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# Coherent-state quantization of constrained fermion systems

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**Abstract.** The quantization of systems with first- and second-class constraints within the coherent-state path-integral approach is extended to quantum systems with fermionic degrees of freedom. As in the bosonic case the importance of path-integral measures for Lagrange multipliers, which in this case are in general expected to be elements of a Grassmann algebra, is emphasized. Several examples with first- and second-class constraints are discussed.

## 1 Introduction

The quantization of constrained systems has recently been reexamined [1–4] from the point of view of coherent-state path integrals, which revealed significant differences from the standard operator and path-integral approaches. The aim of this contribution is to extend this approach, formulated for bosonic degrees of freedom, to fermionic systems. That is, we will discuss the generalization of the approach of [1] to constrained quantum systems with fermionic degrees of freedom. As in the bosonic case we will utilize the (fermion) coherent-state path-integral approach. In essence the basic idea of inserting projection operators via proper path-integral measures for Lagrange multipliers is the same as in the bosonic case [1]. Therefore, we will closely follow the approach of [1] and put more emphasize on the presentation of various examples with first- as well as second-class constraints. We will omit a discussion of the classical version of such systems, that is, the so-called pseudomechanics [5,6] which is the classical dynamics of Grassmann degrees of freedom. Also the quantization of such systems (without constraints) is well discussed in the literature [7,6]. Note that due to the Grassmannian nature, the classical dynamics formulated in phase-space always exhibits second-class constraints which, however, can easily be removed [6]. For these reasons we will exclusively concentrate our attention on fermionic quantum systems with operator-valued constraints.

The outline of this paper is as follows. In Sect. 2 we will review some basic concepts of quantum systems consisting of $N$ fermionic degrees of freedom. In particular, we discuss several properties of fermion coherent states and the associated path-integral approach. In doing so we shall also give a minimal review Grassmann theory. Section 3 is devoted to a general discussion of first-class constraints including a construction method for projection operators following [1]. In Sect. 4 several examples with first-class constraints are discussed. In Sect. 5 we briefly outline the generalization of the treatment of second-class constraints of [1] to fermion systems. Section 6 presents a discussion for a wide range of odd second-class constraints on the basis of typical examples. Finally, in Sect. 7 we consider an example of a constrained boson-fermion system.

## 2 Basic concepts of fermionic degrees of freedom

### 2.1 Grassmann numbers

It is well-known that Grassmann numbers may serve as classical analogues of fermionic degrees of freedom. To be more explicit, the "classical phase space" of $N$ fermions may be identified with the Grassmann algebra $\mathbb{C}B_{2N}$ over the field of complex numbers [8,9], which is generated by the set $\{\bar\psi_1,\ldots,\bar\psi_N,\psi_1,\ldots,\psi_N\}$ of $2N$ independent Grassmann numbers obeying the anticommutation relations

$$\begin{aligned}\{\psi_i,\psi_j\} &:= \psi_i\psi_j+\psi_j\psi_i = 0\,,\\ \{\psi_i,\bar\psi_j\} &= 0\,,\quad \{\bar\psi_i,\bar\psi_j\}=0\,.\end{aligned} \tag{1}$$

This algebra allows for a natural $\mathbb{Z}_2$ grading by appointing a degree (also called Grassmann parity) to all homogeneous elements (monomials) of $\mathbb{C}B_{2N}$:

$$\deg\left(\bar\psi_{j_1}\cdots\bar\psi_{j_m}\psi_{i_1}\cdots\psi_{i_n}\right) := \begin{cases}0 & \text{for } m+n \text{ even}\\ 1 & \text{for } m+n \text{ odd}\end{cases}\,. \tag{2}$$

In other words, the even elements of $\mathbb{C}B_{2N}$ are commuting and the odd elements are anticommuting numbers. For further details we refer to the textbooks by Cornwell [8] and Constantinescu and de Groote [9]. Here we close by giving the convention of Grassmann integration and differentiation used in this paper:

$$\int d\psi\, 1 = 0\,, \quad \int d\psi\, \psi = 1\,, \quad \frac{d}{d\psi}\, 1 = 0\,, \quad \frac{d}{d\psi}\, \psi = 1\,. \tag{3}$$

Here $\psi$ stands for any of the $2N$ generators of $\mathbb{C}B_{2N}$, and the integration and differentiation operators are treated like odd Grassmann quantities according to the $\mathbb{Z}_2$ grading (2).

## 2.2 Fermion coherent states

Throughout this paper we will consider quantum systems with a finite number, say $N$, of fermionic degrees of freedom which are characterized by annihilation and creation operators $f_i$ and $f_i^\dagger$, $i = 1, 2, \ldots, N$, obeying the canonical anticommutation relations

$$\{f_i, f_j\} = 0\,, \quad \{f_i^\dagger, f_j^\dagger\} = 0\,, \quad \{f_i^\dagger, f_j\} = \delta_{ij}\,. \tag{4}$$

The corresponding Hilbert space is the $N$-fold tensor product of the two-dimensional Hilbert spaces $\mathcal{H}_i \equiv \mathbb{C}^2$ for a single degree of freedom,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N = \mathbb{C}^{2^N}\,. \tag{5}$$

A standard basis in this "$N$-fermion" Hilbert space $\mathcal{H}$ is the simultaneous eigenbasis of the number operators $f_i^\dagger f_i$:

$$f_i^\dagger f_i |n_1 n_2 \ldots n_N\rangle = n_i |n_1 n_2 \ldots n_N\rangle\,, \quad n_i = 0, 1\,, \tag{6}$$

where

$$|n_1 n_2 \ldots n_N\rangle := |n_1\rangle_1 \otimes |n_2\rangle_2 \otimes \cdots \otimes |n_N\rangle_N \tag{7}$$

with $|n\rangle_i$ being a vector in the one-fermion Hilbert space $\mathcal{H}_i$ on which the operators $f_i$ and $f_i^\dagger$ are acting via

$$\begin{aligned} f_i|0\rangle_i &= 0\,, \quad f_i|1\rangle_i = |0\rangle_i\,, \\ f_i^\dagger|0\rangle_i &= |1\rangle_i\,, \quad f_i^\dagger|1\rangle_i = 0\,. \end{aligned} \tag{8}$$

Fermion coherent states are defined in analogy to the canonical (boson) coherent states [10–12]. They qualitatively differ, however, from the latter as the basic quantities labeling these states are not ordinary c-numbers but rather are odd Grassmann numbers. To be more precise, they are the generators of the classical phase space $\mathbb{C}B_{2N}$. For simplicity let us consider in the following discussion only one fermionic degree of freedom, that is, we set $N = 1$ and subscripts will be omitted. Then the fermion coherent states are defined [10–12] as follows:

$$|\psi\rangle := \exp\{-\tfrac{1}{2}\bar\psi\psi\} e^{f^\dagger\psi}|0\rangle = \exp\{-\tfrac{1}{2}\bar\psi\psi\}\Big(|0\rangle - \psi|1\rangle\Big)\,. \tag{9}$$

The corresponding adjoint states read

$$\langle\psi| := \exp\{-\tfrac{1}{2}\bar\psi\psi\}\langle 0|e^{\bar\psi f} = \exp\{-\tfrac{1}{2}\bar\psi\psi\}\Big(\langle 0| + \bar\psi\langle 1|\Big)\,. \tag{10}$$

The normalized states (9) form an overcomplete set in the one-fermion Hilbert space $\mathbb{C}^2$, that is,

$$\begin{aligned} \langle\psi_1|\psi_2\rangle &= \exp\{-\tfrac{1}{2}\bar\psi_1\psi_1\}\exp\{-\tfrac{1}{2}\bar\psi_2\psi_2\}\exp\{\bar\psi_1\psi_2\} \\ &= \exp\{-\tfrac{1}{2}\bar\psi_1(\psi_1 - \psi_2) + \tfrac{1}{2}(\bar\psi_1 - \bar\psi_2)\psi_2\} \end{aligned} \tag{11}$$

and provide a resolution of the identity $\mathbf{1}$ via

$$\begin{aligned} &\int d\bar\psi d\psi\, |\psi\rangle\langle\psi| \\ &= \int d\bar\psi d\psi \Big[|0\rangle\langle 0| - \psi|1\rangle\langle 0| + \bar\psi|0\rangle\langle 1| - \bar\psi\psi|1\rangle\langle 1|\Big] \\ &= \int d\bar\psi d\psi \Big[|0\rangle\langle 0| + |1\rangle\langle 0|\psi + \bar\psi|0\rangle\langle 1| + \psi\bar\psi|1\rangle\langle 1|\Big] = \mathbf{1}\,. \end{aligned} \tag{12}$$

In the above we have already made use of a $\mathbb{Z}_2$ grading in analogy to that of Grassmann numbers. That is, we have appointed even and odd Grassmann degrees to the fermion coherent states and the operators [11]:

$$\begin{aligned} \deg(|0\rangle) &= \deg(|\psi\rangle) = \deg(\langle\psi|) = 0\,, \\ \deg(|1\rangle) &= \deg(f) = \deg(f^\dagger) = 1\,, \end{aligned} \tag{13}$$

from which follow rules like

$$\psi|0\rangle = |0\rangle\psi\,, \quad \psi|1\rangle = -|1\rangle\psi\,, \quad \psi f = -f\psi\,, \quad \text{etc.} \tag{14}$$

