BERNT MICHAEL HOLMBOE'S TEXTBOOKS

and the development of mathematical analysis in the 19th century

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ABSTRACT

Bernt Michael Holmboe (1795–1850) was a teacher at Christiania Kathedralskole in Norway, from 1818 till 1826. After that he was lecturer at the University of Christiania until 1834, when he was appointed professor in pure mathematics, a position he held until his death in 1850. Holmboe wrote textbooks in arithmetic, geometry, stereometry and trigonometry for the learned schools in Norway, and one textbook in higher mathematics, and some of them came in several editions. This paper will focus on the first three editions of the textbook in arithmetic, and the way he dealt with the rigour in mathematical analysis.

In the first half of the 19th century there was a growing demand for rigour in the foundations and methods of mathematical analysis, and this led to a thorough reconceptualisation of the foundations of analysis. Niels Henrik Abel (1802–1829) complained in a letter to Christopher Hansteen (1784–1873) in 1826 that mathematical analysis totally lacked any plan and system, and that very few theorems in the higher analysis had been proved with convincing rigour.

The teaching of mathematics was an important motivation for this rigourisation, and I will try to demonstrate that the textbooks of Holmboe reflected the contemporary development in mathematical analysis.

1 Introduction

The 19th century was an important century in the development of the modern number concept. After the introduction of new methods like analytic geometry, differential and integral calculus, and algebraic transformation of equations in the 17th century, the 18th century gave the world influential results by the use of these methods. Calculations with *negative numbers* became unproblematic in this century, and we also saw the start of definitions of what later developed into what we now know as irrational numbers. What characterized the 19th century was the demand for rigour with the basis of the methods and fundamental concepts. From the beginning of the century the concept of number was thoroughly reinvestigated. The demand for purity in methodology did *not* allow the use of geometrical or purely intuitive arguments as basis of the number concepts, and the development led finally to the definition of the *real numbers* in 1872 by Richard Dedekind (1831–1916) and other equivalent definitions by mathematicians like Cantor, Weierstrass, Meré etc. There were two mathematicians who made important contributions in the definition of *irrational numbers* in the beginning of the 19th century, namely Bernard Bolzano (1781–1848) and Martin Ohm (1792–1872). (Gericke 1996)

The kingdom Denmark/Norway introduced a new school reform around year the 1800 which in many ways strengthened the position of the discipline of mathematics, and from now on may we talk about proper teaching in mathematics in the higher education (Piene 1937). There was much work done in the last decades of the 18th century to reintroduce mathematics as a subject in school.

The motivation and background for my research is to make an analysis of the development of mathematics education – and the didactical debate – in the first half of the 19th century in Norway, in view of the development of mathematical analysis, and to throw some light on Bernt Michael Holmboe's significance.

This paper is based on an oral presentation given at the ESU–6 conference in Vienna. It is part of an ongoing PhD project about Bernt Michael Holmboe and his textbooks in mathematics, and it is work in progress.

1.1 Bernt Michael Holmboe

Bernt Michael Holmboe was born on the 23rd of March 1795 in Vang in Valdres, centrally situated in Southern Norway, and he died on the 28th of March 1850, at the age of 55 years and 5 days. Holmboe was a teacher at *Christiania Kathedralskole* from 1818 till 1826, and after that he was lecturer at the *University of Christiania* until 1834, when he was appointed professor in mathematics, a position he had until his untimely death. Holmboe's home burnt down shortly after his death, and some of his letters and works were lost, in addition to some of Abel's letters and works. (Bjerknes 1925: 56,79)

Among Holmboe's students we find great mathematicians like Niels Henrik Abel, Ole Jacob Broch and Carl Anton Bjerknes. Holmboe proclaimed that in no other subject did novices complain more than in mathematics (Piene 1937), and his aim was to make the students familiar with mathematical signs *before* a more methodical study. He further stated that unless pupils engaged in «uninterrupted practice»[1] even persons with several years of education would find that mathematics is «something of a mind consuming and boring matter».[2] The lecture notes of Carl Anton Bjerknes shows, however, that Holmboe's teaching was characterized by *pre-abelian* times, in spite of his knowledge of Abel and his works (Bjerknes 1925).

