E_8, THE MOST EXCEPTIONAL GROUP

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ABSTRACT. The five exceptional simple Lie algebras over the complex number are included one within the other as $\mathfrak{g}_2 \subset \mathfrak{f}_4 \subset \mathfrak{e}_6 \subset \mathfrak{e}_7 \subset \mathfrak{e}_8$. The biggest one, $\mathfrak{e}_8$, is in many ways the most mysterious. This article surveys what is known about it including many recent results, focusing on the point of view of Lie algebras and algebraic groups over fields.

CONTENTS

1. INTRODUCTION

The Lie algebra $\mathfrak{e}_8$ or Lie group E_8 was first sighted by a human being sometime in summer or early fall of 1887, by Wilhelm Killing as part of his program to classify the semisimple finite-dimensional Lie algebras over the complex numbers [94, pp. 162–163]. Since then, it has been a source of fascination for mathematicians and others in its role as the largest of the exceptional Lie algebras. (It appears, for example, as part of the fictional Beard-Einstein Conflation in the prize-winning novel *Solar* [120].) Killing's classification is now considered the core of a typical graduate course on Lie algebras, and the paper containing the key ideas, [104], has

even been called "the greatest mathematical paper of all time"[1], see [35] and [96]. Killing showed that the simple Lie algebras make up four infinite families together with just five[2] others, called *exceptional*. They are ordered by inclusion and $\mathfrak{e}_8$ is the largest of them. When someone says "E_8" today, they might be referring to the simple Lie algebra $\mathfrak{e}_8$, an algebraic group or Lie group, a specific rank 8 lattice, or a collection of 240 points in $\mathbb{R}^8$ (the root system E_8). This article gives an introduction to these objects as well as a survey of some of the many results on E_8 that have been discovered since the millennium and indications of the current frontiers.

There are many reasons to be interested in E_8. Here is a pragmatic one. The famous Hasse-Minkowski Principle for quadratic forms and the Albert-Brauer-Hasse-Noether Theorem for division algebras over number fields can be viewed as special cases of the more general Hasse Principle for semisimple algebraic groups described and proved in [128, Ch. 6]. This is a powerful theorem, which subsumes not just those famous results but also local-global statements for objects that don't have ready descriptions in elementary language. Its proof involves case-by-case considerations, with the case of E_8 being the most difficult, eventually proved by Chernousov in [27], building on work by Harder in [91] and [92]. That E_8 was the most difficult is typical. In the words of [160]: "Exceptional groups, and in particular E_8, appear to have a more dense and complicated structure than classical ones, making computational problems more challenging for them.... Thus E_8 serves as a 'gold standard': to judge the effectiveness of the implementation of a general algorithm, one looks to how it performs for E_8."

I have heard this sentiment expressed more strongly as: We understand semisimple algebraic groups only as far as we understand E_8. If we know some statement for all groups except E_8, then we do not really know it.

Here is a more whimsical reason to be interested in E_8: readers who in their youth found quaternions and octonions interesting will naturally gravitate to the exceptional Lie algebras and among them the largest one, $\mathfrak{e}_8$. Indeed, one might call E_8 *the Monster of Lie theory*, because it is the largest of the exceptional groups, just as the Monster is the largest of the sporadic finite simple groups. Another take on this is that "E_8 is the most non-commutative of all simple Lie groups" [116].

Remarks on exceptional Lie algebras. The exceptional[3] Lie algebras mentioned above form a chain $\mathfrak{g}_2 \subset \mathfrak{f}_4 \subset \mathfrak{e}_6 \subset \mathfrak{e}_7 \subset \mathfrak{e}_8$. There are various surveys and

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[1]Any such delcaration is obviously made with an intent to provoke. One objection I have heard to this particular nomination is that Killing made a serious mathematical error — made famous by Cartan's quote "Malheureusement les recherches de M. Killing manquent de rigueur, et notamment, en ce qui concerne les groupes qui ne sont pas simples, il fait constamment usage d'un théorème qu'il ne démontre pas dans sa généralité" [23, p. 9] — but this error is from a different paper and is irrelevant to the current discussion. See [94] for details.

[2]Since Killing's paper contains so many new and important ideas, produced by someone working in near-isolation, it feels churlish to mention that it actually claims that there are *six* exceptional ones. Killing failed to notice that two of the ones in his list were isomorphic.

[3]We are following the usual definition of "exceptional", but there are alternatives that are appealing. For example, [42] considers a longer chain of inclusions

$$\mathfrak{sl}_2 \subset \mathfrak{sl}_3 \subset \mathfrak{g}_2 \subset \mathfrak{so}_8 \subset \mathfrak{f}_4 \subset \mathfrak{e}_6 \subset \mathfrak{e}_7 \subset \mathfrak{e}_8$$

such that, at the level of simply connected Lie groups, all of the inclusions are unique up to conjugacy. One could alternatively define all the Lie algebras appearing in that chain to be exceptional.

references focusing on some or all of these, such as [8], [56], [1], [99], [106], [63], and [147]. Certainly, the greatest amount of material is available for $\mathfrak{g}_2$, corresponding to the octonions, where one can find discussions as accessible as [49]. Moving up the chain, less is available and what exists is somewhat less accessible. This note takes the approach of jumping all the way to $\mathfrak{e}_8$ at the end, because it is the case with the least existing exposition, or, to say the same thing differently, the most opportunity.

Another feature of exceptional groups is that, by necessity, various ad hoc constructions are often employed in order to study one group or another. These can appear inexplicable at first blush. As much as we can, we will explain how these peculiar construction arise naturally from the general theory of semisimple Lie algebras and groups. See §4 for an illustration of this.

2. What is E_8?

To explain what Killing was doing, suppose you want to classify Lie groups, meaning for the purpose of this section, a smooth complex manifold G that is also a group and that the two structures are compatible in the sense that multiplication $G \times G \to G$ and inversion $G \to G$ are smooth maps. The tangent space to G at the identity, $\mathfrak{g}$, is a vector space on which G acts by conjugation, and this action gives a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ denoted $(x, y) \mapsto [x, y]$ such that $[x, x] = 0$ and $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$. The vector space $\mathfrak{g}$ endowed with this "bracket" operation is called the *Lie algebra* of G. Certainly, if two Lie groups are isomorphic, then so are their Lie algebras; the converse, or something like it, also holds and is known as "Lie's third theorem".

Consequently, for understanding Lie groups, it is natural to classify the finite-dimensional Lie algebras over the complex numbers, which was Killing's goal. Suppose L is such a thing. Then, among the ideals I such that the sequence $I \supseteq [I, I] \supseteq [[I, I], [I, I]] \supseteq \cdots$ eventually reaches zero, there is a unique maximal one, called the radical of L and denoted $\operatorname{rad} L$. The possible such ideals $\operatorname{rad} L$ are unclassifiable if $\dim(\operatorname{rad} L)$ is large enough, see [11], so we ignore it; replacing L by $L/\operatorname{rad} L$ we may assume that $\operatorname{rad} L = 0$. Then L is a direct sum of simple Lie algebras, which we now describe how to analyze.

The root system. To classify the simple Lie algebras, Killing observed that we can extract from a simple L a piece of finite combinatorial data called a *root system*, and he classified the possible root systems. The simple root system called E_8 is a 240-element subset R of $\mathbb{R}^8$, whose elements are called *roots*. It consists of the short vectors in the lattice Q generated by the *simple roots*

$$\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7 + e_8), \quad \alpha_2 = e_1 + e_2, \quad \alpha_3 = -e_1 + e_2,$$

$$\alpha_4 = -e_2 + e_3, \quad \alpha_5 = -e_3 + e_4, \quad \alpha_6 = -e_4 + e_5,$$

$$\alpha_7 = -e_5 + e_6, \quad \alpha_8 = -e_6 + e_7$$

where e_i denotes the i-th element of an orthonormal basis of $\mathbb{R}^8$. This information is encoded in the graph
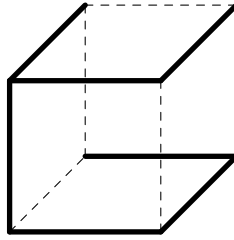
(2.1)
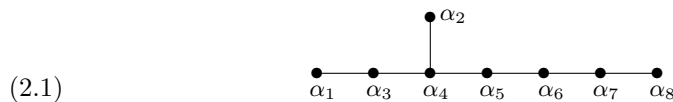
FIGURE 1. Dynkin diagram of E_8 obtained by deleting 5 edges from a cube in $\mathbb{R}^3$

where vertices correspond to simple roots, a single bond joining α_i to α_j indicates that $\alpha_i \cdot \alpha_j = -1$, and no bond indicates that $\alpha_i \cdot \alpha_j = 0$. Such a graph is called a *Dynkin diagram*, and the root system is commonly described in this language today, but they were not invented for more than 50 years after Killing's paper. The graph determines the root system up to isomorphism. Evidently the particular embedding of R in $\mathbb{R}^8$ and the labeling of the simple roots is not at all unique; here we have followed [19]. One could equally well draw it as suggested in [116], with the edges lying on along the edges of a cube as in Figure 1.
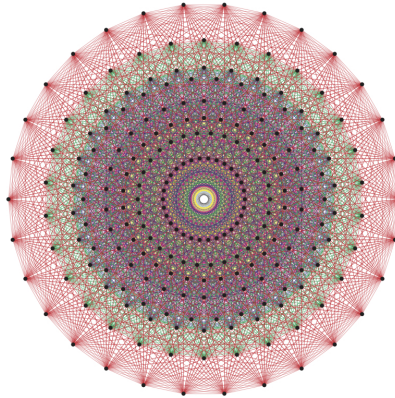

FIGURE 2. E_8's publicity photo, courtesy of J. Stembridge

We can draw a picture of the full set R by projecting $\mathbb{R}^8$ onto a well-chosen plane[4], see Figure 2. In the picture, the 240 roots are the black dots, which lie, 30 each, on 8 concentric circles — this much follows from general theory as in [19, VI.1.11, Prob. 33(iv)]. The edges in the picture join the images of roots that are nearest neighbors. The colors don't have any mathematical significance.

