M. Dirac; Proc. Roy. Soc. London A **133** 60-72 (1931)

- Classical mechanics

- Newton; laid the base of classical mechanics and calculus.

- When a physicist knows that a certain quantity is an observable?

- In any answer: an observable u is known when also a function $F(\cdot)$ of this observable is known. In particular, u^n should be known.

- This feature of observables was, probably, a motivation for Newton to develop calculus and to use it in his laws of motion.

- We will repeat this question in the framework of Statistical Mechanics.

- However, some 150 years later, Cauchy and Riemann gave a rigorous definition of derivative and integral respectively. Subsequently, in the second half of XIX century, Classical Mechanics was fully developed.

- Hamilton, Lagrange, Liouville.

- To a system, Γ - phase space, was associated,

- and classical mechanics was built on $< C(\Gamma)$, measure, dynamics $>$.

- a selection of a family of observables (functions on Γ) encodes individual features of a physical system!, eg $C(\Gamma)$ or L^{∞}, etc

- Quantum mechanics

- Heisenberg (as Newton) wrote a dynamical equation (in the new setting and now with the first noncommutative derivative)

- canonical quantization; no via bounded operators - so Hilbert space should be of infinite dimension. Consequently, "quantum" is not equivalent to "non-commutative" - a description of a quantum system should be compatible with the canonical quantization. In particular, $M_n(\mathbf{C})$, is well suited for a description of a non-commutative system, BUT not for a quantum one.

- uniqueness of quantization (von Neumann theorem); small and large systems (Fock, van Hove, Schwinger, Wightmann, Dyson); uniqueness only for small systems!

- Powers; factors III (relations to large systems, statistical physics)

- we will use *noncommutative integration theory*; Segal, Dixmier, Nelson, Haagerup, Araki-Masuda, Connes, Hilsum, Dodds, Dodds de Pachter; von Neumann.

- individual characteristic of a system will be taken into account - Haag-Kastler approach.

- $(\mathcal{A}, \phi)$ where $\mathcal{A}$ - C^*-algebra, ϕ a state , gives noncommutative probability space.

- Consequently, non-commutative calculus, stemming from C^*-algebra theory, will be used and all basic features of quantization will be taken into account.

- Classical statistical physics

- Maxwell, Boltzmann - laid the base for statistical mechanics (classical).

- to select a function f which can describe a probability (velocity distribution function) it was necessary to assume: $\int f dx = 1 < \infty$; consequently L^1 space was appeared and subsequently, to have a dual pair (observables and states) the pair of Banach spaces $< L^\infty, L^1 >$ was also appeared.

- Boltzmann theory: model of gas, Boltzmann equation, H-theorem; there are no paradoxes - cf Kac explanations.

- Let me term by "Boltzmann dream" the following problems:

1. existence of solutions of Boltzmann equation,
2. properties of solutions; asymptotic

3. return to equilibrium

- NOTE: the standard approach based on $< L^{\infty}, L^{1} >$ seems to be not effective.

- observables - stochastic variables.

- entropy - the function $x \mapsto xlogx$ is appearing; to support this choice we note

  1. McKean - natural Lyapunov functional for simplified (by Kac) model of Boltzmann gas.
  2. Stirling-type bounds - essential for calculations of various probabilities.

- problems:

- The set of "good" density matrices $\{\varrho : S(\varrho) < \infty\}$ (Wehrl, Rev. Mod. Phys, (1978)) is a meager set only (we assume that the dimension of the underlying Hilbert space is infinite!).

- For $H(f)$ see Bourbaki, *Éléments de Mathématique*, Livre VI, Intégration (1955); for $f \in L^1$, $H(f)$ is not well defined!

- Entropic problems lead to serious problems with the explanation of the phenomenon of return to equilibrium and with the second law of thermodynamics (entropy should be a state function which is increasing in time) (Ray Streater).

- Pistone-Sampi: The **Orlicz space** based on an exponentially growing function $cosh - 1$ is a "good" space for a description of regular observables!