Holmboe wrote in a letter to the then 24 years old Carl Anton Bjerknes (Holmboe 1849) that he is inspired by the great French mathematician Joseph-Louis Lagrange (1736–1813). Bjerknes had asked Holmboe to advise him about studies in mathematics, and Holmboe wrote «*The best I have to state in this respect is to inform you about some notes from Lagrange and some rules and remarks by him regarding the study of mathematics, which I found in Lindemanns and Bohnebergers* Zeitschrift für Astronomie *about 30 years ago … Those who really want, should read Euler, because in his works all is clear, well said, well calculated, because there is an abundance of good examples, and because one should always study the sources*».[3]

[1] «idelig øvelse»

[2] «noget åndsfortærende og kjedsommelig tøi»

[3] «Som det bedste jeg her i denne Henseende veed at anføre, vil jeg meddelle nogle Notiser om Lagrange og en Deel Regler og Bemærkninger av ham angaaende Mathematikens Studium, hvilke jeg for omtrent 30 Aar siden fandt i Lindmann's (sic!) og Bohnebergers *Zeitschrift für Astronomie* … De som virkelig vil, bør lese Euler, fordi i hans skrifter alt er klart, godt sagt, godt regnet, fordi de vrimler av gode eksempler og fordi man altid bør studere kildene.»

1.2 Holmboe's textbooks

These are the textbooks that Holmboe wrote, and I will in this paper focus on the three first editions of the Arithmetic (Holmboe 1825, 1844, 1850), the three editions that came out in Holmboe's lifetime.

Title	Edition	Edited by	Publisher
Lærebog i Mathematiken. Første Deel, Inneholdende Indledning til Mathematiken samt Begyndelsesgrundene til Arithmetiken	1825 1844 1850 1855 1860		Jacob Lehmann J. Lehmanns Enke J. Chr. Abelsted R. Hviids Enke R. Hviids Enke
Lærebog i Mathematiken. Anden Deel, Inneholdende Begyndelsesgrundene til Geometrien	1827 1833 1851 1857	 Jens Odén Jens Odén	Jacob Lehmann Jacob C. Abelsted R. Hviids Enke J. W. Cappelen
Stereometrie	1833 1859	 C. A. Bjerknes	C. L. Rosbaum J. Chr. Abelsted
Plan og sphærisk Trigonometrie	1834		C. L. Rosbaum
Lærebog i den høiere Mathematik. Første Deel	1849		Chr. Grøndahl

2 The development of mathematical analysis

I will now take a closer look at three important participants in the development of mathematical analysis.

2.1 Bernard Bolzano

An intuitive and geometrical interpretation of real numbers did not satisfy the 19th century mathematicians' demand for purity in methodology, and the need for a new understanding of real numbers did arise in connection with the proof of the *intermediate value theorem*. This theorem was proven by Bolzano (1817), and both continuity of functions and convergence of infinite series are defined and used correctly in this paper, as it is understood in a modern sense. Bolzano's comprehensive paper was later translated into English (Russ 1980, 2004), and the latest of these two translations forms the basis of the following short description.

According to Russ (1980: 157), Bolzano's paper includes «*the criterion for the (pointwise) convergence of an infinite series, although the proof of its sufficiency, prior to any definition or construction of the real numbers, is inevitably inadequate*». This criterion is, however, *not* concerning the definition of convergence, but the Bolzano-Weierstrass theorem, which states that every bounded sequence has a convergent subsequence.

Bolzano's only strict requirement is «*that examples never be put forward instead* *of* proofs *and that the essence* [Wesenheit] *of a deduction never be based on the merely figurative use of phrases or on associated ideas, so that the deduction itself becomes void as soon as these are changed*»[4] (Russ 2004: 256).