The E_8 lattice. The lattice Q together with the quadratic form $v \mapsto \frac{1}{2}\|v\|^2$ is the unique positive-definite, even, unimodular lattice of rank 8. It has been known for a long time that it is the lattice with the densest sphere packing in $\mathbb{R}^8$, and the

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[4]Precisely: take a Coxeter element w in the Weyl group of R. It is unique up to conjugacy and its minimal polynomial has a simple factor $x^2 - 2\cos(2\pi/h) + 1$, for $h = 30$ the Coxeter number for the root system E_8. That is, there is a unique plane in $\mathbb{R}^8$ on which the projection of w has that minimal polynomial, so that w acts on the plane by rotations by $2\pi/h$.

sphere packing with the highest kissing number [38]. Recently, Viazovska showed that it gives the densest sphere packing in $\mathbb{R}^8$, among all possible packings both lattice and non-lattice [162]. At the time of her work, the densest packing was unknown in $\mathbb{R}^n$ for all $n \geq 4$!

The roots R — the short vectors in Q — are a 7-design on the unit sphere in $\mathbb{R}^8$, i.e., every polynomial on $\mathbb{R}^8$ of total degree at most 7 has the same average on R as on the entire sphere, with the minimal number of elements [9]; such a thing is very rare. The theta-series for this lattice, $\theta(q) = \sum_{v \in Q} q^{\|v\|^2/2}$, is the 4-th Eisenstein series, which provides a connection with the j-invariant via the formula $j(q) = \theta(q)^3/\eta(q)^{24}$, where η denotes Dedekind's eta-function.[5] For more on this lattice and how it fits into the rest of mathematics, see [38] or [52].

Another view on the lattice is that it is a maximal order in the (real) octonions, as described in [39]. In particular, this turns Q into a non-associative ring with 240 units, namely the roots R.

From the root system to the Lie algebra. Starting from $R \subset \mathbb{R}^8$, one can write down a basis and multiplication table for the Lie algebra $\mathfrak{e}_8$ over $\mathbb{C}$ as in [117] or [77]. (This is a recent development. The old way to do this is via the Chevalley relations as in, for example, [20, §VIII.4.3]. Some signs have to be chosen — see [155], [25], or [60, §2.3] — but the isomorphism class of the resulting algebra does not depend on the choices made. Explicit multiplication tables are written in [24, p. 328] and [161].) In this way, one can make the Lie algebra from the root system.

3. E_8 AS AN AUTOMORPHISM GROUP

To study $\mathfrak{e}_8$, then, we can view it as a specific example of a simple Lie algebra described by generators and relations given by the root system. We will exploit this view below. However, that is not what is commonly done for other simple Lie algebras and groups! It is typical to describe the "classical" (meaning not exceptional) Lie algebras and groups not in terms of their root systems, but rather as, for example, SL_n, the n-by-n determinant 1 matrices; SO_n, the n-by-n orthogonal matrices of determinant 1; and Sp_{2n}, the $2n$-by-$2n$ symplectic matrices. Killing explicitly asked, in his 1889 paper, for similar descriptions of the exceptional Lie algebras and groups.

Looking back at the descriptions of the classical groups in the previous paragraph, each of them arises from a faithful irreducible representation $G \hookrightarrow \mathrm{GL}(V)$ where $\dim V$ is smaller than $\dim G$. Using now the classification of irreducible representations of simple Lie algebras, the Weyl dimension formula, and other general-purpose tools described entirely in terms of root systems (and which post-date Killing and can be found in standard textbooks such as [98]), we find that the smallest faithful irreducible representations of $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, and $\mathfrak{e}_7$ (alternatively, the simply connected Lie groups G_2, F_4, E_6, and E_7) have dimensions 7, 26, 27, and 56, which are much smaller than the dimensions 14, 52, 78, and 133 of the corresponding algebra. In the years since Killing posed his problem, all of these algebras have found descriptions using the corresponding representation, as we now relate.

The smallest nontrivial representation of G_2, call it V, has dimension 7. Calculating with weights, one finds that V has G_2-invariant linear maps $b: V \otimes V \to \mathbb{C}$

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[5] For connections of the coefficients of $j(q)^{1/3}$ with dimensions of irreducible representations of $E_{8,\mathbb{C}}$, see [95, esp. §4.1].

(a bilinear form, which is symmetric) and $\times : V \otimes V \to V$ (product, which is skew-symmetric[6]), which are unique up to multiplication by an element of $\mathbb{C}^{\times}$. To see this, one decomposes the G_2-modules $V^* \otimes V^*$ and $V^* \otimes V^* \otimes V$ into direct sums of irreducible representations and notes that in each case there is a unique 1-dimensional summand. The subgroup of $\mathrm{GL}(V)$ preserving these two structures is G_2. Alternatively, we can define a G_2 invariant bilinear form t and product on $\mathbb{C} \oplus V$ by setting

$$t((x,c),(x',c')) := xx' + b(c,c') \text{ and}$$
$$(x,c) \bullet (x',c') := (xx' + b(c,c'), xc' + x'c + c \times c').$$

With these definitions and scaling $\times$ so that $b(u \times v, u \times v) = b(u,u)b(v,v) - b(u,v)^2$, $\mathbb{C} \oplus V$ is isomorphic to the complex octonions, a nonassociative algebra — see [49, §10.3] for details — and G_2 is the automorphism group of that algebra, compare [147, §2.3]. Equivalently, $\mathfrak{g}_2$ is the Lie algebra of $\mathbb{C}$-linear derivations of $\mathbb{C} \oplus V$.

In a similar way, the smallest nontrivial representation of F_4, call it V, has unique F_4-invariant linear maps $b \colon V \otimes V \to \mathbb{C}$ (a symmetric bilinear form) and $\times \colon V \otimes V \to V$ (product, which is symmetric). Extending in a similar manner, we find that $\mathbb{C} \oplus V$ has the structure of an Albert algebra — a type of Jordan algebra — and F_4 is the automorphism group of that algebra, see [32] or [147, §7.2].

The smallest nontrivial representation of E_6, call it V, has an E_6-invariant cubic polynomial $f \colon V \to \mathbb{C}$ that is unique up to multiplication by an element of $\mathbb{C}^{\times}$. The isometry group of f, i.e., the subgroup of $g \in \mathrm{GL}(V)$ such that $f \circ g = f$, is E_6. This was pointed out in [23], but see [147, §7.3] for a proof.

For E_7, the smallest nontrivial representation has an E_7-invariant quartic polynomial whose isometry group is generated by E_7 and the group $\langle e^{\pi i/2} \rangle$ of fourth roots of unity [61]. That is, it has two connected components, and the connected component of the identity is E_7.

Answers for E_8. The same approach, applied to E_8, is problematic. The smallest nontrivial representation of E_8 has dimension 248; it is the action on its Lie algebra $\mathfrak{e}_8$. In this context E_8 is the automorphism group of the Lie algebra[7] $\mathfrak{e}_8$. This is not specific to E_8 — it is a typical property of semisimple Lie algebras, see [148] for a precise statement —and only serves to re-cast problems about the algebras as problems about the group and vice versa.

However, there is a general result that, in the case of E_8, says that for each faithful irreducible representation V, there is a homogeneous, E_8-invariant polynomial f such that E_8 is the identity component of the stabilizer of f in $\mathrm{GL}(V)$.

In the case of the smallest faithful irreducible representation, there is a degree 8 homogeneous polynomial on the Lie algebra $\mathfrak{e}_8$ whose automorphism group is generated by E_8 and the eighth roots of unity $\langle e^{\pi i/4} \rangle$. (For experts: one takes a degree 8 generator for the Weyl-group-invariant polynomials on a Cartan subalgebra and pulls it back to obtain an E_8-invariant polynomial on all of $\mathfrak{e}_8$. The paper [26] gives a formula for it in terms of invariants of the D_8 subgroup of E_8.)

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[6]The product can be viewed as a 7-dimensional analogue of the usual cross-product in $\mathbb{R}^3$, compare for example [49, §10.3].