- An argument, in favour of Orlicz spaces, was provided by Cheng and Kozak (J. Math. Phys; 1972). Namely, it seems that the framework within which certain non-linear integral equation of Statistical Mechanics can be studied is that provided by Orlicz spaces.

- mathematical problems associated with Boltzmann's equation; e.g C. Villani in *Handbook of mathematical fluid dynamics, vol I*, 2002; R. DiPerna, P.L. Lions, *Commun. Math. Phys.* **120** 1-23 (1988) and *Ann. Math* **130** 312-366 (1989); *where are solutions of Boltzmann equation?, existence?*

- To solve both outlined above problems we propose to replace the pair of Banach spaces

$$< L^{\infty}(X, \Sigma, m), L^{1}(X, \Sigma, m) > \qquad (1)$$

- by the pair of Orlicz spaces (or equivalent pairs).

$$\langle L^{\cosh -1},\ L \log(L+1) \rangle. \qquad (2)$$

- The second Orlicz space $L \log(L+1)$, is the space defined by the Young's function $x \mapsto x \log(x+1),\ x \geq 0$.

- The pair of Orlicz spaces we explicitly use are respectively built on the exponential function (for the description of regular observables) and on an entropic type function (for the corresponding states).

- They form a dual pair (both for classical and quantum systems).

- This pair has the advantage of being general enough to encompass regular observables, and specific enough for the latter Orlicz space to select states with a well-defined entropy function.

- Moreover for small quantum systems this pair is shown to agree with the classical pairing of bounded linear operators on a Hilbert space, and the trace-class operators.

- The proposed quantization differs between "large systems" and "small systems".

- Consequently, the (Köthe) dual space consist of more regular states. This is a new way of removing "non-physical" states which lead to infinities. Thus a kind of renormalization is proposed.

Orlicz spaces (Bennet, Sharpley; Krasnosielsky, Rutickij)

- Basic idea: whereas $L^1(m)$, $L^2(m)$, $L^\infty(m)$ and the interpolating $L^p(m)$ spaces may be regarded as spaces of measurable functions conditioned by the functions t^p $(1 \leq p < \infty)$, the more general category of Orlicz spaces are spaces of measurable functions conditioned by a more general class of convex functions; the so-called Young's functions

- **Definition 1.** *Let* $\psi : [0, \infty) \to [0, \infty]$ *be an increasing and left-continuous function such that* $\psi(0) = 0$. *Suppose that on* $(0, \infty)$ ψ *is neither identically zero nor identically infinite. Then the function* Ψ *defined by*

$$\Psi(s) = \int_0^s \psi(u)du, \qquad (s \geq 0) \tag{3}$$

is said to be a Young's function.

- $x \mapsto |x|^p$, $x \mapsto cosh(x) - 1$, $x \mapsto xln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + 1$, $x \mapsto xln(x+1)$ are Young's functions while $x \mapsto xlnx$ not. We will assume that Young's functions are equal to 0 for $x = 0$.

- **Definition 2.** *1. A Young's function Ψ is said to satisfy the Δ_2-condition if there exist $s_0 > 0$ and $c > 0$ such that*

$$\Psi(2s) \leq c\Psi(s) < \infty, \qquad (s_0 \leq s < \infty). \tag{4}$$

2. A Young's function Φ is said to satisfy ∇_2-condition if there exist $x_0 > 0$ and $l > 1$ such that

$$\Phi(x) \leq \frac{1}{2l}\Phi(lx) \tag{5}$$

for $x \geq x_0$.

- **Definition 3.** *Let* Ψ *be a Young's function, represented as in (3) as the integral of* ψ*. Let*

$$\phi(v) = \inf\{w : \psi(w) \geq v\}, \qquad (0 \leq v \leq \infty). \tag{6}$$

Then the function

$$\Phi(t) = \int_0^t \phi(v)dv, \qquad (0 \leq t \leq \infty) \tag{7}$$

is called the complementary Young's function of Ψ*.*

- We note that if the function $\psi(w)$ is continuous and increasing monotonically then $\phi(v)$ is a function exactly inverse to $\psi(w)$.