The comprehension of the concepts of real numbers is clear when Bolzano emphasizes that «*between any two nearby values of an* independent variable*, such as the root x of a function, there are always infinitely many* intermediate values.» He continues that for any continuous function, there is no *last* x which makes it negative, and there is no *first* x which makes it positive (Russ 2004: 258). Bolzano states in a theorem (Russ 2004: 266–67) that if in a series[5] of quantities, the difference between the nth term and every later term is smaller than any given quantity if n has been taken large enough, then there is always a constant quantity, and only one, which the terms of the series approach, and to which it can come as near as desired if the series is continued far enough. In a following note (Russ 2004: 268), he explains that if one is trying to determine the value of such a quantity as described above, namely by using one of the terms of which the given series is composed, then the value cannot be determined entirely accurately unless all terms after a certain term are equal to one another. One cannot, however, conclude that the value of a quantity is irrational if it cannot be determined accurately by the terms of a certain series.

It is worthwhile to emphasize the fact that Bolzano's notion of «irrational» is *not* the classical definition by the Greeks. Russ (2004: 2) writes that «*In the 1830s, he* [Bolzano] *began an elaborate and original construction of a form of real numbers – his so-called 'measurable numbers'. ... He formulated and proved (1817) the greatest lower bound property of real numbers which is equivalent to what was to be called the Bolzano-Weierstrass theorem. He later gave a superior proof with the aid of his measurable numbers.*»

Bolzano introduces *measurable numbers* in his *Pure Theory of Numbers*[6] (Russ 2004: 347–49, 360–61). According to the definition, S is *measurable* when

$$\forall q \in \mathbb{N} \quad \exists p \in \mathbb{Z} \quad \text{such that} \quad S \;=\; \frac{p}{q} + p_1 \;=\; \frac{p+1}{q} - p_2 \quad \text{where} \quad p_1 \geq 0, \quad p_2 > 0$$

In other words

$$\frac{p}{q} \;\leq\; S \;<\; \frac{p+1}{q}$$

Bolzano explains that p_1 and p_2 denotes a pair of strictly positive number expressions, the former possibly denoting zero (Russ 2004: 361).[7]

[4]Bolzano also criticised Gauss's original proof of the fundamental theorem of algebra of 1799, because Gauss here used geometrical considerations to prove an algebraic theorem (Otte 2009: 53). Bolzano did not doubt the validity of the theorem, but he criticised the «impurity» of the method.

[5]Russ uses the word «*series*» in this English translation of Bolzano's theorem, where the text might indicate that «*sequence*» would be correct. An equivalent use of «*series [Reihe]*» is found in Bolzano's *Paradoxien des Unendlichen*, (Russ 2004: 602).

[6]«Reine Zahlenlehre». Was not published until late 20th century (Russ 2004: 681). Bolzano wrote in a letter, dated 5th of April, 1835, to one of his former students, that he had «*one book near completion with the title* Pure Theory of Numbers *consisting of two volumes: an* Introduction to Mathematics*, the first concepts of the general theory of quantity, and then the* Theory of Numbers *itself*» (Russ 2004: 347).

[7]Bolzano denoted them p^1 and p^2, but the superscripts only distinguished, they where not powers (Russ 2004: 349). I have chosen to use subscripts to avoid confusion.

The measurable number may be used to measure, or determine by approximation, the magnitude or quantity. Bolzano called the fraction $\frac{p}{q}$ the *measuring fraction*, and the fraction $\frac{p+1}{q}$ the *next greater fraction*. p_1 is called the *completion* of the measuring fraction since $S = \frac{p}{q} + p_1$. Every *rational number* is a measurable number where $p_1 = 0$, and indeed a complete measure.

Abel is the only known reference that Bolzano was known already in the 1820s, as he mentions Bolzano in his Paris notes (Schubring 1993: 45). Schubring (1993: 50) writes that during the four months Niels Henrik Abel stayed in Berlin in 1825, he was in close contact with August Leopold Crelle and his mathematical circle, where he was engaged in intensive conversations on all mathematical issues. Crelle had Bolzano's three booklets in his personal library, and Abel's reading of Bolzano was part of this process of communication.

2.2 Augustin Louis Cauchy

The major work of AUGUSTIN-LOUIS CAUCHY (1789–1857), *Cours d'analyse* from 1821 (Bradley and Sandifer 2009), was designed for students at *École Polytechnique*, and Cauchy was concerned with developing the basic theorems of the calculus as rigorously as possible.