Finally, we mention that the fermion coherent states are eigenstates of the annihilation and creation operators

$$f|\psi\rangle = \psi|\psi\rangle = |\psi\rangle\psi\,, \quad \langle\psi|f^\dagger = \bar\psi\langle\psi| = \langle\psi|\bar\psi \tag{15}$$

and as a consequence the coherent-state matrix element of a normal-ordered operator $G(f^\dagger, f) = :G(f^\dagger, f):$ reads

$$\langle\psi_1|G(f^\dagger, f)|\psi_2\rangle = G(\bar\psi_1, \psi_2)\langle\psi_1|\psi_2\rangle\,. \tag{16}$$

All of the above properties can trivially be generalized to the case of $N > 1$ degrees of freedom. In this case the fermion coherent states are essentially the ordered direct product of $N$ one-fermion coherent states [12]. For example, in the case of two degrees of freedom these fermion coherent states read

$$\begin{aligned} |\Psi\rangle &:= |\psi_1\rangle \otimes |\psi_2\rangle \\ &= e^{-\bar\Psi\cdot\Psi/2}\Big(|00\rangle + |10\rangle\psi_1 + |01\rangle\psi_2 - |11\rangle\psi_1\psi_2\Big)\,, \\ \langle\Psi| &:= \langle\psi_1| \otimes \langle\psi_2| \\ &= e^{-\bar\Psi\cdot\Psi/2}\Big(\langle 00| + \bar\psi_1\langle 10| + \bar\psi_2\langle 01| - \bar\psi_1\bar\psi_2\langle 11|\Big)\,, \end{aligned} \tag{17}$$

where we have set $\bar\Psi\cdot\Psi := \bar\psi_1\psi_1 + \bar\psi_2\psi_2$. This notation naturally generalizes to cases with even more fermions, for example,

$$\langle\Psi''|\Psi'\rangle = e^{-\bar\Psi''\cdot\Psi''/2}\, e^{-\bar\Psi'\cdot\Psi'/2}\, e^{\bar\Psi''\cdot\Psi'}\,. \tag{18}$$

and we will adopt this obvious generalization throughout this paper.

### 2.3 Fermion coherent-state path integrals

As in the standard canonical case one can represent the fermion-coherent-state matrix element of the time-evolution operator $\exp\{-itH\}$ in terms of a coherent-state path integral [10–12]. For convenience we again consider a quantum system with a single degree of freedom which is completely characterized by an even normal-ordered Hamiltonian $H = H(f^\dagger, f) = :H(f^\dagger, f):$. Hence, the coherent-state matrix element of the evolution operator (or propagator) is given by

$$\langle\psi''|\mathrm{e}^{-\mathrm{i}tH}|\psi'\rangle = \langle\psi''|\mathrm{e}^{-\mathrm{i}\varepsilon H}\mathrm{e}^{-\mathrm{i}\varepsilon H}\cdots\mathrm{e}^{-\mathrm{i}\varepsilon H}|\psi'\rangle \qquad (19)$$

where $\varepsilon := t/N$. Inserting the completeness relation (12) $N-1$ times and taking the limit $\varepsilon \to 0$, that is $N \to \infty$ such that $N\varepsilon = t = const.$, one obtains the time-lattice definition ($\psi_N := \psi''$, $\psi_0 := \psi'$, $\Delta\psi_n := \psi_n - \psi_{n-1}$, $\Delta\bar{\psi}_n := \bar{\psi}_n - \bar{\psi}_{n-1}$)

$$\begin{aligned}
&\langle\psi''|\mathrm{e}^{-\mathrm{i}tH}|\psi'\rangle \\
&= \lim_{\varepsilon\to 0}\prod_{n=1}^{N-1}\int d\bar{\psi}_n d\psi_n \prod_{n=1}^{N}\langle\psi_n|\mathrm{e}^{-\mathrm{i}\varepsilon H}|\psi_{n-1}\rangle \\
&= \lim_{\varepsilon\to 0}\prod_{n=1}^{N-1}\int d\bar{\psi}_n d\psi_n \prod_{n=1}^{N}\langle\psi_n|[1-\mathrm{i}\varepsilon H]|\psi_{n-1}\rangle \\
&= \lim_{\varepsilon\to 0}\prod_{n=1}^{N-1}\int d\bar{\psi}_n d\psi_n \prod_{n=1}^{N}\mathrm{e}^{-\mathrm{i}\varepsilon H(\bar{\psi}_n,\psi_{n-1})}\langle\psi_n|\psi_{n-1}\rangle \\
&= \lim_{\varepsilon\to 0}\prod_{n=1}^{N-1}\int d\bar{\psi}_n d\psi_n \prod_{n=1}^{N}\exp\Big\{-\frac{1}{2}\bar{\psi}_n\Delta\psi_n \\
&\quad +\frac{1}{2}\Delta\bar{\psi}_n\psi_{n-1} - \mathrm{i}\varepsilon H(\bar{\psi}_n,\psi_{n-1})\Big\}
\end{aligned} \qquad (20)$$

for the formal coherent-state path-integral representation of the propagator

$$\begin{aligned}
\langle\psi''|\mathrm{e}^{-\mathrm{i}Ht}|\psi'\rangle &= \int\mathcal{D}\bar{\psi}\mathcal{D}\psi \\
&\times\exp\left\{\mathrm{i}\int_0^t d\tau\left[\tfrac{\mathrm{i}}{2}\left(\bar{\psi}\dot{\psi}-\dot{\bar{\psi}}\psi\right) - H(\bar{\psi},\psi)\right]\right\}.
\end{aligned} \qquad (21)$$

Similar path-integral expressions may also be derived for other matrix elements of the time-evolution operator [11–13]. The above path-integral formulation is easily extended to several fermionic [11] and additional bosonic degrees of freedom [13].

The aim of this paper is to find similar path-integral representations of fermion systems subjected to additional constraints. In doing so we will closely follow the idea of [1], which incorporates proper projection operators via some additional path-integral measure for the Lagrange multipliers.

## 3 First-class constraints

The quantum systems under consideration are characterized by an even self-adjoint and normal-ordered Hamiltonian $H(f^\dagger, f)$, where $f^\dagger$ and $f$ stand for the set $\{f_1^\dagger,\ldots,f_N^\dagger\}$ and $\{f_1,\ldots,f_N\}$, respectively. The quantum dynamics generated by this Hamiltonian is assumed to be subjected to constraints characterized by operator-valued normal-ordered functions of the fermionic annihilation and creation operators. Furthermore, we assume that these constraints have a well-defined Grassmann parity. Then, in the general case, we have two sets of constraints. One consists of even operators denoted by

$$\Phi_a \equiv \Phi_a(f^\dagger, f) = :\Phi_a(f^\dagger, f): = \Phi_a^\dagger\,, \quad \deg\Phi_a = 0\,, \quad (22)$$

and enumerated by Latin characters $a, b, c, \ldots$. The other one consists of odd constraints, for which we will use the notation

$$\chi_\alpha \equiv \chi_\alpha(f^\dagger, f) = :\chi_\alpha(f^\dagger, f): = \chi_\alpha^\dagger\,, \quad \deg\chi_\alpha = 1\,. \quad (23)$$

They will be enumerated by Greek letters $\alpha, \beta, \gamma, \ldots$. With these constraints the physical Hilbert space is determined by the conditions

$$\Phi_a|\varphi\rangle_{\text{phys}} = 0\,, \quad \chi_\alpha|\varphi\rangle_{\text{phys}} = 0\,, \qquad (24)$$

for all $a$ and $\alpha$. Note that here we have assumed that the constraint operators are self-adjoint. If they are not self-adjoint we will assume that they appear in pairs such as $(\chi,\chi^\dagger)$ which in turn allows us to generate self-adjoint constraints via proper linear combinations like $\chi+\chi^\dagger$ and $\mathrm{i}\chi - \mathrm{i}\chi^\dagger$.

Following Dirac [14] we group the constraints into two classes. For first-class constraints the above conditions (24) need to be enforced only initially at $t = 0$ as the quantum evolution guarantees that a physical state will always remain in the physical Hilbert space as time evolves. If this is not the case there exists at least one constraint which is of second class.