- Define (another Young's function)

$$xln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + 1 = \int_0^x arsinh(v)dv. \tag{8}$$

- **Corollary 4.** $xln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + 1$ *and* $coshx - 1$ *are complementary Young's functions.*

- Let L^0 be the space of measurable functions on some σ-finite measure space (X, Σ, μ). We will always assume, that the considered measures are σ-finite.

- **Definition 5.** *The Orlicz space L^Ψ (being a Banach space) associated with Ψ is defined to be the set*

$$L^\Psi \equiv L^\Psi(X, \Sigma, \mu) = \{f \in L^0 : \Psi(\lambda|f|) \in L^1 \quad \textit{for some} \quad \lambda = \lambda(f) > 0\}. \tag{9}$$

- Luxemburg-Nakano norm

$$\|f\|_\Psi = \inf\{\lambda > 0 : \|\Psi(|f|/\lambda)\|_1 \leq 1\}.$$

- An equivalent - Orlicz norm, for a pair (Ψ, Φ) of complementary Young's functions is given by

$$\|f\|_\Phi = \sup\{\int |fg| d\mu : \int \Psi(|g|) d\mu \leq 1\}.$$

- L_p-spaces are nice examples of Orlicz spaces. Further, Zygmund spaces:

- – $LlogL$ is defined by the following Young's function

$$slog^+ s = \int_0^s \phi(u) du$$

  where $\phi(u) = 0$ for $0 \leq u \leq 1$ and $\phi(u) = 1 + logu$ for $1 < \infty$, where $log^+ x = max(logx, 0)$

  – L_{exp} is defined by the Young's function

$$\Psi(s) = \int_0^s \psi(u) du,$$

  where $\psi(0) = 0$, $\psi(u) = 1$ for $0 < u < 1$, and $\psi(u)$ is equal to e^{u-1} for $1 < u < \infty$. Thus $\Psi(s) = s$ for $0 \leq s \leq 1$ and $\Psi(s) = e^{s-1}$ for $1 < s < \infty$.

- To understand the role of Zygmund spaces the following result will be helpful

Theorem 6. *Take (Y, Σ, μ) for the measure space with $\mu(Y) = 1$. The continuous embeddings*

$$L^\infty \hookrightarrow L_{exp} \hookrightarrow L^p \hookrightarrow LlogL \hookrightarrow L^1 \tag{10}$$

hold for all p satisfying $1 < p < \infty$. Moreover, L_{exp} may be identified with the Banach space dual of $LlogL$.

- for any classical Orlicz space X being a rearrangement-invariant Banach function space (over a resonant measure space) one has (Bennet, Sharpley; $Interpolation\ of\ operators$, 1988)

$$L^1 \cap L^\infty \hookrightarrow X \hookrightarrow L^1 + L^\infty \tag{11}$$

- More generally, for a pair (Ψ, Φ) of complementary Young's functions with the function Ψ satisfying Δ_2-condition there is the following relation $(L^{\Psi})^* = L^{\Phi}$. In particular, $(Llog(L+1))^* = L^{cosh-1}$. $(Llog(L+1))$ is defined by e.g $x \mapsto xlog(x+1)$.

- Finally, we will write $F_1 \succ F_2$ if and only if $F_1(bx) \geq F_2(x)$ for $x \geq 0$ and some $b > 0$, and we say that the functions F_1 and F_2 are equivalent, $F_1 \approx F_2$, if $F_1 \prec F_2$ and $F_1 \succ F_2$.

- **Example 7.** *Consider, for $x > 0$*

  – $F_1(x) = xln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + 1 = \int_0^x ln(s + \sqrt{1+x^2})ds$,
  – $F_2 = kxlnx = k\int_0^x (lns + 1)ds$, $k > e$.

  Then $F_1 \succ F_2$.

- **Remark 8.** *1. Recall, $x \mapsto x ln x$ is not a Young's function. Therefore, it is difficult to speak about Orlicz space$L^{x ln x}$.*
  2. If $\Psi \succ F$, Ψ is a Young's function satisfying Δ_2-condition, the function F is bounded below by $-c$, then for $f \in L^{\Psi}$ the integral $\int F(f)(u)dm(u)$ is finite provided that the measure m is finite.