[7]Or, what is the same, the group of linear transformations preserving both the Killing form $\kappa \colon \mathfrak{e}_8 \times \mathfrak{e}_8 \to \mathbb{C}$ and the alternating trilinear form $\wedge^3 \mathfrak{e}_8 \to \mathbb{C}$ given by the formula $x \wedge y \wedge z \mapsto \kappa(x, [y, z])$.

The second smallest faithful irreducible representation V has dimension 3875. Combinatorial calculations with "weights" of the representation show that it has a nonzero, E_8-invariant symmetric bilinear form b and trilinear form t, both of which are unique up to scaling by an element of $\mathbb{C}^\times$. These give a commutative and E_8-invariant "product", i.e., a bilinear map, $\bullet\colon V \times V \to V$ defined by $t(x,y,z) = b(x, y \bullet z)$. (Close readers of Chevalley will recognize this construction of a product from [31, §4.2].) The automorphism group of $(V, \bullet)$ is E_8, see [70].

These results are recent and have not yet been seriously exploited.

As the automorphism group of a variety. As with other semisimple groups, one can view E_8 as the group of automorphisms of a projective variety on which it acts transitively — a *flag variety* — see [43] for the general statement. In the case of E_8, the smallest such variety has dimension 57 (and can be obtained by quotienting out by the maximal parabolic subgroup determined by the highest root [23, p. 152]); see [89] for an alternative treatment.

4. Constructing the Lie algebra via gradings

Earlier, we described constructing the Lie algebra $\mathfrak{e}_8$ by generators and relations using a Chevalley basis. There are various refinements on this that we now discuss. This situation is quite general and is often applied to study other exceptional groups, so we write in a slightly more general context. Consider an irreducible root system R with set Δ of simple roots, and let $\mathfrak{g}$ be the Lie algebra generated from this using a Chevalley basis.

Suppose that $\mathfrak{g}$ has a grading by an abelian group Γ, meaning that as a vector space $\mathfrak{g}$ is a direct sum of subspaces $\mathfrak{g}_\gamma$ for $\gamma \in \Gamma$ and that $[g_\gamma, g_{\gamma'}] \in \mathfrak{g}_{\gamma+\gamma'}$ for $\gamma, \gamma' \in \Gamma$. Then $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}$ and each $\mathfrak{g}_\gamma$ is a $\mathfrak{g}_0$-module. Roughly speaking, one can typically write down a formula for the bracket on $\mathfrak{g}$ in terms of the action of $\mathfrak{g}_0$ on the $\mathfrak{g}_\gamma$, see [1, Ch. 6] or [64, §22.4] for discussion. The literature contains a profusion of such constructions, which are sometimes called a "$\mathfrak{g}_0$ construction of $\mathfrak{g}$". (I will give some references below, but it is hopeless to attempt to be comprehensive.)

To produce such a construction of $\mathfrak{g}$, then, we should look for gradings of $\mathfrak{g}$. These are controlled by subgroups of $\mathrm{Aut}(\mathfrak{g})$ in the following sense. If $F = \mathbb{C}$ and $\mathfrak{g}$ has a $\mathbb{Z}/n$ grading, then the map $\sum_\gamma y_\gamma \mapsto \sum e^{2\pi i \gamma/n} y_\gamma$ exhibits the n-th roots of unity $\mu_n(\mathbb{C}) = \langle e^{2\pi i/n} \rangle$ as a subgroup of $\mathrm{Aut}(\mathfrak{g})$ and conversely every homomorphism $\mu_n \to \mathrm{Aut}(\mathfrak{g})$ gives a $\mathbb{Z}/n$ grading on $\mathfrak{g}$ by the same formula. We generalize this example somewhat to the language of diagonalizable group schemes as in [44] or [167]. Write Γ^D for the Cartier dual of Γ, the (concrete) group of homomorphisms from Γ to $\mathbb{G}_\mathrm{m}$, the algebraic group with K-points $K^\times$ for every field K. In that notation, we have the following folklore classification of gradings:

Proposition 4.1. *Let S be a commutative ring and $\mathfrak{g}$ a finitely generated Lie algebra over S. Then for each finitely generated abelian group Γ, there is a natural bijection between the set of Γ-gradings on $\mathfrak{g}$ and the set of morphisms $\Gamma^D \to \mathrm{Aut}(\mathfrak{g})$ of group schemes over S.*

Sketch of proof. The group Γ is naturally identified with the dual $\mathrm{Hom}(\Gamma^D, \mathbb{G}_\mathrm{m})$, so given a Γ-grading on $\mathfrak{g}$, setting $t \cdot y = \gamma(t)y$ for $y \in \mathfrak{g}_\gamma$ and $t \in \Gamma^D$ defines a homomorphism $\Gamma^D \to \mathrm{Aut}(\mathfrak{g})$. Conversely, a homomorphism $\Gamma^D \to \mathrm{Aut}(\mathfrak{g})$ gives $\mathfrak{g}$ the structure of a comodule under the coordinate ring of Γ^D, i.e., the group ring

$S[\Gamma]$, which amounts to an S-linear map $\rho\colon A \to A \otimes S[\Gamma]$ that is compatible with the Hopf algebra structure on $S[\Gamma]$. One checks that $\mathfrak{g}_\gamma := \{y \in \mathfrak{g} \mid \rho(y) = y \otimes \gamma\}$ gives a Γ-grading on $\mathfrak{g}$. $\square$

As is clear from the proof, the hypothesis that $\mathfrak{g}$ is a Lie algebra is not necessary, and the result applies equally well to other sorts of algebraic structures over R. One could also replace R by a base scheme.

Examples: gradings by free abelian groups. By the proposition, a grading of $\mathfrak{g}$ by a free abelian group $\mathbb{Z}^r$ corresponds to a homomorphism of a rank r torus $\mathbb{G}_{\mathrm{m}}^r \to \mathrm{Aut}(\mathfrak{g})$. The image is connected, so it lies in the identity component of $\mathrm{Aut}(\mathfrak{g})$, which is the adjoint group, call it G^{ad}, with Lie algebra $\mathfrak{g}$. Up to conjugacy, all homomorphisms $\mathbb{G}_{\mathrm{m}}^r \to G^{\mathrm{ad}}$ have image in a given maximal torus T, and we conclude that all gradings of $\mathfrak{g}$ by $\mathbb{Z}^r$ are obtained from a rank r sublattices of $\mathrm{Hom}(\mathbb{G}_{\mathrm{m}}, T)$, i.e., the weight lattice for the root system dual to R. (E_8 is self-dual and so in that case it is the same as Q.)

A natural grading of this sort is obtained by choosing some r-element subset Δ' of Δ and defining the degree of a root vector x_β to be the coefficients c_δ of elements of Δ' in the expression $\beta = \sum_{\delta \in \Delta} c_\delta \delta$. Basic facts concerning this kind of grading can be found in [6] or [133]; most importantly, the $\mathfrak{g}_\gamma$ are irreducible representations of $\mathfrak{g}_0$ with an open orbit. This sort of grading on $\mathfrak{e}_8$ (and $\mathfrak{e}_7$ and $\mathfrak{e}_6$) with $r = 2$ was used in [15] to reconstruct the quadrangular algebras studied in [168] for studying the corresponding Moufang polygons.

The simplest example of a grading as in the previous paragraph is to take Δ' to be a singleton $\{\delta'\}$, in which case one finds a $\mathbb{Z}$-grading of $\mathfrak{g}$ with support $\{-n, \ldots, n\}$ where n is the coefficient of δ' in the highest root. Famously, in the case $n = 1$, one finds a 3-term grading in which the ± 1 components make up a Jordan pair, and one can use the Jordan pair to recover the bracket on $\mathfrak{g}$, see for example [125]. But that case does not occur for E_8, because its highest root is

$$(4.2) \qquad \omega_8 = e_7 + e_8 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8,$$

where the smallest coefficient is 2 for $\delta' = \alpha_1$ or α_8. Those δ' give a 5-term grading, where the ± 1 components are structurable algebras as in [4] or [3, Th. 4] or a Kantor pair as in [2], depending on your point of view. Starting with a structurable algebra, one can also recover the bracket on $\mathfrak{g}$, as was done for $\mathfrak{e}_8$ in [3, §8].

Examples: $\mathbb{Z}/n$ gradings. By the proposition, a $\mathbb{Z}/n$ grading on $\mathfrak{g}$ is given by a homomorphism $\mu_n \to \mathrm{Aut}(\mathfrak{g})$. The ones with image in G^{ad} (as is necessarily true for E_8, because $\mathrm{Aut}(\mathfrak{g}) = G^{\mathrm{ad}}$ in that case) are classified by their Kac coordinates, see [102] or [145], or see [111] for an expository treatment. Therefore in principle one has a list of all of the corresponding gradings.