Cauchy makes a clear distinction between *number* and *quantity* (Bradley and Sandifer 2009: 5–6,267–68), where *numbers* arise from the absolute measure of magnitudes, and *quantities* are real positive or negative quantities, that is to say numbers preceded by the signs «+» or «−». Quantities are intended to express *increase* or *decrease*,[8] and to indicate this intention, we represent the sizes that ought to serve as increase or decrease as numbers preceded by a sign. The *numerical value*, or absolute value, of a quantity is the number that forms the basis of the quantity, and we may perform arithmetical operations on a quantity. Absolute numbers are considered to be only the *proper* numbers. Negative numbers are quantities, and the difference between a number and a quantity is the difference that is between an absolute number, and a number with an additional quality, for instance a sign. A positive number may therefore be *both* a number and a quantity (Schubring 2005: 446).

Cauchy says that a variable is called *variable* if it is able to take on successively many different values, as opposed to a *constant* when it takes on a fixed and determined value. A variable quantity becomes *infinitely small* when its numerical value decreases indefinitely so that it converges towards the limit zero. Corresponding, a variable quantity becomes *infinitely large* when its numerical value increases indefinitely so that it converges towards the limit ∞ (Bradley and Sandifer 2009: 6,21-22).

He had to leave France for political reasons, and he met Bolzano in Prague (Schubring 2005: 429). Some historians (*see* Grabiner 2005: 11) assert that Cauchy was very influenced by Bolzano, and especially his proof of the *intermediate value theorem* (Bolzano 1817), when he wrote *Cours d'analyse*. His works made, however, a new standard for the demands of rigour in connection with concepts like *limit* and *convergence*. Cauchy neither started nor completed the rigourization of analysis, but he was, according to Grabiner (2005: 166), more than any other mathematician responsible for the first great revolution in mathematical rigourization since the Greeks in the Antiquity.

[8] *accroissements* or *diminuitions* (Schubring 2005: 446)

2.3 Martin Ohm

Martin Ohm was autodidact in mathematics, and a teacher, which may have lead him to the idea that mathematics needed to be completely revised, in order to reach a broader public and to achieve a higher logical clarity. He made an attempt to make a completely consistent foundation for mathematics (Ohm 1832) – to found a concept of numbers free of contradictions, and to provide an appropriate conception of negative numbers for it. Prior to that, he wrote an elementary book on number theory (Ohm 1816), and a book in pure elementary mathematics in two volumes.[9] The separation of number and quantity was the core of Ohms number concept, and there are no other numbers than «absolute integer numbers». Zero, negative numbers, rational and irrational numbers are extensions. (Schubring 2005: 521pp)

Ohm defines the arithmetical operations *addition* and *multiplication* traditionally in the first volume of pure elementary mathematics (Gericke 1996: 136). One adds two number by imagining a number that has as many units as the two numbers together, and one multiplies a number a by a number m by adding a, m times to itself. The quotient $\frac{a}{b}$ is defined as the number that gives a when it is multiplied by b.

3 Textbooks and irrational numbers

In the following sections, I will show some of my findings concerning the development of Holmboe's definitions of irrational numbers, and compare that with definitions found in the other textbooks in arithmetics and algebra in the Danish language, that may have been used or known in the learned schools in Norway in the first half of the 19th century. The focus will be on their way of handling the rigour in mathematical analysis.

The mathematical notation in the English translations are in some cases slightly modernized by me, but the original quotation in the footnotes are unaltered.

3.1 Bernt Michael Holmboe

In the first edition of Holmboe's textbook in arithmetics (Holmboe 1825), irrational numbers are defined as

> *Any number, that can be expressed neither as a whole number nor as a fraction, whose numerator and denominator are whole and finite numbers, is called an* irrational *number.*[10]

This definition tells us what an irrational number *isn't*, it doesn't tell us what it *is*. Opposed to irrational numbers, all whole numbers and fractions, whose numerator and denominator are whole and finite numbers, are called *rational* numbers. The definition has some corollaries, or supplements, and the following corollary explains intuitively that