- These results lead to

  Corollary 9. *Let (X, Σ, m) be a probability space. Putting $\Psi(x) = x\log(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + 1$ and $F(x) = kx \log x$ where $k > e$ is a fixed positive number we obtain: $H(f)$ is finite provided that $f \in L^{\Psi}_{+}$.*

- One has

- **Theorem 10.** *Let Φ_i, $i = 1, 2$ be a pair of equivalent Young's function. Then $L^{\Phi_1} = L^{\Phi_2}$.*

Proposition 11. *Let* (Y, Σ, μ) *be a* σ*-finite measure space and* $L\log(L+1)$ *be the Orlicz space defined by the Young's function* $x \mapsto x\log(x+1)$, $x \geq 0$. *Then* $L\log(L+1)$ *is an equivalent renorming of the Köthe dual of* $L^{\cosh -1}$.

Proposition 12. *For finite measure spaces* $(\mathcal{X}, \Sigma, m)$ *one has*

$$L^{\cosh -1} = L_{exp}. \tag{12}$$

Consequently, for the finite measure case, $\langle L^{\cosh -1}, L\log(L+1)\rangle$ *is an equivalent renorming of* $\langle L_{exp}, L\log L\rangle$.

- Note also: the functions $xlog(x+1)$ and $x\log(x+\sqrt{1+x^2})-\sqrt{1+x^2}+1$ are equivalent.

Regular classical systems

- Let $\{\Omega, \Sigma, \nu\}$ be a measure space; ν will be called the reference measure. The set of densities of all the probability measures equivalent to ν will be called the state space $\mathcal{S}_\nu$, i.e.

$$\mathcal{S}_\nu = \{f \in L^1(\nu) : f > 0 \quad \nu - a.s., E(f) = 1\}, \tag{13}$$

$E(f) \equiv \int f d\nu$. $f \in \mathcal{S}_\nu$ implies that $f d\nu$ is a probability measure.

- **Definition 13.** *The classical statistical model consists of the measure space* $\{\Omega, \Sigma, \nu\}$*, state space* $\mathcal{S}_\nu$*, and the set of measurable functions* $L^0(\Omega, \Sigma, \nu)$*.*

- We define for a stochastic variable u on $(\Omega, \Sigma, f d\nu)$

$$\hat{u}_f(t) = \int exp(tu) f d\nu, \qquad t \in \mathbb{R}. \tag{14}$$

- and to have a selection procedure:

Definition 14. *The set of all random variables on (Ω, Σ, ν) such that for a fixed $f \in S_\nu$*

1. *$\hat{u}_f$ is well defined in a neighborhood of the origin 0,*
2. *the expectation of u is zero,*

will be denoted by $L_f \equiv L_f(f \cdot \nu)$ and called the set of regular random variables (these conditions imply that all moments are finite!).

- It was proved

Theorem 15. *(Pistone-Sempi) L_f is the closed subspace of the Orlicz space $L^{\cosh -1}(f \cdot \nu)$ of zero expectation random variables.*

- Note that there is the relation $\succ$ between the Young's function $xln(x + \sqrt{1+x^2}) - \sqrt{1+x^2} + 1$ and the entropic function $c \cdot xlnx$ where c is a positive number. Consequently, the condition $f \in L^{xln}(f \cdot \nu)$ guarantees (for finite measure case) that the continuous entropy is well defined.

Corollary 16.

$$\langle L^{\cosh -1}, L \log(L+1) \rangle$$

or equivalently

$$\langle L_{exp}, L \log L \rangle$$

provides the proper framework for the description of classical regular statistical systems (based on probability measures).

- *the proposed approach is compatible with a rigorous analysis of Boltzmann's equation* (infinite measure case).

- Why?

