Landau and Lifshitz (1975), *The Classical Theory of Fields*, Fourth Revised English Edition, Chapter 13, Annotated by R. M L Baker, Jr.

Pergamon Press Offices:

U. K.	Pergamon Press Ltd., Headington Hill Hall, Oxford, England
U. S. A.	Pergamon Press Inc., Maxwell House, Fairview Park, Elmsford, New York 10523, U.S.A.
CANADA	Pergamon of Canada Ltd., 207 Queen's Quay West, Toronto 1, Canada
AUSTRALIA	Pergamon Press (Aust.) Pty. Ltd., 19a Boundary Street, Rushcutters Bay, N.S.W. 2011, Australia
FRANCE	Pergamon Press SARL, 24 rue des Ecoles, 75240 Paris, Cedex 05, France
WEST GERMANY	Pergamon Press GmbH, 3300 Braunschweig, Postfach 2923, Burgplatz 1, West Germany

First English edition 1951
Second English edition 1962
Third English edition 1971
Fourth English edition 1975

Library of Congress Cataloging in Publication Data

Landau, Lev Davidovich, 1908-1968.
The classical theory of fields.
(Course of theoretical physics; v. 2)
Translated from the 6th rev. ed. of Teoriia polia.
Includes bibliographical references.
1. Electromagnetic fields. 2. Field theory (Physics) I. Lifshits, Evgeniĭ Mikhaĭlovich, joint author. II. Title.
QC665. E4L3713 1975 530.1'4 75-4737
ISBN 0-08-018176-7

Translated from the 6th revised edition of Teoriya Pola, Nauka, Moscow, 1973

Printed in Great Britain by Billing & Sons Ltd. Guildford and London

CHAPTER 13

GRAVITATIONAL WAVES

§ 107. Weak gravitational waves

Just as in electrodynamics, in the relativistic theory of gravitation the finite velocity of propagation of interactions results in the possibility of the existence of free gravitational fields that are not linked to bodies—gravitational waves.

We consider the weak gravitational field in vacuum. As in § 105, we introduce the tensor h_{ik}, describing a weak perturbation of the galilean metric:

$$g_{ik} = g_{ik}^{(0)} + h_{ik}. \tag{107.1}$$

Then, to terms of first order in the h_{ik}, the contravariant metric tensor is:

$$g^{ik} = g^{ik(0)} \sim h^{ik}, \tag{107.2}$$

and the determinant of the tensor g_{ik}:

$$g = g^{(0)}(1+h), \tag{107.3}$$

where $h \equiv h_i^i$; all operations of raising and lowering tensor indices are done with the unperturbed metric $g^{(0)}$.

As already pointed out in § 105, the condition that the h_{ik} be small leaves the possibility of arbitrary transformations of reference system of the form $x'^i = x^i + \xi^i$, with small ξ^i; then

$$h'_{ik} = h_{ik} - \frac{\partial \xi_i}{\partial x^k} - \frac{\partial \xi_k}{\partial x^i}. \tag{107.4}$$

Using this arbitrariness of gauge for the tensor h_{ik}, we impose on it the supplementary condition

$$\frac{\partial \psi_i^k}{\partial x^k} = 0, \quad \psi_i^k = h_i^k - \tfrac{1}{2}\delta_i^k h, \tag{107.5}$$

after which the Ricci tensor takes the simple form (105.11):

$$R_{ik} = \tfrac{1}{2}\Box h_{ik}, \tag{107.6}$$

where $\Box$ denotes the d'Alembertian operator:

$$\Box = -g^{lm(0)}\partial^2/\partial x^l \partial x^m = \triangle - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}.$$

The conditions (107.5) still do not fix a unique choice of reference frame: if certain h_{ik}

satisfy these conditions, then so will the h'_{ik} of (107.4), if only the ξ^i are solutions of the equations

$$\Box \xi^i = 0. \tag{107.7}$$

Equating (107.6) to zero, we thus find the equations for the gravitational field in vacuum in the form

$$\Box h_i^k = 0. \tag{107.8}$$

This is the ordinary wave equation. Thus gravitational fields, like electromagnetic fields, propagate in vacuum with the velocity of light.

Let us consider a plane gravitational wave. In such a wave the field changes only along one direction in space; for this direction we choose the axis $x^1 = x$. Equation (107.8) then changes to

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) h_i^k = 0, \tag{107.9}$$

the solution of which is any function of $t \pm x/c$ (§ 47).

Consider a wave propagating in the positive direction along the x axis. Then all the quantities h_i^k are functions of $t - x/c$. The auxiliary condition (107.5) in this case gives $\dot{\psi}_i^1 - \dot{\psi}_i^0 = 0$, where the dot denotes differentiation with respect to t. This equality can be integrated by simply dropping the sign of differentiation—the integration constants can be set equal to zero since we are here interested only (as in the case of electromagnetic waves) in the varying part of the field. Thus, among the components ψ_i^k that are left, we have the relations

$$\psi_1^1 = \psi_1^0, \qquad \psi_2^1 = \psi_2^0, \qquad \psi_3^1 = \psi_3^0, \qquad \psi_0^1 = \psi_0^0. \tag{107.10}$$

As we pointed out, the conditions (107.5) still do not determine the system of reference uniquely. We can still subject the coordinates to a transformation of the form $x'^i = x^i + \xi^i(t - x/c)$. These transformations can be employed to make the four quantities $\psi_1^0, \psi_2^0, \psi_3^0, \psi_2^2 + \psi_3^3$ vanish; from the equalities (107.10) it then follows that the components $\psi_1^1, \psi_2^1, \psi_3^1, \psi_0^0$ also vanish. As for the remaining quantities $\psi_2^3, \psi_2^2 - \psi_3^3$, they cannot be made to vanish by any choice of reference system since, as we see from (107.4), these components do not change under a transformation $\xi_i = \xi_i(t - x/c)$. We note that $\psi = \psi_i^i$ also vanishes, and therefore $\psi_i^k = h_i^k$.