The above characterization of first-class constraints is equivalent to the requirement that they obey the following commutation and anticommutation relations.

$$\begin{aligned}
&[\Phi_a,\Phi_b] := \Phi_a\Phi_b - \Phi_b\Phi_a = \mathrm{i}c_{ab}{}^{c}\Phi_c\,, \\
&[\Phi_a,\chi_\alpha] = \mathrm{i}d_{a\alpha}{}^{\beta}\chi_\beta\,, \quad \{\chi_\alpha,\chi_\beta\} = \mathrm{i}g_{\alpha\beta}{}^{a}\Phi_a\,.
\end{aligned} \qquad (25)$$

$$[\Phi_a, H] = \mathrm{i}h_a{}^{b}\Phi_b\,, \quad [\chi_\alpha, H] = \mathrm{i}k_\alpha{}^{\beta}\chi_\beta\,. \qquad (26)$$

In other words, the constraints together with the Hamiltonian form a Lie superalgebra [8] defined by the structure constants $c, d, g, h$ and $k$. In general these structure constants could be operator-valued quantities depending on the fermion operators. Throughout this paper we will, however, consider only thoses cases where the structure constants are complex valued numbers. Let us also note that the first-class constraints alone define a Lie superalgebra (25) which is an ideal of the total algebra including (26). This ideal generates a Lie supergroup (via the usual exponential map) which in turn would enable us to construct in combination with the associated invariant Haar measure [15] a proper projection operator in analogy to the approach of [1]. However, things are much simpler in this case. In particular, with the help of the last anticommutation relation in (25) one can easily show that the first condition in (24), that is, $\Phi_a|\psi\rangle_{\rm phys} = 0$ for all $a$, implies the second one. In other words, in the case of first-class constraints the odd constraints are implied by the even constraints. This argument holds only for the case when even constraints are present. If this would not be the case, then the algebra of the constraints reduces to $\{\chi_\alpha, \chi_\beta\} = 0$ for all $\alpha$ and $\beta$. This algebra, however, does not have a non-trivial (in)finite-dimensional realization. Actually, such an algebra implies $\chi_\alpha|\psi\rangle = 0$ for all $\alpha$ and all $\psi \in \mathcal{H}$. Or in other words, the only possible self-adjoint realization of purely odd first-class constraints are given by $\chi_\alpha \equiv 0$, and hence does not represent any constraints.

### 3.1 The projection operator

Because of the above mentioned properties it suffices to consider only the ordinary Lie algebra spanned by the even constraints $\{\Phi_a\}$ with structure constants $c_{ab}{}^c$. We may construct a proper projection operator via the invariant Haar measure of the corresponding Lie group following [1]. Let us be more explicit. The general group element generated by the even constraints is given by

$$\exp\{-{\rm i}\xi^a\Phi_a(f^\dagger, f)\}\ , \tag{27}$$

where $\{\xi^a\}$ are real group parameters. To be more precise, (27) is a $2^N$-dimensional unitary fully reducible representation of this Lie group in $\mathcal{H}$. For simplicity, we consider here only the case of a compact group. For the treatment in cases of non-compact groups see [16]. For a compact group, let us denote the corresponding invariant normalized Haar measure by $d\mu(\xi)$. Then a proper projection operator may be defined by [17]

$$\mathbb{E} := \int d\mu(\xi)\, \exp\{-{\rm i}\xi^a\Phi_a\} \tag{28}$$

which due to the invariance of the Haar measure and the group-composition law obviously obeys the properties $\mathbb{E} = \mathbb{E}^2 = \mathbb{E}^\dagger$ of an orthogonal projector. It projects onto the physical Hilbert space since by construction the physical states are the eigenstates of $\mathbb{E}$ with eigenvalue one, $\mathbb{E}\,|\psi\rangle_{\rm phys} = |\psi\rangle_{\rm phys}$. Furthermore, we note that

$$\exp\{-{\rm i}\xi^a\Phi_a\}\mathbb{E} = \mathbb{E} \tag{29}$$

for any set $\{\xi^a\}$ and

$${\rm e}^{-{\rm i}tH}\mathbb{E} = \mathbb{E}\,{\rm e}^{-{\rm i}tH} = \mathbb{E}\,{\rm e}^{-{\rm i}tH}\mathbb{E} = \mathbb{E}\,{\rm e}^{-{\rm i}t(\mathbb{E}H\mathbb{E})}\mathbb{E}\ , \tag{30}$$

which is the (constrained) time-evolution operator in the physical subspace. As an aside we mention that this operator may be viewed as an element of the Lie group, associated with the Lie algebra spanned by the Hamiltonian and the even constraints, which is averaged over the subgroup associated with the subalgebra of the even constraints. In other words, it is invariant under right and left multiplication of this subgroup and, hence, belongs to the corresponding two-sided coset.

Finally, let us mention that the $N$-fermion Hilbert space is finite dimensional and, hence, the spectrum of the constraints is pure point. Therefore, technical difficulties arising from a possible continuous spectrum of the constraints (see [1]) do not occur.

### 3.2 Path-integral representations for the constrained propagator

Let us now construct a path-integral representation for the constrained propagator, that is, the coherent-state matrix element of the constrained time-evolution operator (30):

$$\begin{aligned}\langle\psi''|{\rm e}^{-{\rm i}tH}\mathbb{E}\,|\psi'\rangle &= \langle\psi''|{\rm e}^{-{\rm i}tH}{\rm e}^{-{\rm i}\xi^a\Phi_a}\mathbb{E}\,|\psi'\rangle \\ &= \int d\bar\psi_0 d\psi_0\,\langle\psi''|{\rm e}^{-{\rm i}tH}{\rm e}^{-{\rm i}\xi^a\Phi_a}|\psi_0\rangle\langle\psi_0|\mathbb{E}\,|\psi'\rangle\ .\end{aligned} \tag{31}$$

Making use of the group composition law, which follows from the algebra of the even constraints, setting again $\varepsilon = t/N$ and inserting the resolution (12) of the identity we find

$$\begin{aligned}&\langle\psi''|{\rm e}^{-{\rm i}tH}{\rm e}^{-{\rm i}\xi^a\Phi_a}|\psi_0\rangle \\ &= \langle\psi_N|\overleftarrow{\prod_{n=1}^{N}}\left({\rm e}^{-{\rm i}\varepsilon H}{\rm e}^{-{\rm i}\varepsilon\eta^a_n\Phi_a}\right)|\psi_0\rangle \\ &= \prod_{n=1}^{N-1}\int d\bar\psi_n d\psi_n\,\prod_{n=1}^{N}\langle\psi_n|{\rm e}^{-{\rm i}\varepsilon H}{\rm e}^{-{\rm i}\varepsilon\eta^a_n\Phi_a}|\psi_{n-1}\rangle\ ,\end{aligned} \tag{32}$$

where $\{\eta^a_n\}$ are appropriate real numbers. Taking, as in Sect. 2.3, the limit $\varepsilon \to 0$ one ends up with the following time-lattice definition of a constrained fermion coherent-state path integral (notation as in Sect. 2.3 except $\psi' \neq \psi_0$)

$$\begin{aligned}\langle\psi''|{\rm e}^{-{\rm i}tH}\mathbb{E}\,|\psi'\rangle &= \lim_{\varepsilon\to 0}\prod_{n=0}^{N-1}\int d\bar\psi_n d\psi_n \int d\mu(\xi) \\ &\times\exp\left\{-\sum_{n=1}^{N}\left[\frac{1}{2}\bar\psi_n\Delta\psi_n - \frac{1}{2}\Delta\bar\psi_n\psi_{n-1}\right.\right. \\ &\left.\left. + {\rm i}\varepsilon H(\bar\psi_n, \psi_{n-1}) + {\rm i}\varepsilon\eta^a_n\Phi_a(\bar\psi_n, \psi_{n-1})\right]\right\} \\ &\times\langle\psi_0|\exp\{-{\rm i}\xi^a\Phi_a(f^\dagger, f)\}|\psi'\rangle\ .\end{aligned} \tag{33}$$

Hence, we arrive at the formal path-integral representation of the constrained propagator

$$\begin{aligned}\langle\psi''|{\rm e}^{-{\rm i}tH}\mathbb{E}\,|\psi'\rangle &= \int \mathcal{D}\bar\psi\mathcal{D}\psi\int d\mu(\xi) \\ &\times\exp\left\{{\rm i}\int_0^t d\tau\left[\frac{{\rm i}}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi) - H(\bar\psi,\psi)\right.\right. \\ &\left.\left. - \eta^a\Phi_a(\bar\psi,\psi)\right]\right\}\exp\left\{-{\rm i}\xi^a\Phi_a(\bar\psi',\psi')\right\}\ .\end{aligned} \tag{34}$$

Despite the fact that in this path integral the time-dependent real-valued functions $\{\eta^a\}$ explicitly appear, which may be interpreted as Lagrange multipliers, it is completely independent of them as is clearly shown by the left-hand side. Hence, as in [1], we are free to average the right-hand side over the functions $\{\eta^a\}$ with an arbitrary in general complex-valued measure $C(\eta)$ which is normalized, $\int \mathcal{D}C(\eta) = 1$. The only requirement we impose on this measure is, that such an average will introduce at least one projection operator $\mathbb{E}$ to account for the initial value equation (24). If it puts in two or more of these projection operators the result will be the same since $\mathbb{E}^2 = \mathbb{E}$. Hence, there are many forms for this measure which will be admissible. For an example see the Appendix. In doing so we have derived yet another path-integral representation of the constrained propagator.

$$\langle\psi''|\mathrm{e}^{-\mathrm{i}tH}\mathbb{E}\,|\psi'\rangle = \int \mathcal{D}\bar\psi\mathcal{D}\psi \int \mathcal{D}C(\eta)\exp\left\{\mathrm{i}\int_0^t d\tau \right. \\ \left. \times\left[\frac{\mathrm{i}}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi) - H(\bar\psi,\psi) - \eta^a\Phi_a(\bar\psi,\psi)\right]\right\} . \tag{35}$$

In essence, formulas (33), (34) and (35) resemble the fermionic counter parts of the results (64), (65) and (66) in [1] where the bosonic case has been studied.