- (spatially homogeneous) Boltzmann's equation:

$$\frac{\partial f_1}{\partial t} = \int d\Omega \int d^3 v_2 I(g,\theta)|\mathbf{v}_2 - \mathbf{v}_1|(f_1' f_2' - f_1 f_2) \tag{15}$$

where $f_1 \equiv f(\mathbf{v}_1, t)$, $f_2' \equiv f(\mathbf{v}_2', t)$, etc, are velocity distribution functions, with $\mathbf{v}$ standing for velocities before collision, and $\mathbf{v}'$ for velocities after collision. $I(g,\theta)$ denotes the differential scattering cross section, $d\Omega$ is the solid angle element, and $g = |\mathbf{v}|$.

- The natural Lyapunov functional for this equation is the continuous entropy with opposite sign, i.e.

$$H_+(f) = \int f(x) \log f(x) dx$$

where f is supposed to be a solution of Boltzmann's equation.

- The important point to note here is the fact that DiPerna-Lions, Villani showed that the estimates

$$f \in L_t^\infty([0,T]; L^1_{x,v}((1+|v|^2+|x|^2)dxdv) \cap L\log(L+1)) \tag{16}$$

and

$$D(f) \in L^1([0,T] \times \mathbb{R}^N_x), \tag{17}$$

where $D(f) = \frac{1}{4}\int d\Omega \int d^3v_1 d^3v_2 I(g,\theta)|\mathbf{v}_2 - \mathbf{v}_1|(f_1' f_2' - f_1 f_2)\log\frac{f_1' f_2'}{f_1 f_2}$, are sufficient to build a mathematical theory of weak solutions (faithful citation; x stands for coordinate - is appearing in non-spatially homogeneous Boltzmann equation).

- Furthermore, Villani announced that for particular cross sections (collision kernels in Villani's terminology) weak solutions of Boltzmann equation are in $L\log(L+1)$.

- Also, $H_+(f)$ is nicely defined, provided that $f \in L\log(L+1)$.

- Consequently, the scheme for classical statistical mechanics based on the two distinguished Orlicz spaces $\langle L^{\cosh -1}, L\log(L+1)\rangle$ does work. However, the basic theory for Nature is Quantum Mechanics. Therefore the question of a quantization of the given approach must be considered.

- However, there is a problem:

- W. Thirring, *Quantum Mathematical Physics. Atoms, Molecules and large systems*, Springer (2002) p.322 (second edition; vol. III and IV of "Mathematical Physics")

- " To the malicious delight of many mathematicians the initial impression that type III is the rule for infinite systems has panned out with the passage of time. Types I and II turns out to be peripheral possibilities"

- see also: R. Haag's book: *Local Quantum Physics; Fields, Particles, Algebras*, 1996

Non-commutative Orlicz spaces

- Let $\mathcal{A}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with normal faithful semifinite weight φ

- Generate a bigger algebra $\mathcal{M}$ on $L^2(\mathbb{R}, \mathcal{H})$, so called cross product $\mathcal{M} = \mathcal{A} \times_\sigma \mathbb{R}$

- $\mathcal{M}$ is generated by $\pi(x),\ x \in \mathcal{A}$ and $\lambda_s,\ s \in \mathbb{R}$.

- $(\pi(x)\xi)(t) = \sigma^{\varphi}_{-t}\xi(t)$ and $(\lambda_s \xi)(t) = \xi(t-s)$

- $\mathcal{M}$ is a semifinite von Neumann algebra equipped with an fns (faithful normal semifinite) trace τ,

Non-commutative measurable functions

- Let a be a densely defined closed operator on $L^2(\mathbb{R}, \mathcal{H})$ with domain $\mathcal{D}(a)$ and let $a = u|a|$ be its polar decomposition.

- a is affiliated with $\mathcal{M}$ (denoted $a\eta\mathcal{M}$) if u and all the spectral projections of $|a|$ belong to $\mathcal{M}$.

- a is τ-measurable if $a\eta\mathcal{M}$ and there is, for each $\delta > 0$, a projection $e \in \mathcal{M}$ such that $eL^2(\mathbb{R}, \mathcal{H}) \subset \mathcal{D}(a)$ and $\tau(1 - e) \leq \delta$.

- Denote by $\widetilde{\mathcal{M}}$ the set of all τ-measurable operators.

- Haagerup's approach to non-commutative integration.