A popular choice for such a grading is to start by picking a simple root $\delta' \in \Delta$, thus obtaining as in the preceding subsection an inclusion $\iota\colon \mathbb{G}_{\mathrm{m}} \to G^{\mathrm{ad}}$ such that, for $\delta \in \Delta$, the composition $\delta\iota\colon \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ is the identity for $\delta = \delta'$ and trivial otherwise. The desired $\mathbb{Z}/n$ grading is obtained by composing ι with the natural inclusion $\mu_n \hookrightarrow \mathbb{G}_{\mathrm{m}}$ for n the coefficient of δ' in the highest root. One finds again in this case that each $\mathfrak{g}_\gamma$ is an irreducible representation of $\mathfrak{g}_0$, see [163]. The identity component of the centralizer of $C_{G^{\mathrm{ad}}}(\mu_n)$ is semisimple and its Dynkin diagram is obtained by taking the so-called *extended Dynkin diagram* of R — obtained by adding the negative of the highest root as a new vertex — and deleting the vertex

δ' and all edges connected to it. (This is the first step in the iterative procedure described in [16] for calculating the maximal-rank semisimple subgroups of G^{ad}. See [19, Ch. VI, §4, Exercise 4] or [111] for treatments in the language of root systems or group schemes, respectively.)

For example, doing this for R the root system of type G_2 and δ' the simple root that is orthogonal to the highest root, one finds $n = 3$, $C_{G^{\mathrm{ad}}}(\mu_3)$ has identity component SL_3, and the ± 1-components of $\mathfrak{g}_2$ are the tautological representation and its dual. The action of G_2 on the octonions, restricted to this μ_3 subgroup, gives the direct sum decomposition of the octonions known as the Zorn vector matrix construction as in [134].

Example 4.3 ($D_8 \subset E_8$). Doing this with E_8 and $\delta' = \alpha_1$, we find $n = 2$ by (4.2), $C_{E_8}(\mu_2)$ has identity component Spin_{16}/μ_2, and the 1-component of $\mathfrak{e}_8$ is the half-spin representation. This grading was surely known to Cartan, and is the method used to construct $\mathfrak{e}_8$ in [1] and [58].

Example 4.4 ($A_8 \subset E_8$). Doing this with E_8 and $\delta' = \alpha_2$, we find $n = 3$, $C_{E_8}(\mu_3)$ has identity component SL_9/μ_3, and we have

$$\mathfrak{e}_8 \cong \mathfrak{sl}_9 \oplus (\wedge^3 \mathbb{C}^9) \oplus (\wedge^3 \mathbb{C}^9)^*$$

as modules under $\mathfrak{sl}_9$. One can use this operation to define the Lie bracket on all of $\mathfrak{e}_8$, see for example [62], [64, Exercise 22.21], and [55].

Example 4.5 ($A_4 \times A_4 \subset E_8$). Taking $\delta' = \alpha_5$, we find $n = 5$, and $C_{E_8}(\mu_5)$ is isomorphic to $(\mathrm{SL}_5 \times \mathrm{SL}_5)/\mu_5$. See [65, §14] or [82, §6] for concrete descriptions of the embedding, or [143, 1.4.2] for how it provides a non-toral elementary abelian subgroup of order 5^3.

Dynkin [48, Table 11] lists 75 different subsystems of E_8 constructed in this way, and each of these gives a way to construct the Lie algebra E_8 from a smaller one. Here is such an example:

Example 4.6 ($D_4 \times D_4 \subset E_8$). There is a subgroup $\mu_2 \times \mu_2$ of E_8 such that each of the three involutions has centralizer Spin_{16}/μ_2 as in Example 4.3, and the centralizer of $\mu_2 \times \mu_2$ is the quotient of $\mathrm{Spin}_8 \times \mathrm{Spin}_8$ by a diagonally embedded copy of $\mu_2 \times \mu_2$. This gives a grading of $\mathfrak{e}_8$ by the Klein four-group, with $(\mathfrak{e}_8)_0 = \mathfrak{so}_8 \times \mathfrak{so}_8$ and the other three homogeneous components being tensor products of an 8-dimensional representation of each of the copies of $\mathfrak{so}_8$. The corresponding construction of E_8 is the one described in [166, Ch. IV], [113], or [110].

We have described some large families of gradings, but there are of course others, see for example [51]. One can also construct $\mathfrak{e}_8$ without using a grading. The famous *Tits construction* of E_8 is of this type, and we will describe it in detail in §7 below.

5. E_8 OVER THE REAL NUMBERS

A Lie algebra L over the real numbers, i.e., a vector space over $\mathbb{R}$ with a bracket that satisfies the axioms of a Lie algebra, is said to be of type E_8 if $L \otimes_{\mathbb{R}} \mathbb{C}$ is isomorphic to the Lie $\mathbb{C}$-algebra $\mathfrak{e}_8$ studied by Killing. (The notation $L \otimes_{\mathbb{R}} \mathbb{C}$ is easy to understand concretely. Fix a basis $\ell_1, \ldots, \ell_n$ for the real vector space L. Then $L \otimes_{\mathbb{R}} \mathbb{C}$ is a complex vector space with basis $\ell_1, \ldots, \ell_n$ and the bracket on $L \otimes_{\mathbb{R}} \mathbb{C}$ is defined by $[\sum_i z_i \ell_i, \sum_j z'_j \ell_j] = \sum_i \sum_j z_i z'_j [\ell_i, \ell_j]$ for $z_i, z'_j \in \mathbb{C}$.)

Cartan proved in [24] that there are exactly three Lie $\mathbb{R}$-algebras of type E_8. One can reduce the problem of determining the isomorphism class of such algebras to determining the orbits under the Weyl group of elements of order 2 in a maximal torus in the compact real form of E_8, see [144, §III.4.5]. Given such an involution and the description of E_8 in terms of the Chevalley relations, one can do calculations, so this is an effective construction. Two of the three E_8's over $\mathbb{R}$ have names: split and compact, where the automorphism group of the compact Lie algebra is a real Lie group that is compact. The third one does not have a standard name; we call it *intermediate* following [142], but others say *quaternionic*. This is summarized in Table A.

Tits index or Satake diagram	name	signature of Killing form

	compact	-248
	intermediate	-24
	split	8

TABLE A. Real forms of E_8

Remarks on the Tits index. The first column of Table A indicates the Tits index of the group as defined in [154, 2.3], which is the Dynkin diagram as in (2.1) — the same for all groups of type E_8 — with the possible addition of circles around some vertices.[8] At this point in the survey we are only concerned with the field $\mathbb{R}$, in which special case the Tits index is sometimes called a Satake diagram and one can close the discussion by saying that the circles are drawn around the non-compact simple roots. But the theory surrounding the Tits index of a semisimple algebraic group G over an arbitrary field F is much richer and we will use the more general notion later, so let's take a moment to talk about it now. A *split torus* is a product of copies of $\mathbb{G}_{\mathrm{m}}$, the algebraic group with K-points $K^{\times}$ for every field K. All maximal split tori in G are conjugate over F, and the circles in the Tits index amount to precise formulas for the embedding of such a torus in G via the theory in [17, §6]. One coarse fact is that the dimension of a maximal split torus equals the number of circles in the Tits index.

A basic fact connect with the Tits index is Tits' *Witt-type theorem* from [154, 2.7] or [146, 16.4.2], which gives a concrete description of the structure of G in terms of the Tits index and a subgroup corresponding to the uncircled vertices in the diagram. This theorem includes as special cases Wedderburn's theorem that each central simple k-algebra can be written uniquely as $M_r(D)$ for D a central division k-algebra and $r \geq 1$, and Witt's theorem that a nondegenerate quadratic form is an orthogonal sum of a uniquely determined anisotropic nondegenerate quadratic

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[8]More precisely, the Tits index of a group also includes the action of the absolute Galois group on the Dynkin diagram, but the diagram for E_8 has only the identity automorphism, so this action is trivial.

form and a sum of hyperbolic planes. That is, the Witt-type theorem generalizes and unifies both of those classical results in the same way that the Hasse Principle mentioned in the introduction generalized and unified the Albert-Brauer-Hasse-Noether Theorem and the Hasse-Minkowski Principle.

Physics. Returning now to the discussion of real groups of type E_8, various such groups play a role in physics. For example, the compact real E_8 appears in string theory — see [87], [45], and [46] — and the split real form appears in supergravity [119].

On the other hand, E_8 does not play any role in the Grand Unified Theories of the kind described in [7] because every finite dimensional representation of every real form of E_8 is real orthogonal and not unitary. And at least one well-publicized approach to unification based on E_8 does not work, as explained in [47].

A magnet detects E_8 symmetry? A few years ago, experimental physicsts reported finding "evidence for E_8 symmetry" in a laboratory experiment [34]. (You may have seen descriptions aimed at mathematicians in [107] or [18].) This is yet a different role for E_8 than those mentioned before. The context is as follows. A one-dimensional magnet subjected to an external magnetic field can be described with a quantum one-dimensional Ising model — this is standard — and, in the previous millennium and not using E_8, [170] and [41] made numerical predictions regarding what would happen if the magnet were subjected to a second, orthogonal magnetic field. Some difficulties had to be overcome in order to experimentally test these predictions, namely, producing a large enough crystal with the correct properties (such as containing relatively isolated one-dimensional chains of ions) and producing a magnetic field that was large enough (5 tesla) and tunable. In the intervening years, these obstacles have been overcome, and the paper [34] reports on the resulting experiment.