# 4 Examples of first-class constraints

As we have seen in the above discussion, the treatment of first-class constraints for fermionic systems is very much the same as that for bosonic systems [1]. In particular, it is sufficient to consider only even constraints which are bosonic in nature. Therefore, we will discuss below only two examples which demonstrate the minor differences to the bosonic case.

## 4.1 First example of first-class constraints

As a simple example with purely even constraints let us consider an $N$-fermion system subjected to the even constraint

$$\Phi(f^\dagger, f) = \sum_{i=1}^{N} f_i^\dagger f_i - M . \tag{36}$$

Obviously, this constraint fixes the number of fermions to $M \in \mathbb{N}$ with $M \leq N$. In order to make the effects of the constraints more transparent we will consider only the path-integral representation of the coherent-state matrix element of the projection operator

$$\mathbb{E} = \int_0^{2\pi} \frac{d\xi}{2\pi}\,\mathrm{e}^{-\mathrm{i}\xi\Phi} = \delta_{\Phi,0} = \mathbb{E}^2 = \mathbb{E}^\dagger , \tag{37}$$

that is, we will consider a system with a vanishing Hamiltonian, $H = 0$, and limit ourselves to the special case $M = 1$, $N = 2$. Formally, the corresponding path integral is then given by

$$\int \mathcal{D}\bar\Psi\mathcal{D}\Psi \int \mathcal{D}C(\eta) \\ \times\exp\left\{\mathrm{i}\int_0^t d\tau\left[\frac{\mathrm{i}}{2}(\bar\Psi\cdot\dot\Psi - \dot{\bar\Psi}\cdot\Psi) - \eta(\bar\Psi\cdot\Psi - 1)\right]\right\} \tag{38}$$

and leads to the coherent-state matrix element (for details see the Appendix)

$$\langle\Psi''|\mathbb{E}\,|\Psi'\rangle = \mathrm{e}^{-\bar\Psi''\cdot\Psi''/2}\,\mathrm{e}^{-\bar\Psi'\cdot\Psi'/2}\,\bar\Psi''\cdot\Psi' \tag{39}$$

where we have adopted the short-hand notation of (17). We leave it to the reader to verify that this matrix element represents a reproducing kernel in the physical subspace given by the linear span of the two vectors $|01\rangle$ and $|10\rangle$:

$$\int d\bar\Psi d\Psi\,\langle\Psi''|\mathbb{E}\,|\Psi\rangle\langle\Psi|\mathbb{E}\,|\Psi'\rangle = \langle\Psi''|\mathbb{E}\,|\Psi'\rangle , \tag{40}$$

where $d\bar\Psi d\Psi := d\bar\psi_1 d\psi_1 d\bar\psi_2 d\psi_2$.

## 4.2 Second example of first-class constraints

As a second example we will now consider a three-fermion system ($N = 3$) subjected to one even and two odd constraints given by

$$\begin{aligned}\Phi &= 1 - f_1^\dagger f_1 - f_2^\dagger f_2 - f_2^\dagger f_2 + f_1^\dagger f_1 f_2^\dagger f_2 \\ &\quad + f_2^\dagger f_2 f_3^\dagger f_3 + f_3^\dagger f_3 f_1^\dagger f_1 , \\ \chi &= f_1 f_2 f_3 , \quad \chi^\dagger = f_3^\dagger f_2^\dagger f_1^\dagger .\end{aligned} \tag{41}$$

These first-class constraints obey the Lie superalgebra

$$[\chi,\Phi] = 0 = [\chi^\dagger,\Phi] , \quad \{\chi,\chi^\dagger\} = \Phi , \quad \chi^2 = 0 = (\chi^\dagger)^2. \tag{42}$$

Obviously, the six-dimensional physical subspace is characterized by having at least one empty and one occupied fermion state. As in the previous example the spectrum of the even constraint $\Phi$ is integer and therefore the projection operator has the same integral representation.

$$\mathbb{E} = \int_0^{2\pi}\frac{d\xi}{2\pi}\,\mathrm{e}^{-\mathrm{i}\xi\Phi} \tag{43}$$

and can explicitly be expressed in terms of the fermion number operators

$$\begin{aligned}\mathbb{E} &= f_1^\dagger f_1(1 - f_2^\dagger f_2) + f_2^\dagger f_2(1 - f_3^\dagger f_3) + f_3^\dagger f_3(1 - f_1^\dagger f_1) \\ &= 1 - \Phi .\end{aligned} \tag{44}$$

The path integral for the coherent-state matrix element of the projection operator formally reads

$$\langle\Psi''|\mathbb{E}\,|\Psi'\rangle = \int \mathcal{D}\bar\Psi\mathcal{D}\Psi \int \mathcal{D}C(\eta)\exp\left\{\mathrm{i}\int_0^t d\tau\, L\right\} , \tag{45}$$

where

$$\begin{aligned}L &:= \tfrac{\mathrm{i}}{2}\left(\bar\Psi\cdot\dot\Psi - \dot{\bar\Psi}\cdot\Psi\right) - \eta\Big(1 - \bar\Psi\cdot\Psi \\ &\quad + \bar\psi_1\psi_1\bar\psi_2\psi_2 + \bar\psi_2\psi_2\bar\psi_3\psi_3 + \bar\psi_3\psi_3\bar\psi_1\psi_1\Big) .\end{aligned} \tag{46}$$

An explicit path integration then leads to the result

$$\begin{aligned}\langle\Psi''|\mathbb{E}\,|\Psi'\rangle &= \langle\Psi''|\Psi'\rangle\Big[\bar\Psi''\cdot\Psi' - \bar\psi_1''\psi_1'\bar\psi_2''\psi_2' \\ &\quad -\bar\psi_2''\psi_2'\bar\psi_3''\psi_3' - \bar\psi_3''\psi_3'\bar\psi_1''\psi_1'\Big] \\ &= \mathrm{e}^{-(\bar\Psi''\cdot\Psi''+\bar\Psi'\cdot\Psi')/2}\Big[\bar\Psi''\cdot\Psi' + \bar\psi_1''\psi_1'\bar\psi_2''\psi_2' \\ &\quad +\bar\psi_2''\psi_2'\bar\psi_3''\psi_3' + \bar\psi_3''\psi_3'\bar\psi_1''\psi_1'\Big]\,.\end{aligned} \tag{47}$$

## 5 Second-class constraints

Second-class constraints are all those which are not first class. For second-class constraints it is not sufficient to start with an initial state on the physical subspace as in this case the time evolution generated by the Hamiltonian will generally depart from the physical subspace. In other words, after some short time interval (say $\varepsilon$) one may have to project the state back onto the physical subspace. Hence, we are led to consider the constrained propagator

$$\begin{aligned}&\langle\psi''|\mathbb{E}\,\mathrm{e}^{-\mathrm{i}t(\mathbb{E}\,H\mathbb{E}\,)}\mathbb{E}\,|\psi'\rangle \\ &= \lim_{\varepsilon\to0}\langle\psi''|\mathbb{E}\,\mathrm{e}^{-\mathrm{i}\varepsilon H}\mathbb{E}\,\mathrm{e}^{-\mathrm{i}\varepsilon H}\mathbb{E}\,\cdots\mathbb{E}\,\mathrm{e}^{-\mathrm{i}\varepsilon H}\mathbb{E}\,|\psi'\rangle \\ &= \lim_{\varepsilon\to0}\int\prod_{n=1}^{N-1}d\bar\psi_n d\psi_n\prod_{n=1}^{N}\langle\psi_n|\mathbb{E}\,\mathrm{e}^{-\mathrm{i}\varepsilon H}\mathbb{E}\,|\psi_{n-1}\rangle\,.\end{aligned} \tag{48}$$

Again we will closely follow the basic ideas used in the canonical coherent-state path-integral approach [1]. Hence, we start by introducing the unit vectors $|\psi\rangle\rangle := \mathbb{E}\,|\psi\rangle/||\mathbb{E}\,|\psi\rangle||$ and set $M'' := ||\mathbb{E}\,|\psi''\rangle||$, $M' := ||\mathbb{E}\,|\psi'\rangle||$. The path integral for the constrained propagator can then be rewritten as

$$\begin{aligned}&M''M'\lim_{\varepsilon\to0}\int\left[\prod_{n=1}^{N-1}d\bar\psi_n d\psi_n\,\langle\psi_n|\mathbb{E}\,|\psi_n\rangle\right] \\ &\times\prod_{n=1}^{N}\langle\langle\psi_n|\mathrm{e}^{-\mathrm{i}\varepsilon H}|\psi_{n-1}\rangle\rangle\end{aligned} \tag{49}$$

which admits the following formal path-integral representation

$$\begin{aligned}&M''M'\int\mathcal{D}_E\mu(\bar\psi,\psi) \\ &\times\exp\left\{\mathrm{i}\int_0^t d\tau\left[\mathrm{i}\langle\langle\psi|\tfrac{d}{d\tau}|\psi\rangle\rangle - \langle\langle\psi|H|\psi\rangle\rangle\right]\right\}\,.\end{aligned} \tag{50}$$