> *One can always find a rational number, whose value approaches the value of a given irrational root, so that the difference between them is less than any given unit fraction.*[11]

[9] *Die reine Elementar-Mathematik*, 1825 and 1826

[10] «Ethvert Tal, der hverken kan udtrykkes som et heelt Tal eller som en Brøk, hvis Tæller og Nævner ere hele og endelige Tal, kaldes er *irrationalt* Tal.» (Holmboe 1825: 134)

[11] «Man kan altid finde er rationalt tal, hvis Værdie nærmer sig Værdien af en given irrational Rod saameget, at Forskjellen mellem begge er mindre end en given Brøkeenhed.» (Holmboe 1825: 136)

The definition of irrational numbers in Holmboe (1844), and subsequent editions, may indicate an influence by Bolzano and his definition of measurable numbers. It is also noteworthy that Holmboe now specifies *magnitude*, and not number.

> *Any magnitude, that can be expressed neither as a whole number nor as a fraction, whose numerator and denominator are whole and finite numbers, but whose value always falls between two fractions $\frac{t}{n}$ and $\frac{t+1}{n}$, where t and n are whole numbers, and where one can make n larger than any given number, is called an* irrational *number.*[12]

This following corollary from Holmboe (1844) shows how two magnitudes between the same two limits must be the same magnitude. In Holmboe (1850), the specification of *irrational* is taken out – the statement is valid for *real* magnitudes, rational and irrational.

> *If two irrational, positive magnitudes P and Q, both independent of n and between boundaries of the form r and $r+\frac{a}{n}$, are in such a way that $r < P < r+\frac{a}{n}$ and $r < Q < r+\frac{a}{n}$, where one can make n larger than any given number and a is finite. Then is $P = Q$.*[13]

Finally, this definition from Holmboe (1850) shows that if x and y are two irrational magnitudes, squeezed between two pairs of rational limits, then the sum $x+y$ is squeezed between the sums of the lower and of the upper limits, and the difference of these to sums of limits also disappears when n grows infinitely, which makes the sum $x+y$ unique.

> *If one or both of two magnitudes, x and y, are irrational, and where $\frac{t}{n} \leq x < \frac{t+1}{n}$ and $\frac{p}{n} \leq y < \frac{p+1}{n}$, and one can make n larger than any given number. The sum $x+y$ is then to be understood as the common boundary for the sums $\frac{t}{n}+\frac{p}{n}$ and $\frac{t+1}{n}+\frac{p+1}{n}$, whose difference is $\frac{2}{n}$, which disappears when n grows infinitely.*[14]

3.2 Hans Christian Linderup

The Danish teacher of mathematics, HANS CHRISTIAN LINDERUP (1763–1809) published a basic textbook (Linderup 1807), where he proves that a power of an irreducible fraction never can be a whole number. As a corollary to this he claims that *when the root of a whole number is not a whole number, it is also not a fraction, whose numerator and denominator are finite numbers, but as any quantity must be expressed by whole numbers or fractions, must this root necessarily be a fraction; it is consequently a fraction, whose*

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[12] «Enhver Størrelse, der hverken kan udtrykkes som et heelt Tal eller som en Brøk, hvis Tæller og Nævner ere hele og endelige Tal, men hvis Værdie altid falder mellem to Brøker $\frac{t}{n}$ og $\frac{t+1}{n}$, hvor t og n ere hele Tal, og hvor man kan gjøre n større end ethvert givet Tal, kaldes er *irrationalt* Tal. (Holmboe 1844: 128)»

[13] «Falde 2 irrationale positive af n uafhengige Størrelser P og Q mellem Grændser af Formen r og $r+\frac{a}{n}$, saaledes at $P > r$, $P < r+\frac{a}{n}$, $Q > r$, $Q < r+\frac{a}{n}$, hvor stor man gjør n, naar a er en endelig Størrelse: saa er $P = Q$;» (Holmboe 1844: 128–29)