The measurements were as predicted, and therefore the experiment can be viewed as providing evidence for the theory underlying the predictions.[9] Which raises the question: how does E_8 appear in these predictions? Indeed, looking at the original papers, one finds no use of E_8 in making the predictions (although Zamolodchikov did remark already in [170] that numerological coincidences strongly suggest that E_8 should play a role). Those papers, however, in addition to a handful of unobjectionable non-triviality assumptions, rest on a serious assumption that the perturbed conformal field theory would be an integrable theory. It was later discovered that if one views the original conformal field theory as arising from a compact real Lie algebra via the "coset construction", then the perturbed theory is given by the corresponding affine Toda field theory, see [54], [97], and [50]. In this way, invoking the compact real form of E_8 simplifies the theory by removing an assumption, and therefore *any evidence for the theory as a whole — such as that provided by the experiment in* [34] *— may be viewed as "evidence for E_8 symmetry".*

For those wanting further checks on the theory, another experiment to try would be to test the predictions for a thermal perturbation of the tricritical Ising model,

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[9]Some skeptics aver that the evidence is weak, in that the confirmed predictions amount to somewhat noisy measurements of two numbers, namely the ratio of the masses of the two heaviest quasi-particles and the ratio of the corresponding intensities. In Bayesian language, the skeptics' estimate of the (probability of) correctness of the underlying theory was only somewhat increased by learning of the experimental results.

for which the role of E_8 is played instead by E_7. The unperturbed version has already been realized [151].

6. E_8 OVER AN ARBITRARY FIELD

It makes sense to speak of Lie algebras or groups "of type E_8" over any field. Starting with the E_8 root system and the Chevalley relations as in the previous section, one can put a Lie algebra structure on $\mathbb{Z}^{248}$. The isomorphism class of the resulting Lie algebra $\mathfrak{e}_{8,\mathbb{Z}}$ does not depend on the choice of signs, and in this way one obtains, for every field F, a Lie F-algebra $\mathfrak{e}_{8,F} := \mathfrak{e}_{8,\mathbb{Z}} \otimes F$ called the *split* $\mathfrak{e}_8$ *over* F. Its automorphism group $E_{8,F}$ is an algebraic group over F[10] known as the *split group of type E_8 over F*.

Finite fields. For example, if F is a *finite* field with q elements, then the F-points $E_{8,F}(F)$ are a finite simple group, normally denoted $E_8(q)$. These are enormous groups, whose order is a polynomial in q with leading term q^{248}. The smallest such group is $E_8(2)$, which has more than 3×10^{74} elements, about 10^{20} times the size of the Monster [37, p. 242]. A special case of the Inverse Galois Problem is to ask: Does $E_8(q)$ occur as a Galois group over $\mathbb{Q}$? It is known when q is a prime larger than 5 by [90] and [169].

Groups of type E_8. More generally, a Lie algebra L over a field F is said to *have type E_8* or be an *F-form of E_8* if there is a field $K \supseteq F$ such that $L \otimes_F K$ is isomorphic to $L_0 \otimes_\mathbb{Z} K$. A similar definition applies for algebraic groups.

An algebraic group G of type E_8 over F is *split* if it is isomorphic to $E_{8,F}$. It is *isotropic* if it contains a copy of $\mathbb{G}_\mathrm{m}$ (equivalently, if $\mathrm{Lie}(G)$ has a nontrivial $\mathbb{Z}$-grading), and *anisotropic* otherwise. For $F = \mathbb{R}$, there is a unique anisotropic form of E_8, the compact form, but for other fields F there may be many, see the examples in §8.

Fields over which the split E_8 is the only E_8. Over some fields F, the split group $E_{8,F}$ is the only group of type E_8. This is true for algebraically closed fields (trivially by [44]). It is also true for finite fields, fields of transcendence degree 1 over an algebraically closed field, and more generally for fields of cohomological dimension at most 1 [149]. Classically it is also known for p-adic fields and it is also true for every field that is complete with respect to a discrete valuation with residue field of cohomological dimension ≤ 1 [21]. It is also true for global fields of prime characteristic, i.e., finite extensions of $\mathbb{F}_p(t)$ for some p [93].

It is also true if every finite separable extension of F has degree a power of some prime[11] $p \geq 7$. This is argued in an elementary way in [156], but can alternatively be viewed as a special case of older and more general arguments underlying the proof of the Hasse Principle, specifically Propositions 6.19, 6.21 on pages 339, 375 of [128]. (Either way, only the case $p = 7$ is nontrivial, because 7 is the largest prime dividing the order of the Weyl group of E_8.) The analogous statements for $p = 2, 3$, and 5 are false, see Example 8.1 and Theorem 9.1.

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[10]Meaning it is a smooth variety such that the functor {field extensions of F} $\rightarrow$ Sets given by sending $K \mapsto E_{8,F}(K)$ factors through the category of groups.

[11]This sort of field arises naturally when trying to disentangle phenomena related to distinct primes. See for example the discussion in [129, §5].

Serre's "Conjecture II". A bold conjecture of Serre's from 1962 [137] says that, for every field F of cohomological dimension[12] ≤ 2, and G a simply connected semisimple algebraic group over F, $H^1(F, G) = 0$. As $E_{8,F} = \mathrm{Aut}(E_{8,F})$, by descent $H^1(F, E_{8,F})$ is identified with the pointed set of groups of type E_8 over F (with distinguished element $E_{8,F}$ itself). In this case, Serre's conjecture says: the split group $E_{8,F}$ is the only form of E_8 over F. What made this conjecture bold is that it includes totally imaginary number fields and finite extensions of $\mathbb{F}_p(t)$ as special cases and the conjecture was not proved in these special cases until the 1980s [27] and 1970s [93] respectively. Furthermore, the hypothesis on cohomological dimension is very difficult to make use of; a key breakthrough in its application was the Merkurjev–Suslin Theorem [150, Th. 24.8] connecting it with surjectivity of the reduced norm of a central simple associative algebra.

Despite lots of progress during the past 20 years as in [10], and [78], the conjecture remains open; see [81] for a recent survey. It is known to hold for E_8 if:

- every finite extension of k has degree a power of p for some fixed p by [78, §III.2] for $p = 2, 3$ (or see [29] or [74, 10.22] for $p = 3$) and by [28] for $p = 5$ (or see [65, 15.5]). The $p = 2$ case can also be deduced from properties of Semenov's invariant (see Theorem 9.2) if char $F = 0$.
- k is the function field of a surface over an algebraically closed field, see [40] and [81].

Forms of E_8 over $\mathbb{Z}$. One can equally well consider $\mathbb{Z}$-forms of E_8 as in [85] or [36]. We don't go into this here, but note that E_8 is unusual in this context in that the number of $\mathbb{Z}$-forms of the "compact" real E_8 is at least 10^4, in stark contrast to at most 4 for other simple groups of rank ≤ 8, see [85, Prop. 5.3].

7. TITS' CONSTRUCTION

Tits gave an explicit construction of Lie algebras which produces $\mathfrak{e}_8$ as a possible output [153]. Viewing the Lie algebra $\mathfrak{e}_8$ as constructed from the root system E_8 as in §2, one can see that it contains $\mathfrak{g}_2 \times \mathfrak{f}_4$ as a subalgebra. It corresponds to an inclusion of groups $G_2 \times F_4 \hookrightarrow E_8$. Applying Galois cohomology $H^1(F, *)$, we find a map of pointed sets

$$(7.1) \qquad H^1(F, G_2) \times H^1(F, F_4) \to H^1(F, E_8)$$

that is functorial in F; we call this *Tits' construction*. To interpret this, we note that E_8 is its own automorphism group and similarly for F_4 and G_2. Again, descent tells us that $H^1(F, E_8)$ is naturally identified with the set of isomorphism classes of algebraic groups (or Lie algebras) of type E_8 over F, with distinguished element the split group E_8. The same holds for F_4 and G_2. Thus the function (7.1) can be viewed as a construction that takes as inputs a group of type G_2 and a group of type F_4 and gives output a group (or Lie algebra) of type E_8.

Alternatively, viewing G_2 and F_4 as automorphism groups of an octonion and an Albert algebra respectively, Galois descent identifies $H^1(F, G_2)$ and $H^1(F, F_4)$ with the sets of isomorphism classes of octonion F-algebras and Albert F-algebras

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[12]Here *cohomological dimension* means the supremeum of the cohomological p-dimension as p varies over all primes. Cohomological p-dimension is defined via Galois cohomology as in [144] for $p \neq \mathrm{char}\, F$ and via Milne-Kato cohomology if $p = \mathrm{char}\, F$. Furthermore, the conjecture only demands this for a finite list of primes depending on the Killing-Cartan type of G, see [140] or [81] for precise statements. In the case of E_8, the hypothesis only concerns the primes 2, 3, and 5.

respectively, cf. [106]. From this perspective, the construction takes as input two algebras.

Tits expressed his construction as an explicit description of a Lie algebra L made from an octonion F-algebra and an Albert F-algebra, see [153] or [99] for formulas.