- The space of all τ-measurable operators $\widetilde{\mathcal{M}}$ (equipped with the topology of convergence in measure) plays the role of L^0 .

- Xu; Doods, Dodds, de Pagter approach

- BUT

- J. von Neumann, *Some Matrix Inequalities and Metrization of Matrix-Space*, Tomsk. Univ. Rev. **1** , 286-300 (1937)

- $g(s(a))$ gives a nice norm on the matrix algebra, where $s(a)$ is a vector formed from singular values of a while g stands for a symmetric gauge functional. a stands for a $n \times n$ matrix.

- generalized singular values

- $f \in \widetilde{\mathcal{M}}$ and $t \in [0, \infty)$, the generalized singular value $\mu_t(f)$ is defined by $\mu_t(f) = \inf\{s \geq 0 : \tau(1\!\!1 - e_s(|f|)) \leq t\}$ where $e_s(|f|)$ $s \in \mathbb{R}$ is the spectral resolution of $|f|$.

- The function $t \to \mu_t(f)$ will generally be denoted by $\mu(f)$.

- It contains essential information about f - Fack, Kosaki

- Banach Function Space of measurable functions on $(0, \infty)$.

- A function norm ρ on $L^0(0, \infty)$ is defined to be a mapping $\rho : L^0_+ \to [0, \infty]$ satisfying

  - $\rho(f) = 0$ iff $f = 0$ a.e.
  - $\rho(\lambda f) = \lambda \rho(f)$ for all $f \in L^0_+, \lambda > 0$.
  - $\rho(f + g) \leq \rho(f) + \rho(g)$ for all .
  - $f \leq g$ implies $\rho(f) \leq \rho(g)$ for all $f, g \in L^0_+$.

- Such a ρ may be extended to all of L^0 by setting $\rho(f) = \rho(|f|)$.

- Define $L^\rho(0, \infty) = \{f \in L^0(0, \infty) : \rho(f) < \infty\}$. If now $L^\rho(0, \infty)$ turns out to be a Banach space when equipped with the norm $\rho(\cdot)$, we refer to it as a Banach Function space.

- If $\rho(f) \leq \liminf_n(f_n)$ whenever $(f_n) \subset L^0$ converges almost everywhere to $f \in L^0$, we say that ρ has the Fatou Property.

- If this implication only holds for $(f_n) \cup \{f\} \subset L^\rho$, we say that ρ is lower semi-continuous.

- If $f \in L^\rho$, $g \in L^0$ and $\mu_t(f) = \mu_t(g)$ for all $t > 0$, forces $g \in L^\rho$ and $\rho(g) = \rho(f)$, we call L^ρ rearrangement invariant (or symmetric).

- When $\mathcal{M} = L^\infty(X, m)$ and $\tau(f) = \int f dm$ one gets

$$\mu_t(f) = \inf\{s \geq 0; m(\{x \in X; |f(x)| > s\}) \leq t\},$$

- so, $\mu_t(f)$ is exactly the classical non-increasing rearrangement $f^*(t)$.

- Dodds, Dodds and de Pagter formally defined the noncommutative space $L^\rho(\widetilde{\mathcal{M}})$ to be

$$L^\rho(\widetilde{\mathcal{M}}) = \{f \in \widetilde{\mathcal{M}} : \mu(f) \in L^\rho(0,\infty)\}$$

  and showed that if ρ is lower semicontinuous and $L^\rho(0,\infty)$ rearrangement-invariant, $L^\rho(\widetilde{\mathcal{M}})$ is a Banach space when equipped with the norm $\|f\|_\rho = \rho(\mu(f))$.

- For any Young's function Φ, the Orlicz space $L^\Phi(0,\infty)$ is known to be a rearrangement invariant Banach Function space with the norm having the Fatou Property.

- Thus taking ρ to be $\|\cdot\|_\Phi$, the very general framework of Dodds, Dodds and de Pagter presents us with an alternative approach to realizing noncommutative Orlicz spaces.

- Non-commutative regular systems

- 1. $n_\tau = \{x \in \mathcal{M} : \tau(x^* x) < +\infty\}$.
  2. (*definition ideal of the trace* τ) $m_\tau = \{xy : x, y \in n_\tau\}$.
  3. $\omega_x(y) = \tau(xy), \quad x \geq 0$.