[14] «Er af Størrelserne x og y den ene eller begge irrationale, $x =$ eller $> \frac{t}{n}$ og $x < \frac{t+1}{n}$, $y =$ eller $> \frac{p}{n}$ og $y < \frac{p+1}{n}$, hvor stor man gjør n, saa forstaaes ved Summen $x+y$ den fælles Grændse for Summerne $\frac{t}{n}+\frac{p}{n}$ og $\frac{t+1}{n}+\frac{p+1}{n}$, hvilke Summers Forskjel er $\frac{2}{n}$, der forsvinder med $\frac{1}{n}$, det er, naar n voxer i det Uendelige.» (Holmboe 1850: 115–16)

numerator and denominator are infinitely large, and therefore may never be expressed exact. Such root magnitudes are called irrational numbers.[15] The term *infinitely large* must here imply that the fraction is a limit, reached by stepwise expanding the numerator and denominator. This mode of expression is intuitive, and does not have an evident mathematical definition.

Linderup is here talking about *any* root, and he specifies this in the next corollary, by defining that if the square root of a number is a whole number, then the original number is a *perfect square number*,[16] if the cube root of a number is a whole number, then the original number is a *perfect cube number*,[17] and likewise for any power. If the root on the other hand is *not* a whole number, then the number is called an *imperfect square and cube number*,[18] or in general, an *imperfect power*.[19] He concludes that *all* roots of imperfect powers are irrational numbers.

3.3 Ole Jacob Broch

Holmboe's former student, and successor as professor in mathematics at the university, Ole Jacob Broch (1818–1889), published in 1839 a collection of exercises and examples (Broch 1839) to Holmboe's textbook. He also published in 1860 a textbook in arithmetic and algebra which succeeded Holmboe's textbook in the learned schools. In this book he defines in the introduction that *mathematics* is the science about *magnitudes* and their connections. The study of magnitudes separate from any matter is called *pure mathematics*, and the study of magnitudes belonging to material objects is called *applied mathematics*.[20] He continues to say that if we look at several magnitudes of the same kind, and we focus first on one specific magnitude, and then on the whole collection of magnitudes, then we have the conception of *one* magnitude and of *several* magnitudes. The word *unit* is used to characterize any of these homogeneous magnitudes, and the word *number* is used to characterize a collection of units as well as the unit itself.[21]

He shows in the section about powers and roots that when the root value of a natural number is *not* a natural number, it can *not* be a fraction either, where both numerator and denominator are whole and finite numbers (Broch 1860: 184). He continues that *one may always approximate its value by a fraction, such that the difference between the root value and the fraction is smaller than any given finite positive magnitude, however*

[15] «Naar altsaa Roden til et heelt Tal ikke er et helt Tal, er det heller ingen Brøk, hvis Tæller og Nævner ere endelige Tal; men da enhver Mængde maa kunde udtrykkes ved hele Tal og Brøk, maa denne Rod nødvendig være en Brøk; den er altsaa en Brøk, hvis Tæller og Nævner ere uendelig store, og følgelig aldrig nøiagtig kan udtrykkes. Saadanne Rod-Størrelser kaldes irrationale tal.» (Linderup 1807: 99).

[16] fuldkomment Quadrat-Tal

[17] fuldkomment Cubik-Tal

[18] ufuldkomment Quadrat- og Cubic-Tal

[19] ufuldkommen Potens

[20] «*Mathematik* er Læren om *Størrelser* og deres Forbindelser. Forsaavidt den betragter Størrelserne afsondrede fra enhver Materie, kaldes den *reen Mathematik*; betragter den derimod Størrelserne som henhørende til materielle Gjenstande, kaldes den *anvendt Mathematik*.» (Broch 1860: 1)

[21] «Naar vi betragte flere Størrelser af samme Art, og vi henvende vor Opmærksomhed først paa een af disse Størrelser i Særdeleshed, dernæst paa den hele Samling af Størrelser, saa have vi Begrebet om *een* Størrelse og om *flere* Størrelser. Man benytter ordet *Enhed* for at betegne en hvilkensomhelst af disse eensartede Størrelser, og man betegner ved ordet *Tal* saavel den hele samling ef Enheder som ogsaa Enheden selv.» (Broch 1860: 1)

small it is chosen.[22]