Example 7.2. Let G be a group of type E_8 obtained from an octonion algebra C and Albert algebra A via Tits' construction; we give the formula for the isomorphism class of its Killing form. The algebra C is specified by a 3-Pfister quadratic form γ_3 and A has i-Pfister quadratic forms ϕ_i for $i = 3, 5$, see [72]. Formulas for the Killing forms on the subgroups $\mathrm{Aut}(C)$ and $\mathrm{Aut}(A)$ are given in [72, 27.20], and plugging these into the calculation in [99, p. 117, (144)] gives that the Killing form on G is

$$\langle 60 \rangle [8 - (4\gamma_3 + 4\phi_3 + \langle 2 \rangle \gamma_3(\phi_5 - \phi_3))]$$

in the Witt ring of k, exploiting the notation for elements of the Witt ring from [53].

In case $F = \mathbb{R}$, we know that there are two groups of type G_2 or octonion algebras, the compact form (corresponding to the octonion division algebra) and the split form (the split algebra). There are three real groups of type F_4 or Albert $\mathbb{R}$-algebras, the compact form (the Albert algebra with no nilpotents), the split form (the split algebra), and a third form we will call "intermediate" (the Albert algebra with nilpotents but not split). The book [99] contains a careful calculation of the Killing form on the Lie algebra of type E_8 produced by this construction for each choice of inputs, and we list the outputs in Table B.

Form of E_8	Rost invariant	Semenov invariant	inputs to Tits' construction	
			G_2	F_4
compact	0	1	compact	compact
intermediate	1	NA	split	intermediate or compact
			compact	split
split	0	0	split	split
			compact	intermediate

TABLE B. Real forms of E_8: cohomological invariants and Tits' construction

One could ask for a description of the Tits index of the E_8 produced by (7.1) in terms of the the inputs, e.g., to know whether and how the output group is isotropic. In one direction, the answer is easy: the groups of type G_2 and F_4 used as inputs to the construction are subgroups of the output, and therefore isotropic inputs give isotropic outputs. However, in the analogous question for Tits' construction of groups of type F_4 or E_6, anisotropic inputs can produce isotropic outputs, see for example [73].

Example 7.3 (Number fields). If G is a group of type E_8 over a number field F, then the natural map

$$H^1(F, G) \to \prod_{\text{real closures } R \text{ of } F} H^1(R, G)$$

is bijective. (This is true more generally for every semisimple simply connected group G [128, p. 286]. It is the Hasse Principle mentioned in the introduction.)

In particular, if F has r real embeddings, then there are 3^r isomorphism classes of groups of type E_8 over F, all of which are obtained from Tits' construction. Two groups of type E_8 over F are isomorphic if and only if their Killing forms are isomorphic, because this is so when $F = \mathbb{R}$.

8. COHOMOLOGICAL INVARIANTS; THE ROST INVARIANT

At this point in the survey, we have assembled only a few tools for determining whether or not two groups or Lie algebras of type E_8 over a field F are isomorphic, or whether they can be obtained as outputs from Tits' construction. Indeed, the only tools we have so far for this are the Tits index and the Killing form. In [156], Tits showed that a "generic"[13] group G of type E_8 has no reductive subgroups other than rank 8 tori (which necessarily exist); such a group cannot arise from Tit's construction because those groups all have a subgroup of type $G_2 \times F_4$. This brings us to the state of knowledge in 1990, as captured in [138].

To address this gap, we will use various invariants in the sense of [72]. Let G be an algebraic group over a field F. Let A be a functor from the category of fields containing F to abelian groups. An *A-invariant f of G* is a morphism of functors $f\colon H^1(*, G) \to A(*)$, i.e., for each field K containing F, there is a function $f_K\colon H^1(K, G) \to A(K)$ and for each morphism $\alpha\colon K \to K'$ the diagram

$$
\begin{array}{ccc}
H^1(K, G) & \xrightarrow{\ f_K\ } & A(K) \\
{\scriptstyle H^1(\alpha)}\big\downarrow & & \big\downarrow{\scriptstyle A(\alpha)} \\
H^1(K', G) & \xrightarrow{\ f_{K'}\ } & A(K')
\end{array}
$$

commutes. One goal of the the theory of cohomological invariants is to describe all the A-invariants of G, for various choices of G and A. Some classical examples of invariants in this sense are the Brauer class of central simple algebras (an $H^2(*, \mathbb{G}_m)$-invariant of PGL_n) and the Stiefel-Whitney classes or Hasse-Witt invariant of a quadratic form when char $F \neq 2$ ($H^\bullet(*, \mathbb{Z}/2)$-invariants of an orthogonal group O_n).

The addition in $A(K)$ for each K gives the structure of an abelian group to the collection of A-invariants of G. Every element $a \in A(F)$ gives a "constant" invariant f_a that sends every element of $H^1(K, G)$ to the image of a in $A(K)$, for every K. An invariant is *normalized* if the distinguished element of the pointed set $H^1(F, G)$ is sent to the zero element of $A(F)$. Evidently every A-invariant can be written uniquely as (constant) + (normalized).

The Rost invariant. Considering the case where A is Galois cohomology with torsion coefficients and G is a simple and simply connected group, one finds that the group of normalized invariants $H^1(*, G) \to H^d(*, \mathbb{Q}/\mathbb{Z}(d-1))$ is zero for $d = 1$ (because G is connected [106, 31.15]) and for $d = 2$ (because G is simply connected, see [12, Th. 2.4]). For $d = 3$, Markus Rost proved that the group is cyclic with a canonical generator r_G, now known as the *Rost invariant*, whose basic properties are developed in [72]. (Rost's theorem has recently been extending to include the case where G is split reductive, see [122] and [108].) For $G = G_2, F_4, E_8$, r_G has order 2, 6, 60 respectively.

---

[13]Meaning a *versal* form as defined in [72]. By definition, one can obtain from a given versal form of E_8 every group of type E_8 over every extension of F by specialization.

Example 8.1. Let G be a versal form of E_8, defined over some field K. Then $r_{E_8}(G)$ has order 60 by [72, p. 150]. For a prime p, let L be the separable algebraic extension of K fixed by a p-Sylow subgroup of the absolute Galois group of K, so every finite separable extension of L has degree a power of p. If $p \geq 7$, then as in §6, the base change G_L of G to L is split. But if $p < 7$, then p divides 60, and $r_{E_8}(G_L)$ is not zero, so G_L is not split. (To see that $r_{E_8}(G_L) \neq 0$, note that $H^3(L, \mathbb{Q}/\mathbb{Z}(2))$ is the colimit $\varinjlim H^3(L_0, \mathbb{Q}/\mathbb{Z}(2))$ as L_0 runs over finite extensions of K contained in L. But if any such L_0 killed $r_{E_8}(G)$, a restriction/corestriction argument as in [144, §I.2.4, Prop. 9] would show that $r_{E_8}(G)$ has order dividing $[L_0 : K]$, a contradiction.)

Example 8.2. The composition

$$(8.3) \qquad H^1(F, G_2) \times H^1(F, F_4) \xrightarrow{\text{Tits}} H^1(F, E_8) \xrightarrow{r_{E_8}} H^3(F, \mathbb{Q}/\mathbb{Z}(2))$$

is $r_{G_2} + r_{F_4}$, by an argument analogous to that in the proof of [75, Lemma 5.8]. This reduces the computation of the Rost invariant for E_8's arising from Tits' construction to the (known, calculable) Rost invariants r_{G_2} and r_{F_4}, and therefore gives a formula for all forms of E_8 over $\mathbb{R}$ or over a number field. We have included these values in Table B. For arbitrary F, the image of the composition (8.3) belongs to the 6-torsion subgroup, $H^3(F, \mathbb{Z}/6(2))$.

Example 8.4. Considering the $\mathbb{Z}/5$-grading on $\mathfrak{e}_8$ from Example 4.5, we find a subgroup $(\mathrm{SL}_5 \times \mathrm{SL}_5)/\mu_5$ of E_8, which in turn contains a subgroup $\mu_5 \times \mu_5 \times \mathbb{Z}/5$. Applying cohomology gives a map

$$(8.5) \qquad H^1(F, \mu_5) \times H^1(F, \mu_5) \times H^1(F, \mathbb{Z}/5) \to H^1(F, E_8).$$

Composing this with r_{E_8}, we obtain a map

$$H^1(F, \mu_5) \times H^1(F, \mu_5) \times H^1(F, \mathbb{Z}/5) \to H^3(F, \mathbb{Z}/5(2))$$

that is, up to sign, the cup product map into $H^3(F, \mathbb{Z}/5(2))$ [82]. Therefore, if $H^3(F, \mathbb{Z}/5(2))$ is not zero, it contains a nonzero symbol s, and from this one can construct a group G of type E_8 with $r_{E_8}(G) = s$. This group G cannot arise from Tits' construction.

We remark that (8.5) is surjective in the special case where every finite separable extension of F has degree a power of 5.