In terms of the original vectors it reads

$$\begin{aligned}&M''M'\int\mathcal{D}_E\mu(\bar\psi,\psi) \\ &\times\exp\left\{\mathrm{i}\int_0^t d\tau\left[\mathrm{i}\frac{\langle\psi|\frac{d}{d\tau}|\psi\rangle}{\langle\psi|\mathbb{E}\,|\psi\rangle} - \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\mathbb{E}\,|\psi\rangle}\right]\right\}\,.\end{aligned} \tag{51}$$

Another relation may be obtained by assuming that the projection operator allows for an integral representation in terms of the even and odd constraints

$$\mathbb{E}\, = \int d\mu_\varepsilon(\eta,\lambda)\,\mathrm{e}^{-\mathrm{i}\varepsilon(\eta^a\Phi_a+\lambda^\alpha\chi_\alpha)} \tag{52}$$

where $d\mu_\varepsilon$ stands for some even Grassmann-valued measure depending on the real variables $\eta^a$ and the odd Grassmann numbers $\lambda^\alpha$ which both may be considered as Lagrange multipliers. Using this relation in the path-integral expression (48) we find the representation (notation as in Sect. 2.3 except $\psi_N\neq\psi''$)

$$\begin{aligned}&\lim_{\varepsilon\to0}\int\left[\prod_{n=1}^{N}d\bar\psi_n d\psi_n d\mu_\varepsilon(\eta_n,\lambda_n)\right]d\mu_\varepsilon(\eta_0,\lambda_0) \\ &\times\langle\psi''|\mathrm{e}^{-\mathrm{i}\varepsilon(\eta_N^a\Phi_a+\lambda_N^\alpha\chi_\alpha)}|\psi_N\rangle \\ &\times\prod_{n=1}^{N}\langle\psi_n|\mathrm{e}^{-\mathrm{i}\varepsilon H}\mathrm{e}^{-\mathrm{i}\varepsilon(\eta_{n-1}^a\Phi_a+\lambda_{n-1}^\alpha\chi_\alpha)}|\psi_{n-1}\rangle\end{aligned} \tag{53}$$

which can formally be written as

$$\begin{aligned}&\int\mathcal{D}\bar\psi\mathcal{D}\psi\mathcal{D}E(\eta,\lambda)\exp\left\{\mathrm{i}\int_0^t d\tau\left[\frac{\mathrm{i}}{2}(\bar\psi\dot\psi-\dot{\bar\psi}\psi)\right.\right. \\ &\left.\left.-H(\bar\psi,\psi)-\eta^a\Phi_a(\bar\psi,\psi)-\lambda^\alpha\chi_\alpha(\bar\psi,\psi)\right]\right\}\,.\end{aligned} \tag{54}$$

Here let us remark that we have assumed that the constraints are self-adjoint. This is typically not the case for odd constraints, which then appear in pairs $(\chi,\chi^\dagger)$. As a consequence the Grassmann-valued Lagrange multipliers also appear in pairs $(\lambda,\bar\lambda)$. In contrast to the first-class constraints, in the present case one cannot neglect the odd constraints. However, the appearance of Grassmann multipliers may be omitted at the expense of no longer having the constraints appear explicitly in the exponent of (52). Actually, because spec$(\mathbb{E}\,)\subseteq\{0,1\}$ we may always choose the following simple integral representation of the projection operator

$$\mathbb{E}\, = \int_0^{2\pi}\frac{d\eta}{2\pi}\,\mathrm{e}^{-\mathrm{i}\eta(1-\mathbb{E}\,)}\,. \tag{55}$$

Again we would like to point out that (48)–(51) are the fermion counterparts of (104)–(106) of [1], and relation (54) corresponds to (109) in [1].

## 6 Examples of second-class constraints

Even fermionic constraints are in essence similar to bosonic constraints which have extensively been discussed in [1]. For this reason we will concentrate our attention in this section exclusively on odd second-class constraints. We will start with two simple examples of constraints linear in fermion operators and then generalize our approach to an arbitrary set of linear constraints. Based on an example of a non-linear odd constraint we will show that all non-linear diagonal odd constraints can be reduced to the linear case.

### 6.1 Linear odd constraints

As mentioned above we will begin our discussion with a simple, that is $N = 1$, fermion system which obeys the constraints

$$\chi = f - \theta\,, \quad \chi^\dagger = f^\dagger - \bar\theta\,. \tag{56}$$

Here $\bar\theta, \theta \in \mathbb{C}B_2$ are odd Grassmann numbers. The constraints (56) obey the following anticommutation relations

$$\{\chi, \chi^\dagger\} = 1\,, \quad \chi^2 = 0 = (\chi^\dagger)^2 \tag{57}$$

and, therefore, one cannot impose both constraint conditions

$$\begin{aligned} \chi|\varphi\rangle_{\rm phys} &= 0\,, \qquad \text{case A} \\ \chi^\dagger|\varphi\rangle_{\rm phys} &= 0\,, \qquad \text{case B} \end{aligned} \tag{58}$$

simultaneously. Such a procedure would clearly lead to an inconsistent quantum theory. There are several ways to relax the conditions in order to formulate a consistent approach. Here we adopt an approach similar to the so-called holomorphic quantization [6] utilized for bosonic models with similar constraint inconsistencies. That is, we will consider only one of the above two conditions to define a proper physical Hilbert subspace. However, both possible cases will be discussed for completeness.

### 6.1.1 Case A

The solution of (58) in case A is obviously given by the fermion coherent state $|\theta\rangle$ and the corresponding projection operator reads

$$\begin{aligned} \mathbb{E}_A = |\theta\rangle\langle\theta| = \chi\chi^\dagger &= \int d\bar\lambda d\lambda\, \mathrm{e}^{-\mathrm{i}\bar\lambda\chi}\mathrm{e}^{-\mathrm{i}\chi^\dagger\lambda} \\ &= \int d\bar\lambda d\lambda\, \mathrm{e}^{\bar\lambda\lambda/2}\, \mathrm{e}^{-\mathrm{i}(\bar\lambda\chi+\chi^\dagger\lambda)}\,. \end{aligned} \tag{59}$$

The diagonal coherent-state matrix element of this operator, needed for example in evaluating the path integral (49), is given by

$$\langle\psi_n|\mathbb{E}_A|\psi_n\rangle = \exp\{-(\bar\psi_n - \bar\theta)(\psi_n - \theta)\}\,. \tag{60}$$

Hence, for a normal-ordered Hamiltonian $H = H(f^\dagger, f)$ we arrive at the formal path-integral expressions for the constrained propagator

$$\begin{aligned} &\int \mathcal{D}\bar\psi\mathcal{D}\psi\mathcal{D}E(\bar\lambda,\lambda)\exp\left\{\mathrm{i}\int_0^t d\tau\left[\frac{\mathrm{i}}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi)\right.\right. \\ &\quad \left.\left. - H(\bar\psi,\psi) - \bar\lambda(\psi-\theta) - (\bar\psi - \bar\theta)\lambda\right]\right\} \\ &= \int \mathcal{D}\bar\psi\mathcal{D}\psi \exp\left\{\mathrm{i}\int_0^t d\tau\left[\frac{\mathrm{i}}{2}(\bar\psi\dot\psi - \dot{\bar\psi}\psi)\right.\right. \\ &\quad \left.\left. + \mathrm{i}(\bar\psi - \bar\theta)(\psi-\theta) - H(\bar\psi,\psi)\right]\right\}\,. \end{aligned} \tag{61}$$

Explicit path integration (see Appendix) will then lead to the final result

$$\begin{aligned} &\langle\psi''|\mathbb{E}_A \mathrm{e}^{-\mathrm{i}t(\mathbb{E}_A H \mathbb{E}_A)}\mathbb{E}_A|\psi'\rangle = \langle\psi''|\theta\rangle\langle\theta|\psi'\rangle\mathrm{e}^{-\mathrm{i}tH(\bar\theta,\theta)} \\ &= \langle\psi''|\psi'\rangle \exp\left\{-(\bar\psi''-\bar\theta)(\psi'-\theta) - \mathrm{i}tH(\bar\theta,\theta)\right\}\,. \end{aligned} \tag{62}$$

### 6.1.2 Case B

For the second choice (case B) the solution of (58) is given by a different kind of coherent states defined by [11,13]

$$|\varphi\rangle_{\rm phys} = |\bar\theta) := \mathrm{e}^{\bar\theta\theta/2}\Big(|1\rangle - \bar\theta|0\rangle\Big)\,. \tag{63}$$

In contrast to the even fermion coherent states introduced in Sect. 2.2, these states are odd. They are eigenstates of the fermion creation operator and are orthogonal to the corresponding even states:

$$f^\dagger|\bar\theta) = \bar\theta|\bar\theta)\,, \quad (\bar\theta|f = (\bar\theta|\theta\,, \quad \langle\theta|\bar\theta) = 0\,. \tag{64}$$