- 1. if $x \in m_\tau$, and $x \geq 0$, then $\omega_x \in \mathcal{M}_*^+$.
  2. If $L^1(\mathcal{M}, \tau)$ stands for the completion of $(m_\tau, || \cdot ||_1)$ then $L^1(\mathcal{M}, \tau)$ is isometrically isomorphic to $\mathcal{M}_*$.
  3. $\mathcal{M}_{*,0} \equiv \{\omega_x : x \in m_\tau\}$ is norm dense in $\mathcal{M}_*$.

  Finally, denote by $\mathcal{M}_*^{+,1}$ ($\mathcal{M}_{*,0}^{+,1}$) the set of all normalized normal positive functionals in $\mathcal{M}_*$ (in $\mathcal{M}_{*,0}$ respectively).

- **Definition 17.** *The noncommutative statistical model consists of a quantum measure space* $(\mathcal{M}, \tau)$*, "quantum densities with respect to* τ*" in the form of* $\mathcal{M}_{*,0}^{+,1}$*, and the set of* τ*-measurable operators* $\widetilde{\mathcal{M}}$*.*

- **Definition 18.**

$$L_x^{quant} = \{g \in \widetilde{\mathcal{M}} : \quad 0 \in D(\widehat{\mu_x^g(t)})^0, \quad x \in m_\tau^+\}, \tag{18}$$

where $D(\cdot)^0$ *stands for the interior of the domain* $D(\cdot)$ *and*

$$\widehat{\mu_x^g(t)} = \int \exp(t\mu_s(g))\mu_s(x)ds, \qquad t \in \mathbb{R}. \tag{19}$$

(Notice that the requirement that $0 \in D(\widehat{\mu_x^g(t)})^0$*, presupposes that the transform* $\widehat{\mu_x^g(t)}$ *is well-defined in a neighborhood of the origin.)*

- We remind that above and in the sequel $\mu(g)$ ($\mu(x)$) stands for the function $[0,\infty) \ni t \mapsto \mu_t(g) \in [0,\infty]$ ($[0,\infty) \ni t \mapsto \mu_t(x) \in [0,\infty]$ respectively).

- To give a non-commutative generalization of Pistone-Sempi theorem we need a generalization of Dodds, Dodds, de Pagter approach i.e. that one which just was presented

- **Definition 19.** *Let $x \in L^1_+(\mathcal{M},\tau)$ and let ρ be a Banach function norm on $L^0((0,\infty),\mu_t(x)dt)$. In the spirit of Dodds, Dodds, de Pagter, we then formally define the weighted noncommutative Banach function space $L^\rho_x(\widetilde{\mathcal{M}})$ to be the collection of all $f \in \widetilde{\mathcal{M}}$ for which $\mu(f)$ belongs to $L^\rho((0,\infty),\mu_t(x)dt)$. For any such f we write $\|f\|_\rho = \rho(\mu(f))$.*

- **Remark 20.** *Comparing commutative and non-commutative regular statistical models, we note that $\mu_t(x)$ (the Lebesgue measure dt) in Definition 19 stands for f ($d\nu$, respectively).*

- The mentioned generalization of Dodds, Dodds, de Pagter approach is contained in:

Theorem 21. *Let $x \in L^1_+(\mathcal{M}, \tau)$. Let ρ be a rearrangement-invariant Banach function norm on $L^0((0,\infty), \mu_t(x)dt)$ which satisfies the Fatou property, $\rho(\chi_E) < \infty$ and $\int_E f d\mu \leq C_E \rho(f)$ for $E : \mu(E) < \infty$. Then $L^\rho_x(\widetilde{\mathcal{M}})$ is a linear space and $\| \cdot \|_\rho$ a norm. Equipped with the norm $\| \cdot \|_\rho$, $L^\rho_x(\widetilde{\mathcal{M}})$ is a Banach space which injects continuously into $\widetilde{\mathcal{M}}$.*