Example 8.6. We can focus our attention on the 3-torsion part of the image of r_{E_8} by considering $20 r_{E_8}$, which has image in $H^3(*, \mathbb{Z}/3(2))$. What is its image? Given a nonzero symbol s in $H^3(F, \mathbb{Z}/3(2))$, there is an Albert division algebra A constructed by the first Tits construction such that $r_{F_4}(A) = s$. The inclusions $F_4 \subset E_6 \subset E_8$ induce a map $H^1(F, F_4) \to H^1(F, E_8)$ such that the image G of A has Tits index

$$(8.7) \qquad \text{}$$

and $r_{E_8}(G) = r_{F_4}(A) = s$. (Indeed, the uncircled vertices in the Tits index indicate a subgroup of type E_6 that is the group of isometries of the norm on A.) Conversely, given an F-form G of E_8 over any field F such that $20 r_{E_8}(G)$ is a symbol in $H^3(F, \mathbb{Z}/3(2))$, [74] shows that there is a finite extension K of F of degree not divisible by 3 such that $G \times K$ has Tits index (8.7).

Example 8.8. There exists a field F and an F-form G of E_8 such that $20r_{E_8}(G)$ is *not* a symbol in $H^3(F, \mathbb{Z}/3(2))$, nor does it become a symbol over any finite extension of F of degree not divisible by 3. (In particular, such a group does not arise from Tits' construction.) To see this, one appeals to the theory of the J-invariant from [127] that there exists such a G with $J_3(G) = (1, 1)$; [74, Lemma 10.23] shows that such a group has the desired property.

In contrast to the case of 3 and 5 torsion as in the preceding two examples, the 2-primary part of $r_{E_8}(G)$ may have a longer symbol length and a more subtle relationship with the isotropy of G. As in Table B, the compact real form has Rost invariant 0, yet is anisotropic. In the other direction, Appendix A.6 of [65] gives an example of an isotropic group G of type E_8 over a field F such that $r_{E_8}(G)$ belongs to $H^3(F, \mathbb{Z}/2(2))$, and $r_{E_8}(G)$ is not a sum of fewer than three symbols.

Underlying the results used in Examples 8.6 and 8.8 are analyses of the possible decompositions of the Chow motives with $\mathbb{F}_p$-coefficients of the flag varieties for a group of type E_8. This is a way to study the geometry of these varieties despite their lack of rational points in cases of interest, and it has been widely exploited over the last decade for analyzing semisimple groups over general fields, see for example the book [53] for applications to quadratic form theory.

Example 8.9. Suppose G is a group of type E_8 such that $r_{E_8}(G)$ has order divisible by 15. (For example, if G is versal as in Example 8.1.) Then G has no proper reductive subgroups other than rank 8 tori, by the argument for [68, Th. 9.6]. Essentially, employing the Rost invariant simplifies Tits' proof of the fact about versal groups mentioned at the beginning of this section.

9. The kernel of the Rost invariant; Semenov's invariant

The examples in the previous section show that the Rost invariant can be used to distinguish groups of type E_8 from each other and for stating some results about such groups. But it remains a coarse tool. For example: *Given a field F, what are the groups of type E_8 that are in the kernel of the Rost invariant?*

One way to approach this question is to simplify the problem by making an assumption about the arithmetic of F. For example, in §6 we listed several kinds of fields F over which every E_8 is split; in these cases the answer to the question is trivial. To that list we add the following.

Theorem 9.1 (Chernousov)**.** *Suppose that every finite extension of F has degree a power of p for $p = 3$ or 5. Then the kernel of the Rost invariant $r_{E_8}: H^1(F, E_8) \to H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ is zero.*

Proof. This combines the main results of [28] and [29]. (Those papers make some hypothesis on char F, but such an assumption is harmless in view of [74, Prop. 9.3].) For alternative proofs, see [65, Prop. 15.5] for $p = 5$ and [74, Prop. 10.22] for $p = 3$. $\square$

As the case $p \geq 7$ was treated in §6, only $p = 2$ remains. The conclusion of Theorem 9.1 is false in that case, as can be seen already for $F = \mathbb{R}$, where the compact E_8 has Rost invariant zero. Recently, Semenov produced a newer, finer invariant in [136]. Put $H^1(F, E_8)_{15}$ for the kernel of $15r_{E_8}$, a subset of $H^1(F, E_8)$.

Theorem 9.2 (Semenov). *If* $\operatorname{char} F = 0$, *there exists a nonzero invariant*

$$s_{E_8} \colon H^1(*, E_8)_{15} \to H^5(*, \mathbb{Z}/2)$$

such that $s_{E_8}(G) = 0$ *if and only if* G *is split by an odd-degree extension of* F.

For the case $F = \mathbb{R}$, s_{E_8} is defined only for the split and compact real forms, and it is nonzero on the compact form.

This style of invariant, which is defined only on a kernel of another invariant, is what one finds for example in quadratic form theory. The Rost invariant $H^1(*, \operatorname{Spin}_n) \to H^3(F, \mathbb{Z}/2)$ amounts to the so-called Arason invariant e_3 of quadratic forms. There is an invariant e_4 defined on the kernel of e_3 with values in $H^4(F, \mathbb{Z}/2)$, an invariant e_5 defined on the kernel of e_4 with values in $H^5(F, \mathbb{Z}/2)$, and so on, see [53, §16].

Example 9.3. Let G be a group of type E_8 arising from Tits' construction as in Example 7.2. Then $15r_{E_8}(G) = 0$ if and only if $\phi_3 = \gamma_3$ in the notation of that example. In that case, $s_{E_8}(G)$ is the class $e_5(\phi_5) \in H^5(F, \mathbb{Z}/2)$, see [76, Th. 3.10].

It is natural to wonder what other cohomological invariants of E_8 may exist.

Question 9.4. [131, p. 1047] *Do there exist nonzero invariants mapping* $H^1(*, E_8)$ *into* $H^5(*, \mathbb{Z}/3)$ *and* $H^9(*, \mathbb{Z}/2)$?

The parameters in the question were suggested by the existence of certain non-toral elementary abelian subgroups in $E_8(\mathbb{C})$. (Such subgroups are described in [83].)

10. Witt invariants

We introduced the notion of A-invariant as a tool for distinguishing groups of type E_8. In the previous two sections we discussed the case where A was a Galois cohomology group, but one could equally well take A to be the functor W that sends a field F to its Witt group of non-degenerate quadratic forms $W(F)$ as defined in, for example, [53] or [109]. In this section, for ease of exposition, we assume that $\operatorname{char} F \neq 2$. In particular, $W(F)$ is naturally a ring via the tensor product.

Killing form. For G an algebraic group, put $\kappa(G)$ for the Killing form on the Lie algebra of G. (If $\operatorname{char} k$ divides 60, then the Killing form is identically zero, and in that case one should define $\kappa(G)$ to be the reduced Killing quadratic form as defined in [67]. We elide the details here.) The function $G \mapsto \kappa(G)$ is a morphism of functors $H^1(*, E_8) \to W(*)$, and we define

$$f \colon H^1(*, E_8) \to W(*) \quad \text{via} \quad G \mapsto \kappa(G) - \kappa(E_8)$$

to obtain a W-invariant that sends the distinguished class to 0.

The Witt ring has a natural filtration by powers of the "fundamental" ideal I generated by even-dimensional quadratic forms. It is natural to ask: *What is the largest n such that $f(G)$ belongs to I^n?*

Example 10.1. In case G arises by Tits' construction, the formula in Example 7.2 shows that $f(G) \in I^5$. Concretely we can see this over the real numbers, where the signature defines an isomorphism $W(\mathbb{R}) \cong \mathbb{Z}$ that identifies I^5 with $32\mathbb{Z}$, and Table A shows that $f(G) \in \{0, -32, -248\}$.

The action of the split group E_8 on its Lie algebra $\mathfrak{e}_8$ is by Lie algebra automorphisms, so it preserves the Killing form $\kappa(E_8)$, giving a homomorphism $E_8 \to O(\kappa(E_8))$. As E_8 is simply connected, this lifts to a homomorphism $E_8 \to \mathrm{Spin}(\kappa(E_8))$. By Galois descent, it is clear that the image of a group G under the map $H^1(F, E_8) \to H^1(F, O(\kappa(E_8)))$ is $\kappa(G)$, whence $f(G)$ necessarily belongs to I^3 because the map factors through $H^1(F, \mathrm{Spin}(\kappa(E_8)))$ [106, p. 437].

In fact, we cannot do better than the exponent in the previous paragraph, because: $f(G)$ *belongs to I^4 iff $30r_{E_8}(G) = 0$*. To see this, we observe on the one hand that $f(G)$ belongs to I^4 if and only if $f(G)$ is in the kernel of the Arason invariant $e_3 \colon I^3(*) \to H^3(*, \mathbb{Z}/2)$ by [123] or [132], i.e., the Rost invariant $r_{\mathrm{Spin}(\kappa(E_8))}$ vanishes on the image of G. On the other hand, the map $E_8 \to \mathrm{Spin}(\kappa(E_8))$ has Dynkin index 30, so composing it with $r_{\mathrm{Spin}(\kappa(E_8))}$ gives $30r_{E_8}$ by [72, p. 122]. This completes the proof of the claim. A versal group G of type E_8 has $30r_{E_8}(G) \neq 0$, so $f(G) \in I^3 \setminus I^4$.