For case B the projection operator is given by the orthogonal complement of (59)

$$\begin{aligned} \mathbb{E}_B = |\bar\theta)(\bar\theta| = \chi^\dagger\chi &= \int d\bar\lambda d\lambda\, \mathrm{e}^{\mathrm{i}\chi^\dagger\lambda}\mathrm{e}^{\mathrm{i}\bar\lambda\chi} \\ &= \int d\bar\lambda d\lambda\, \mathrm{e}^{-\bar\lambda\lambda/2}\,\mathrm{e}^{\mathrm{i}(\bar\lambda\chi+\chi^\dagger\lambda)} = \mathbf{1} - \mathbb{E}_A \end{aligned} \tag{65}$$

whose diagonal coherent-state matrix element reads

$$\langle\psi_n|\mathbb{E}_B|\psi_n\rangle = (\bar\psi_n - \bar\theta)(\psi_n - \theta)\,. \tag{66}$$

Explicit path integration will then lead to the constrained propagator

$$\begin{aligned} &\langle\psi''|\mathbb{E}_B \mathrm{e}^{-\mathrm{i}t(\mathbb{E}_B H \mathbb{E}_B)}\mathbb{E}_B|\psi'\rangle = \langle\psi''|\bar\theta)(\bar\theta|\psi'\rangle\mathrm{e}^{-\mathrm{i}th(\theta,\bar\theta)} \\ &= \langle\psi''|\psi'\rangle(\bar\psi''-\bar\theta)(\psi'-\theta)\mathrm{e}^{-\mathrm{i}th(\theta,\bar\theta)}\,, \end{aligned} \tag{67}$$

where $h(\theta,\bar\theta) := (\bar\theta|H|\bar\theta)$. Note that for an *anti*-normal ordered Hamiltonian $H = H(f, f^\dagger)$ we have $h(\theta,\bar\theta) = H(\theta,\bar\theta)$.

### 6.1.3 A second example

As a second example of linear constraints let us consider an $N = 2$ fermion system subjected to the two odd constraints

$$\chi = \frac{1}{\sqrt{2}}(f_1 - f_2)\,, \quad \chi^\dagger = \frac{1}{\sqrt{2}}(f_1^\dagger - f_2^\dagger)\,, \tag{68}$$

which also obey the algebra (57). In analogy to the previous example we may again consider two different physical subspaces according to case A and B in (58).

For case A the physical Hilbert space is the two-dimensional subspace spanned by the fermion number eigenstates $|00\rangle$ and $(|01\rangle + |10\rangle)/\sqrt{2}$. The corresponding projection operator is given by $\mathbb{E}_A = \chi\chi^\dagger$ and admits integral representations as given in (59). The path integral for its matrix element (for simplicity we consider here the system $H = 0$) leads to

$$\begin{aligned} &\langle\psi_1''\psi_2''|\mathbb{E}_A|\psi_1'\psi_2'\rangle \\ &= \langle\psi_1''\psi_2''|\psi_1'\psi_2'\rangle\left[1 - \tfrac{1}{2}(\bar\psi_1''-\bar\psi_2'')(\psi_1'-\psi_2')\right] \\ &= \mathrm{e}^{-\bar\Psi''\cdot\Psi''/2}\,\mathrm{e}^{-\bar\Psi'\cdot\Psi'/2}\left[1 + \tfrac{1}{2}(\bar\psi_1''+\bar\psi_2'')(\psi_1'+\psi_2')\right]\,. \end{aligned} \tag{69}$$

In case B we are dealing with the projection operator $\mathbb{E}_B = \mathbf{1} - \mathbb{E}_A = \chi^\dagger\chi$ and its integral representations are the same as in (65). This operator projects onto the orthogonal complement of the previous case, that is, onto the subspace spanned by $|11\rangle$ and $(|01\rangle - |10\rangle)/\sqrt{2}$. Here the result of path integration for the coherent-state matrix element of $\mathbb{E}_B$ reads

$$\begin{aligned}\langle\psi_1''\psi_2''|\mathbb{E}_B|\psi_1'\psi_2'\rangle &= \langle\psi_1''\psi_2''|\psi_1'\psi_2'\rangle\tfrac{1}{2}(\bar\psi_1''-\bar\psi_2'')(\psi_1'-\psi_2') \\ &= \mathrm{e}^{-(\bar\Psi''\cdot\Psi''+\bar\Psi'\cdot\Psi')/2} \\ &\quad\times\left[\bar\psi_1''\psi_1'\bar\psi_2''\psi_2' + \tfrac{1}{2}(\bar\psi_1''-\bar\psi_2'')(\psi_1'-\psi_2')\right] .\end{aligned} \tag{70}$$

#### 6.1.4 Generalization

The above discussion may easily be generalized to a set of diagonal linear second-class constraints obeying the anti-commutation relations

$$\{\chi_\alpha,\chi_\beta\} = 0 = \{\chi_\alpha^\dagger,\chi_\beta^\dagger\} , \quad \{\chi_\alpha,\chi_\beta^\dagger\} = \delta_{\alpha\beta} , \tag{71}$$

where $\alpha,\beta \in \{1,2,\ldots,M\}$, $M \leq N$. Clearly, for each $\alpha$ one has two choices for a projection operator, $\mathbb{E}_A^{(\alpha)} = \chi_\alpha\chi_\alpha^\dagger$ or $\mathbb{E}_B^{(\alpha)} = \chi_\alpha^\dagger\chi_\alpha$. Therefore, for the total physical subspace the corresponding projection operator is not unique and we have to choose one out of the following $2^M$ possible operators,

$$\mathbb{E} = \mathbb{E}_{i_1}^{(1)}\mathbb{E}_{i_2}^{(2)}\cdots\mathbb{E}_{i_M}^{(M)} , \quad i_\alpha \in \{A,B\} , \tag{72}$$

leading to $2^M$ pairwise orthogonal $2^{N-M}$-dimensional subspaces of the $N$-fermion Hilbert space $\mathcal{H} = \mathbb{C}^{2^N}$.

In fact, we may be even more general and assume some non-diagonal linear odd constraints obeying the algebra

$$\{\chi_\alpha,\chi_\beta\} = w_{\alpha\beta} = w_{\beta\alpha} , \quad w_{\alpha\beta} \in \mathbb{R} . \tag{73}$$

For simplicity we have chosen here self-adjoint odd second-class constraints. This system of constraints can easily be reduced to the above diagonal case. To be explicit, let $D \in SO(M)$ denote the orthogonal matrix which diagonalizes the symmetric matrix $W$, $(W)_{\alpha\beta} = w_{\alpha\beta}$. That is, we choose $D$ such that

$$(D^TWD)_{\alpha\beta} = v_\alpha\delta_{\alpha\beta} . \tag{74}$$

Then we may define new constraints via $\chi'_\alpha = (D^T)_\alpha{}^\beta\chi_\beta/\sqrt{v_\alpha}$ which are diagonal

$$\{\chi'_\alpha,\chi'_\beta\} = \delta_{\alpha\beta} , \tag{75}$$

and can be treated as discussed above. Note that $v_\alpha > 0$ as we are dealing with second-class constraints.

In essence, the conclusion of this section is, that any set of linear odd second-class constraints is reducible to the diagonal case and in turn can be incorporated into the path integral.

### 6.2 Nonlinear odd constraints

Let us now consider odd constraints which are not linear in the fermion operators. Again we will begin our discussion with an elementary example which is an $N = 4$ fermion system with constraints given by

$$\chi = f_1 - f_2f_3f_4^\dagger , \quad \chi^\dagger = f_1^\dagger - f_4f_3^\dagger f_2^\dagger . \tag{76}$$

Note that $\chi^2 = 0 = (\chi^\dagger)^2$ as before, however, the anti-commutator is no longer proportional to the identity. To be explicit, it is given by

$$\{\chi,\chi^\dagger\} = X \tag{77}$$

where

$$X := \mathbf{1} + f_2f_2^\dagger f_3f_3^\dagger f_4^\dagger f_4 + f_2^\dagger f_2f_3^\dagger f_3f_4f_4^\dagger . \tag{78}$$

Note that $\mathrm{spec}(X) = \{1,2\}$ and therefore its inverse is well-defined

$$X^{-1} = \mathbf{1} - \tfrac{1}{2}f_2f_2^\dagger f_3f_3^\dagger f_4^\dagger f_4 - \tfrac{1}{2}f_2^\dagger f_2f_3^\dagger f_3f_4f_4^\dagger . \tag{79}$$

As in the linear case we cannot impose both conditions, case A and B in (58), simultaneously. Hence, we again have to choose either case A or B. Which will lead us to two orthogonal eight-dimensional subspaces of $\mathcal{H} = \mathbb{C}^{16}$. Here, however, because of the non-linearity of the constraints, the projection operators are given by

$$\mathbb{E}_A = X^{-1}\chi\chi^\dagger , \quad \mathbb{E}_B = \mathbf{1} - \mathbb{E}_A = X^{-1}\chi^\dagger\chi . \tag{80}$$

Note that $[X,\chi] = 0 = [X,\chi^\dagger]$. In essence, because $X > 0$ one simply replaces the original constraints by new ones,

$$\chi \to \chi' = \chi/\sqrt{X} , \tag{81}$$

which by construction are "linear", i.e., constraints equivalent to linear, and can be treated as shown in the previous section.