$$p \ < \ \sqrt[n]{a} \ \ < \ p + \frac{1}{x}$$

p is here simply called a *number*,[23] but it is obvious that Broch means a rational number, and $\frac{1}{x}$ is a *unit fraction*.[24] An *irrational magnitude* is then defined as a magnitude that cannot be expressed by a finite number of digits, either as a whole number or as a fraction, whose numerator and denominator are whole and finite numbers, but whose value always may be given within two limits, specified by a finite number of digits, and whose difference may be made smaller than any positive magnitude, or may approach zero as much as one wants. If the limits that the irrational magnitude is bounded by are positive, then the magnitude is an *irrational number* (Broch 1860: 186–87).[25] Broch's definition is general, since the irrational magnitude does not need to be a root. The irrational magnitude may also be a magnitude that cannot be expressed by a root with a finite number of digits under the radical sign.

Unlike a rational number, an irrational may never be transformed into a periodic decimal fraction. He also shows that two irrational magnitudes are equal if they are bounded by the same two limits, whose difference may be made smaller than any given positive magnitude, however small it is chosen. In other words

$$p \ < \ a \ < \ p + \frac{1}{x} \quad \wedge \quad p \ < \ b \ < \ p + \frac{1}{x} \quad \Longrightarrow \quad a \ = \ b$$

Furthermore, by adding or multiplying these limits of two irrational numbers respectively, the difference between the sum or product may be made smaller than any given positive magnitude, however small it is chosen. This extends the concept of addition and multiplication to include irrational numbers. All theorems regarding addition and multiplication are only proven for rational numbers, and must therefore also be valid for the rational limits of irrational numbers. As the difference of these limits may be made smaller than any finite positive magnitude, and these limits determine the irrational numbers, then must consequently all these theorems also be valid for irrational numbers.

The doctrines about subtraction and division are based on the doctrines about addition and multiplication, and consequently must all theorems valid for rational numbers also be valid for irrational numbers. Likewise must the doctrines about powers be valid if the root is irrational. (Broch 1860: 189–90)

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[22] «Men man kan isaafald dog stedse tilnærmelsesviis udtrykke dens Værdi ved en saadan Brøk, saaledes at dennes forskjel fra Rodstørrelsen bliver mindre end enhver given endelig positiv Størrelse, hvor liden denne end er valgt.» (Broch 1860: 185)
Broch's original symbolic notation is « $\sqrt[n]{a} \ \ > \ p$ og (*and*) $< \ p + \frac{1}{x}$»

[23] Tal

[24] Brøkenhed

[25] «En Størrelse, som ikke kan udtrykkes med et endelig Antal Ziffre, hverken som et heelt Tal eller som en Brøk, hvis Tæller og Nævner ere hele og endelige Tal, men for hvis Værdi der altid kan angives to med et endeligt Antal Ziffre udtrykte Grændser, hvis Differents kan gjøres mindre end enhver given endelig positiv Størrelse, eller bringes til at nærme sig Nul saameget man vil, kaldes en *irrational Størrelse*, eller, hvis dens Grændser ere positive, et *irrationalt Tal.*»

4 A brief summary

The following is an overview of the significant events mentioned in this paper.

1816–17: Bernhard Bolzano publishes *binomische Lehrsatz*, and proof of the *intermediate value theorem*
1816: Martin Ohm publishes elementary number theory
1821: Cauchy publishes «*Cours d'analyse*»
1822–: Martin Ohm works on «*Complete system ...* »
1825: First edition of Arithmetiken
1825–26: Abel is in Berlin where he meets Crelle and Ohm, and in Paris where he meets Cauchy.
1830ies: Bolzano writes about *measurable* numbers (unpubl.)
1844: Second edition of Arithmetiken
1850: Third edition of Arithmetiken

Abel, who was Holmboe's student at Christiania Kathedralskole and who remained Holmboe's confident friend the rest of his life, is the only known reference that Bolzano was known already in the 1820ies. He mentions Bolzano in his Paris notes. August Leopold Crelle *had* the three books that Bolzano published in 1816–17. The way Holmboe writes about irrational numbers in the editions from 1844 and later is very similar to Bolzano's description of measurable numbers, and not similar to other contemporary textbook authors.

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