Inspired by [139], we may still ask: *If $r_{E_8}(G)$ has odd order, is it necessarily true that $f(G) \in I^8$?* If the answer is yes and $\mathrm{char}\, F = 0$, one can ask for more: *Does Semenov's invariant $s(G) \in H^5(F, \mathbb{Z}/2)$ divide $e_8(f(G)) \in H^8(F, \mathbb{Z}/2)$?* The answer to both of these questions is "yes" for groups arising from Tits' construction, by Example 7.2. See [66] for more discussion.

Witt invariants in general. For every field K containing F, the ring $W(K)$ is a $W(F)$-algebra with identity element q_0, and $W(*)$ is a functor to the category of $W(F)$-algebras. Therefore, the collection of invariants $H^1(*, E_8) \to W(*)$ is itself a $W(F)$-algebra.

Question 10.2. *What are the invariants $H^1(*, E_8) \to W(*)$?*

Here is a natural way to construct such invariants. For each dominant weight λ of E_8, there is a Weyl module $V(\lambda)$ with highest weight λ defined for $\mathfrak{e}_{8,\mathbb{Z}}$ as in [101], that has an indivisible, $E_{8,\mathbb{Z}}$-invariant quadratic form $q_{\mathbb{Z}}$ on it, uniquely determined up to sign. Base change to a field P gives a quadratic form $q_{\mathbb{Z}} \otimes P$ on $V(\lambda) \otimes P$ that has a radical, namely the unique maximal proper submodule of $V(\lambda) \otimes P$, hence we find a quadratic form on the irreducible quotient of $V(\lambda) \otimes P$, call the form $q_{\lambda,P}$. Applying H^1, we obtain a function $H^1(P, E_8) \to H^1(P, O(q_{\lambda,P}))$, and therefore for every group G of type E_8 over P, we obtain a corresponding quadratic form we denote by $q_\lambda(G)$. For example, the "constant" invariant that sends every G to the 1-dimensional quadratic form $x \mapsto x^2$ can be viewed as q_0, and the invariant κ can be viewed as q_λ for λ the highest root. *Do the q_λ, as λ varies over the dominant weights, generate the $W(F)$-algebra of invariants $H^1(*, E_8) \to W(*)$?*

11. Connection with division algebras

Groups of type E_8 are closely related to the smallest open cases of two of the main outstanding problems in the study of (associative) division algebras. To recall the terminology, the center of a division ring D is a field, call it F. We say that D is a division algebra if $\dim_F D$ is finite, in which case $\dim_F D$ is a square, and its square root is called the *degree* of D. Hamilton's quaternions are an example of such a D with $F = \mathbb{R}$ and $\deg D = 2$.

One knows that, among all fields K such that $F \subset K \subset D$, the maximal ones always have $\dim_F K = \deg D$. In case there exists a maximal K that is Galois over F (resp., Galois over F with cyclic Galois group), one says that D is *a crossed*

product (resp., is *cyclic*). For a crossed product, one can write down a basis and multiplication rules in a relatively compact way, and the description is even simpler if D is cyclic.

Every division algebra D with $\deg D = 2$ or 3 is known to be cyclic, and the principal open problem in the theory of division algebras is: *If $\deg D = p$ for a prime $p \geq 5$, must D be cyclic?* (See [5] for context and discussion.) Philippe Gille showed that this question, for the case $p = 5$, can be rephrased as a statement about groups of type E_8, see [79].

Along with the dimension, another property of a division algebra D is its *period*, which is the smallest number p such that a tensor product of p copies of D is isomorphic to a matrix algebra over F. (It is a basic fact that p divides $\deg D$ and that the two numbers have the same prime factors.) Another open problem about division algebras is: *Determine whether every division algebra F of period p and degree p^2 is a crossed product.* For period 2 and degree 2^2, the answer is "yes" and is due to Adrian Albert. (It is also "yes" for period 2 and degree 2^3 by Louis Rowen [100].) The smallest open case, then, is where $p = 3$, and this is where E_8 may play a role. As in [164], using the $\mathbb{Z}/3$-grading on $\mathfrak{e}_8$ from Example 4.4, we find that SL_9 / μ_3 acts transitively on certain 4-dimensional subspaces of $\wedge^3(F^9)$. This gives a surjection in Galois cohomology $H^1(F, N) \to H^1(F, \mathrm{SL}_9 / \mu_3)$ for some subgroup N of SL_9 / μ_3 and one can hope that analyzing this surjection would give insight into whether algebras of degree 9 and period 3 are crossed products. See [114] for more discussion of this general setup and [65] for examples where similar surjections are exploited.

12. OTHER RECENT RESULTS ON E_8

Torsion index. Grothendieck [88] defined an invariant of a compact Lie group G called the *torsion index*, which has interpretations for the Chow groups and motivic cohomology of the classifying space of BG as well as for the étale cohomology of torsors under analogues of G over arbitrary fields, see [159] for precise statements. Jacques Tits' brief paper [156] contained a proof that the torsion index for the compact E_8 is divisible by $60 = 2 \cdot 3 \cdot 5$ and divides $2^9 \cdot 3^2 \cdot 5$, and wrote that one could hope ("hypothése optimiste !") that the correct answer is 60. In [157] and [158], Burt Totaro proved that it is $2^6 \cdot 3^2 \cdot 5$.

Kazhdan-Lusztig-Vogan polynomials. Some readers will remember a flurry of news coverage in 2007 about a "calculation the size of Manhattan". This referred to the calculation of the Kazhdan-Lusztig-Vogan polynomials for the split real forms of simple Lie algebras, where the final step was for $\mathfrak{e}_{8,\mathbb{R}}$. For more on this see [165] and [160]. We have also omitted the related topic of infinite-dimensional unitary representations of E_8, for constructions of which see example [103], [86], and [22].

Finite simple subgroups. We now know which finite simple groups embed in the (infinite) simple group $E_8(\mathbb{C})$, or more generally $E_8(k)$ for k algebraically closed, see [143], [84], and [33] for surveys. One would like to know also how many conjugacy classes there are for each of these finite subgroups, for which the interesting case of the alternating group on 5 symbols was resolved in [115]. These questions can be viewed as a case of a natural generalization of the classification of the finite simple subgroups of $\mathrm{SO}(3)$, which amounts to a classification of the Platonic solids as in [105, Chap. I]. Specifically, the Platonic solids correspond to the embeddings of the

alternating group on 4 or 5 letters, $\mathrm{PSL}(2,3)$ and $\mathrm{PSL}(2,5)$, and the symmetric group on 5 letters, $\mathrm{PGL}(2,3)$, in $\mathrm{SO}(3)$, and these embeddings are part of a series of embeddings of such subgroups in simple Lie groups including the case of $E_8(k)$, see [141] and [143].

Vanishing of trace forms. For a representation $\rho\colon \mathfrak{e}_{8,F} \to \mathfrak{gl}(V)$ for some V, the map $b_\rho\colon (x,y) \mapsto \mathrm{Tr}(\rho(x)\rho(y))$ defines an $\mathfrak{e}_{8,F}$-invariant symmetric bilinear form on $\mathfrak{e}_{8,F}$. (When ρ is the adjoint representation, b_ρ is the Killing form.) In the 1960s, motivated by then-current approaches to studying Lie algebras over fields of prime characteristic, it was an open problem to determine whether b_ρ is identically zero for all ρ when $\mathrm{char}\, F = 5$, see [135, p. 48], [13, p. 554], or [14, p. 544]. It is indeed always zero, and for $\mathrm{char}\, F = 2, 3$ as well, see [67].

Essential dimension. The essential dimension $\mathrm{ed}(G)$ of an algebraic group G is a non-negative integer that declares, roughly speaking, the number of parameters needed to specify a G-torsor, see [130], [129], or [121] for a formal definition and survey of what is known. In the case of E_8, this equals the number of parameters needed to specify a group of type E_8. So far, we only know bounds on $\mathrm{ed}(E_8)$ and the bounds we know are quite weak. Specifically, over $\mathbb{C}$ we have

$$9 \le \mathrm{ed}(E_8) \le 231,$$

where the lower bound is from [131] or [30] and the upper bound is from [112] or [69]. The distance between the upper and lower bounds is remarkable. In contrast, for the other simply connected exceptional groups over $\mathbb{C}$, one knows by [118] and [114] that

$$\mathrm{ed}(G_2) = 3, \quad 5 \le \mathrm{ed}(F_4) \le 7, \quad 4 \le \mathrm{ed}(E_6) \le 8, \quad \text{and} \quad 7 \le \mathrm{ed}(E_7) \le 11,$$

which are all much closer. Determining $\mathrm{ed}(E_8)$ will require new techniques.

Yet more topics. We have furthermore omitted any discussion of:
- Relations with vertex operator algebras as in [57] and [59].
- The adjoint representation of E_8 is in some sense unique among irreducible representations of simple algebraic groups, in that it is the only non-minuscule standard module that is irreducible in all characteristics, see [71].
- The Kneser-Tits Problem as described in [80]. One of the remaining open cases for E_8 was recently settled in [126], and another was settled in some special cases in [152].
- Affine buildings with residues of type E_8 as in [124].

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Acknowledgments

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