Obviously, this procedure can be generalized to a set of non-linear diagonal second-class constraints obeying

$$\{\chi_\alpha,\chi_\beta\} = 0 = \{\chi_\alpha^\dagger,\chi_\beta^\dagger\} , \quad \{\chi_\alpha,\chi_\beta^\dagger\} = X_\alpha\delta_{\alpha\beta} \tag{82}$$

where $X_\alpha \geq 0$ does not vanish as $\chi_\alpha$ is assumed to be second class. Hence, we have $X_\alpha > 0$ and therefore we may redefine the odd constraints $\chi_\alpha \to \chi'_\alpha = \chi_\alpha/\sqrt{X_\alpha}$ which brings us back to the linear case discussed above.

## 7 Application to Bose-Fermi systems

To complete our discussion we finally consider a system of $M$ bosons and $N$ fermions. The $M$ bosonic degrees of freedom are characterized by bosonic annihilation and creation operators $b_i$ and $b_i^\dagger$, respectively, which obey the standard commutation relations

$$[b_i,b_j] = 0 , \quad [b_i^\dagger,b_j^\dagger] = 0 , \quad [b_i,b_j^\dagger] = \delta_{ij} . \tag{83}$$

These operators act on the $M$-boson Hilbert space $L^2(\mathbb{R})\otimes\cdots\otimes L^2(\mathbb{R}) = L^2(\mathbb{R}^M)$. As in the case of fermions we will work in the (boson) coherent-state representation. These are eigenstates of the annihilation operators

$$b_i|z_i\rangle_i = z_i|z_i\rangle_i\ , \quad z_i \in \mathbb{C}\ , \quad |z_i\rangle_i \in L^2(\mathbb{R})\ , \tag{84}$$

and for its $M$-fold tensor product, which represents an $M$-boson state, we will use the notation $|\mathbf{z}\rangle = |z_1\rangle_1 \otimes \cdots \otimes |z_M\rangle_M$. The total Hilbert space of the combined boson fermion system is thus $\mathcal{H} = L^2(\mathbb{R}^M)\otimes\mathbb{C}^{2^N}$ and the boson-fermion coherent states will be denoted by $|\mathbf{z}\Psi\rangle = |\mathbf{z}\rangle\otimes|\Psi\rangle$. The dynamics of such a system is defined by the Hamiltonian which we choose to

$$H := \omega\left[\sum_{i=1}^{M} b_i^\dagger b_i + \sum_{i=1}^{N} f_i^\dagger f_i\right]\ , \quad \omega > 0\ . \tag{85}$$

Note that for $M = N$ this Hamiltonian characterizes a supersymmetric quantum system [18]. The interaction of the bosons and fermions is introduced via the even first-class constraint

$$\Phi := \sum_{i=1}^{M} b_i^\dagger b_i - \sum_{i=1}^{N} f_i^\dagger f_i - p\ , \quad p \in \mathbb{Z}\ , \tag{86}$$

which fixes the fermion number $N_f$ and the boson number $N_b$ to obey the equality $N_f = N_b - p$.

As the spectrum of the constraint is integer we may use the integral representation (37) for constructing the projection operator. In this case the coherent-state matrix element for this operator reads

$$\langle\mathbf{z}''\Psi''|\mathbb{E}\,|\mathbf{z}'\Psi'\rangle = \mathcal{N}\int_0^{2\pi}\frac{d\varphi}{2\pi}\,\mathrm{e}^{\mathrm{i}\varphi p}\exp\{\mathrm{e}^{-\mathrm{i}\varphi}\mathbf{z}''^*\cdot\mathbf{z}' + \mathrm{e}^{\mathrm{i}\varphi}\bar{\Psi}''\cdot\Psi'\}\ , \tag{87}$$

where the normalization factor is given by

$$\mathcal{N} := \exp\left\{-\frac{1}{2}\left[|\mathbf{z}''|^2 + |\mathbf{z}'|^2 + \bar{\Psi}''\cdot\Psi'' + \bar{\Psi}'\cdot\Psi'\right]\right\}\ . \tag{88}$$

Formally, the constrained propagator is represented by the path integral

$$\langle\mathbf{z}''\Psi''|\mathrm{e}^{-\mathrm{i}tH}\mathbb{E}\,|\mathbf{z}'\Psi'\rangle = \int \mathcal{D}z^*\mathcal{D}z\mathcal{D}\bar{\Psi}\mathcal{D}\Psi\mathcal{D}C(\eta) \times\exp\left\{\mathrm{i}\int_0^t d\tau\, L\right\}\ ,$$
$$L := \frac{\mathrm{i}}{2}(\mathbf{z}^*\cdot\dot{\mathbf{z}} - \dot{\mathbf{z}}^*\cdot\mathbf{z} + \bar{\Psi}\cdot\dot{\Psi} - \dot{\bar{\Psi}}\cdot\Psi) -\omega(\mathbf{z}^*\cdot\mathbf{z} + \bar{\Psi}\cdot\Psi) - \eta(\mathbf{z}^*\cdot\mathbf{z} - \bar{\Psi}\cdot\Psi - p)\ , \tag{89}$$

and explicit path integration leads to

$$\begin{aligned}\langle\mathbf{z}''\Psi''|\mathrm{e}^{-\mathrm{i}tH}\mathbb{E}\,|\mathbf{z}'\Psi'\rangle &= \mathcal{N}\int_0^{2\pi}\frac{d\varphi}{2\pi}\,\mathrm{e}^{\mathrm{i}\varphi p} \times\exp\left\{\mathrm{e}^{-\mathrm{i}(\omega t+\varphi)}\mathbf{z}''^*\cdot\mathbf{z}' + \mathrm{e}^{-\mathrm{i}(\omega t-\varphi)}\bar{\Psi}''\cdot\Psi'\right\}\\ &= \mathcal{N}\sum_{m_1=0}^{\infty}\cdots\sum_{m_M=0}^{\infty}\sum_{n_1=0}^{1}\cdots\sum_{n_N=0}^{1}\delta_{\Sigma_N,\Sigma_M+p} \times\frac{\mathrm{e}^{-\mathrm{i}\omega t(\Sigma_M+\Sigma_N)}}{m_1!\cdots m_M!}\\ &\quad\times\overline{(z_1'')^{m_1}\cdots(z_M'')^{m_M}(\psi_1'')^{n_1}\cdots(\psi_N'')^{n_N}}\\ &\quad\times(z_1')^{m_1}\cdots(z_M')^{m_M}(\psi_1')^{n_1}\cdots(\psi_N')^{n_N}\end{aligned} \tag{90}$$

where we have set $\Sigma_M := m_1+\cdots+m_M$, $\Sigma_N := n_1+\cdots+n_N$ and the overbar denotes an involution of the Grassmann algebra defined by $\overline{c\psi_1\psi_2\cdots\psi_N} := c^*\bar{\psi}_N\cdots\bar{\psi}_2\bar{\psi}_1$.

## 8 Conclusions

In this paper we have extended the bosonic coherent-state path-integral approach of constrained systems [1] to those with fermionic degrees of freedom. As in the bosonic case we find that this approach does not involve any $\delta$-functionals of the constraints nor does it require any gauge fixing of first-class or elimination of variables for second-class constraints. In addition we have shown that in the case of first-class constraints for fermion systems it is sufficient to consider only those which have an even Grassmann parity. In other words, for first-class constraints the Lagrange multipliers are ordinary real-valued functions of time. There is no need to introduce either even or odd Grassmann-valued multipliers. In this respect first-class constraints of fermion systems are not much different than those of boson systems and can be incorporated in the path-integral approach in the same way. This also applies to even second-class constraints. It is only in the case of odd second-class constraints where Grassmann-valued Lagrange multipliers may appear in the path-integral approach. For the cases of linear and non-linear diagonal second-class constraints we have been able to reduce the problem to the simpler case of linear diagonal odd constraints which however does not allow for a consistent quantum formulation. Here we have adopted a consistent formulation by imposing only half (case A or B) of the second-class constraints. If one wants to avoid the appearance of Grassmann-valued Lagrange multipliers at all then by virtue of relation (55) one can choose for the projection operators $\mathbb{E}_A^{(\alpha)}$ and $\mathbb{E}_B^{(\alpha)}$ in Sect. 6.1 the simple integral representations

$$\mathbb{E}_A^{(\alpha)} = \int_0^{2\pi}\frac{d\eta}{2\pi}\,\mathrm{e}^{-\mathrm{i}\eta\chi_\alpha^\dagger\chi_\alpha}\ , \quad \mathbb{E}_B^{(\alpha)} = \int_0^{2\pi}\frac{d\eta}{2\pi}\,\mathrm{e}^{-\mathrm{i}\eta\chi_\alpha\chi_\alpha^\dagger}\ . \tag{91}$$

This procedure in effect amounts to replacing the odd second-class constraints $\chi_\alpha$ and $\chi_\alpha^\dagger$ by the even constraints

$\Phi_A^{(\alpha)} := \chi_\alpha^\dagger \chi_\alpha$ and $\Phi_B^{(\alpha)} := \chi_\alpha \chi_\alpha^\dagger$, respectively. Note that from (71) it immediately follows that for $\alpha \neq \beta$

$$[\Phi_A^{(\alpha)}, \Phi_A^{(\beta)}] = 0\,, \quad [\Phi_A^{(\alpha)}, \Phi_B^{(\beta)}] = 0\,, \quad [\Phi_B^{(\alpha)}, \Phi_B^{(\beta)}] = 0\,. \tag{92}$$

In other words, these even constraints are first class. So we finally conclude that any odd first-class constraint and a wide range (linear and diagonal non-linear) of odd second-class constraints appearing in fermion systems can be completely avoided within the approach presented in this paper.