- and the generalization of Pistone-Sempi is given by

Theorem 22. *The set L^{quant}_x coincides with the the weighted Orlicz space $L^{\cosh -1}_x(\widetilde{\mathcal{M}}) \equiv L^\Psi_x(\widetilde{\mathcal{M}})$ (where $\Psi = \cosh -1$) of noncommutative regular random variables.*

- To show that statistics and thermodynamics can be well established for noncommutative regular statistical systems, we note that

Proposition 23. *Let* $\mathcal{M}$ *be a semifinite von Neumann algebra with an fns trace* τ *and let* $f \in L^1 \cap L\log(L+1)(\widetilde{\mathcal{M}})$, $f \geq 0$. *Then* $\tau(f\log(f+\epsilon))$ *is well defined for any* $\epsilon > 0$. *Moreover*

$$\tau(f \log f)$$

is bounded above, and if in addition $f \in L^{1/2}$ *(equivalently* $f^{1/2} \in L^1$*), it is also bounded from below. Thus* $\tau(f\log f)$ *is bounded below on a dense subset of the positive cone of* $L\log(L+1)$.

- Consequently, if the "state" is taken from the "good" noncommutative Orlicz space, then the entropy function is well defined.

- One also has the following "almost" characterization of the elements of $L\log L(\widetilde{\mathcal{M}})^+$ for which $f\log(f)$ is integrable. Zygmunt spaces are employed!

- **Proposition 24.** *As before let $\mathcal{M}$ be a semifinite von Neumann algebra with an fns trace τ. Let $f = f^* \in \widetilde{\mathcal{M}}$ be given. By χ_I will denote spectral projection of f corresponding to the interval I. If $f \in L\log L(\widetilde{\mathcal{M}})^+$ with $\tau(\chi_{[0,1]}) < \infty$, then $\tau(|f\log(f)|)$ exists (i.e. $f\log(f) \in L^1(\widetilde{\mathcal{M}})$).*

  Conversely if $\tau(|f\log(f)|)$ exists , then $f \in L\log L(\widetilde{\mathcal{M}})^+$ with $\tau(\chi_I) < \infty$ for any open subinterval I of $[0,1]$.

- Analogous to the commutative case, we get the following conclusion:

- **Corollary 25.** *Either of the pairs*

$$\langle L^{\cosh -1}, Llog(L+1)\rangle$$

or

$$\langle L_{exp}, L\log L\rangle$$

provides an elegant rigorous framework for the description of non-commutative regular statistical systems, where now the Orlicz (and Zygmund) spaces are noncommutative.

- REMARK:

- There is an exceptional case: $\mathcal{B}(\mathcal{H})$; then $\widetilde{\mathcal{B}(\mathcal{H})} = \mathcal{B}(\mathcal{H})$ and the standard formulation is appearing.

- DYNAMICS:

1. a large class of positive maps defined on $\mathcal{M}$ can be lifted to maps on the corresponding non-commutative Orlicz space (Illinois J. Math)
2. Composition operator can be defined and studied (Illinois J. Math).
3. General recipe for dynamical maps.
   (a) Potential theory: Beurling-Deny, Bakry; Dirichlet forms, Dirichlet spaces, Markov semigroups
   (b) non-commutative generalization: Albeverio-Hoegh-Krohn, Davies-Lindsay, Cipriani-Sauvageot; in particular Dirac's operator is nicely defined!
   (c) The spaces $L^1 \equiv \mathfrak{M}_*$, $L^2 \equiv L^2(\mathfrak{M})$, and $L^\infty \equiv \mathfrak{M}$ are appearing.

(d) (L^∞, L^2) and (L^2, L^1); Calderon couples. Moreover, L^{cosh-1} and $Llog(L+1)$ are nicely "inserted"; Karlovich, Maligranda.

(e) Interpolation techniques give well defined dynamics on the above Orlicz spaces; Bennett-Sharpley; Lindestrauss-Tzafriri.

(f) The given scheme "invites" log-Sobolev "industry". This provides a proper framework to study asymptotic and decay to equilibrium problems.

- Consequently, the proper framework for a solution of "Boltzmann dream" was presented!