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## Appendix

In this appendix we will present the explicit path-integral evaluations of two examples discussed in the main text. The first one is for the system considered in Sect. 4.1 whose formal path integral is given in (38). As measure for the Lagrange multipliers we choose

$$\mathcal{D}C(\eta) = \lim_{\varepsilon\to 0} \prod_{n=1}^{N} d\eta_n\, \delta(\eta_n)\, \frac{d\xi}{2\pi} \langle \Psi_0 | \mathrm{e}^{-\mathrm{i}\xi\Phi} | \Psi' \rangle \tag{A.1}$$

which is normalized (in the $\eta$'s) and also introduces a projection operator at $\tau = 0$. Hence, the time-lattice path integral which we want to evaluate reads

$$\begin{aligned} &\lim_{\varepsilon\to 0} \prod_{n=0}^{N-1} \int d\bar{\Psi}_n d\Psi_n \int_0^{2\pi} \frac{d\xi}{2\pi} \\ &\times \exp\left\{ -\sum_{n=1}^{N} \left[ \frac{1}{2}\bar{\Psi}_n \cdot \Delta\Psi_n - \frac{1}{2}\Delta\bar{\Psi}_n \cdot \Psi_{n-1} \right] \right\} \\ &\times \langle \Psi_0 | \mathrm{e}^{-\mathrm{i}\xi\Phi} | \Psi' \rangle\,. \end{aligned} \tag{A.2}$$

Using the convolution formula

$$\begin{aligned} &\int d\bar{\Psi}_n d\Psi_n\, \mathrm{e}^{-\bar{\Psi}_{n+1}\cdot\Delta\Psi_{n+1}/2 + \Delta\bar{\Psi}_{n+1}\cdot\Psi_n/2} \\ &\quad \times \mathrm{e}^{-\bar{\Psi}_n\cdot\Delta\Psi_n/2 + \Delta\bar{\Psi}_n\cdot\Psi_{n-1}/2} \\ &= \mathrm{e}^{-\bar{\Psi}_{n+1}\cdot(\Psi_{n+1}-\Psi_{n-1})/2}\, \mathrm{e}^{(\bar{\Psi}_{n+1}-\bar{\Psi}_{n-1})\cdot\Psi_{n-1}/2}\,, \end{aligned} \tag{A.3}$$

which follows from the completeness relation $\int d\bar{\Psi}_n d\Psi_n \langle \Psi_{n+1} | \Psi_n \rangle \langle \Psi_n | \Psi_{n-1} \rangle = \langle \Psi_{n+1} | \Psi_{n-1} \rangle$ and (11), the path integral can be reduced to

$$\begin{aligned} &\int d\bar{\Psi}_0 d\Psi_0 \int_0^{2\pi} \frac{d\xi}{2\pi} \\ &\times \exp\left\{ -\frac{1}{2}\bar{\Psi}_N \cdot (\Psi_N - \Psi_0) + \frac{1}{2}(\bar{\Psi}_N - \bar{\Psi}_0)\cdot\Psi_0 \right\} \\ &\times \mathrm{e}^{\mathrm{i}\xi} \langle \Psi_0 | \mathrm{e}^{-\mathrm{i}\xi(f_1^\dagger f_1 + f_2^\dagger f_2)} | \Psi' \rangle\,. \end{aligned} \tag{A.4}$$

The coherent-state matrix element appearing in the above expression is given by

$$\begin{aligned} &\langle \Psi_0 | \mathrm{e}^{-\mathrm{i}\xi(f_1^\dagger f_1 + f_2^\dagger f_2)} | \Psi' \rangle = \mathrm{e}^{-\bar{\Psi}_0\cdot\Psi_0/2}\mathrm{e}^{-\bar{\Psi}'\cdot\Psi'/2} \\ &\quad \times \left[ 1 + \mathrm{e}^{-\mathrm{i}\xi}\bar{\Psi}_0 \cdot \Psi' - \mathrm{e}^{-2\mathrm{i}\xi}\bar{\psi}_1\bar{\psi}_2\psi_1'\psi_2' \right], \end{aligned} \tag{A.5}$$

where we have used the notation $|\Psi'\rangle = |\psi_1'\rangle \otimes |\psi_2'\rangle$ and $\langle \Psi_0| = \langle \psi_1 | \otimes \langle \psi_2 |$. The remaining integrations are straightforward and lead to

$$\begin{aligned} &\int d\bar{\Psi}_0 d\Psi_0 \exp\left\{ -\frac{1}{2}\bar{\Psi}'' \cdot \Psi'' - \frac{1}{2}\bar{\Psi}' \cdot \Psi' \right\} \\ &\quad \times \exp\left\{ (\bar{\Psi}'' - \bar{\Psi}_0)\cdot\Psi_0 \right\} \bar{\Psi}_0 \cdot \Psi' \\ &= \exp\left\{ -\frac{1}{2}\bar{\Psi}'' \cdot \Psi'' - \frac{1}{2}\bar{\Psi}' \cdot \Psi' \right\} \bar{\Psi}'' \cdot \Psi' \end{aligned} \tag{A.6}$$

which is the result presented in (39). The evaluation of the path integral for the second example of first-class constraints (see Sect. 4.2) is similar to that above.

As an example for an explicit path-integral calculation with second-class constraints we choose case A of the linear odd constraint in Sect. 6.1.1. In this case the projection operator is given by $\mathbb{E}_A = |\theta\rangle\langle\theta|$ and the corresponding formal path integral (61) reads in the time-lattice formulation (48)

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int \prod_{n=1}^{N-1} d\bar{\psi}_n d\psi_n \\ &\times \exp\left\{ \mathrm{i} \sum_{n=1}^{N} \left[ \frac{\mathrm{i}}{2}\bar{\psi}_n(\psi_n - \theta) - \frac{\mathrm{i}}{2}(\bar{\psi}_n - \bar{\theta})\theta \right.\right. \\ &\left.\left. + \frac{\mathrm{i}}{2}\bar{\theta}(\theta - \psi_{n-1}) - \frac{\mathrm{i}}{2}(\bar{\theta} - \bar{\psi}_{n-1})\psi_{n-1} - \varepsilon H(\bar{\theta}, \theta) \right] \right\}, \end{aligned} \tag{A.7}$$

where we have made use of the explicit form of the constrained short-time propagator

$$\begin{aligned} &\langle \psi_n | \mathbb{E}_A \mathrm{e}^{-\mathrm{i}\varepsilon H} \mathbb{E}_A | \psi_{n-1} \rangle \\ &= \exp\left\{ -\frac{1}{2}\bar{\psi}_n(\psi_n - \theta) + \frac{1}{2}(\bar{\psi}_n - \bar{\theta})\theta \right\} \\ &\quad \times \exp\left\{ -\frac{1}{2}\bar{\theta}(\theta - \psi_{n-1}) + \frac{1}{2}(\bar{\theta} - \bar{\psi}_{n-1})\psi_{n-1} \right\} \\ &\quad \times \mathrm{e}^{-\mathrm{i}\varepsilon H(\bar{\theta}, \theta)}\,. \end{aligned} \tag{A.8}$$

Rearranging the sum in the exponent the above path integral takes the simple form

$$\begin{aligned} &\mathrm{e}^{-\bar{\psi}''(\psi''-\theta)/2}\mathrm{e}^{(\bar{\psi}''-\bar{\theta})\theta/2}\mathrm{e}^{-\bar{\theta}(\theta-\psi')/2}\mathrm{e}^{(\bar{\theta}-\bar{\psi}')\psi'/2}\mathrm{e}^{-\mathrm{i}tH(\bar{\theta},\theta)} \\ &\quad \times \lim_{\varepsilon\to 0} \prod_{n=1}^{N-1} \left[ \int d\bar{\psi}_n d\psi_n \mathrm{e}^{(\bar{\psi}_n - \bar{\theta})(\theta - \psi_n)} \right]. \end{aligned} \tag{A.9}$$

The remaining $N-1$ integration are easily evaluated providing $N-1$ factors of unity. Hence, we arrive at the result given in (58). The results (67), (69) and (70) given in the main text are derived in a similar fashion.

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