Thus a plane gravitational wave is determined by two quantities, h_{23} and $h_{22} = -h_{33}$. In other words, gravitational waves are transverse waves whose polarization is determined by a symmetric tensor of the second rank in the yz plane, the sum of whose diagonal terms, $h_{22} + h_{33}$, is zero.

For the two independent polarizations we may choose the cases in which one of the two quantities h_{23} and $\frac{1}{2}(h_{22} - h_{33})$ differs from zero. These two polarizations are distinguished from one another by a rotation through $\pi/4$ in the yz plane.

Let us calculate the energy–momentum pseudotensor in a plane gravitational wave. The components t^{ik} are second-order quantities; we must calculate them neglecting terms of still higher order. Since, when $h = 0$, the determinant g differs from $g^{(0)} = -1$ only by terms of second order, we can, in the general formula (96.9), set $\mathfrak{g}^{ik}{}_{,l} \approx g^{ik}{}_{,l} \approx -h^{ik}{}_{,l}$. For a plane wave all the other nonzero terms in t^{ik} are contained in the term

$$\tfrac{1}{2} g^{il} g^{km} g_{np} g_{qr} g^{nr}{}_{,l}\, g^{pq}{}_{,m} = \tfrac{1}{2} h_q^{n,i} h_n^{q,k}$$

in curly brackets in (96.9) (as is easily shown by choosing one of the axes of a galilean system of reference along the direction of propagation of the wave). Thus,

$$t^{ik} = \frac{c^4}{32\pi k} h_{q}^{n,i} h_{n}^{q,k}. \tag{107.11}$$

The energy flux in the wave is given by the quantities $-cgt^{0\alpha} \approx ct^{0\alpha}$. In a plane wave, propagating along the x^1 axis, in which the nonzero quantities h_{23} and $h_{22} = -h_{33}$ depend only on the difference $t - x/c$, this flux is also along x^1 and is equal to

$$ct^{01} = \frac{c^3}{16\pi k}\left[h_{23}^2 + \tfrac{1}{4}(h_{22} - h_{33})^2\right]. \tag{107.12}$$

As initial conditions for the arbitrary field of a gravitational wave we must assign four arbitrary functions of the coordinates: because of the transversality of the field there are just two independent components of $h_{\alpha\beta}$, in addition to which we must also assign their first time derivatives. Although we have made this enumeration here by starting from the properties of a weak gravitational field, it is clear that the result, the number 4, cannot be related to this assumption and applies for any free gravitational field, i.e. for any field which is not associated with gravitating masses.

PROBLEMS

Determine the curvature tensor in a weak plane gravitational wave.

Solution: Calculating R_{iklm} from (105.8), we find the following nonzero components:

$$-R_{0202} = R_{0303} = -R_{1212} = R_{0212} = R_{0331} = R_{3131} = \sigma,$$
$$R_{0203} = -R_{1231} = -R_{0312} = R_{0231} = \mu,$$

where we use the notation

$$\sigma = -\tfrac{1}{2}\ddot{h}_{33} = \tfrac{1}{2}\ddot{h}_{22}, \qquad \mu = -\tfrac{1}{2}\ddot{h}_{23}.$$

In terms of the three-dimensional tensors $A_{\alpha\beta}$ and $B_{\alpha\beta}$ in (92.15), we have:

$$A_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sigma & \mu \\ 0 & \mu & \sigma \end{pmatrix}, \qquad B_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mu & \sigma \\ 0 & \sigma & -\mu \end{pmatrix}.$$

By a suitable rotation of the x^2, x^3 axes, we can make one of the quantities σ or μ vanish (at a given point of four-space); if we make σ vanish in this way, we reduce the curvature tensor to the degenerate Petrov type II (type N).

§ 108. **Gravitational waves in curved space–time**

Just as we have treated the propagation of gravitational waves "on the background" of a flat space–time, we can consider weak perturbations relative to an arbitrary (non-galilean) "unperturbed" metric $g_{ik}^{(0)}$. Also anticipating other possible applications, we shall write the necessary formulas in a more general form.

Again taking the g_{ik} in the form (107.1), we find the first order correction to the Christoffel symbols expressed in terms of the h_{ik}:

$$\Gamma^{i(1)}_{kl} = \tfrac{1}{2}(h^i_{k;l} + h^i_{l;k} - h_{kl}{}^{;i}), \tag{108.1}$$

which can be verified by direct calculation (here, as in the sequel, all tensor operations of raising and lowering of indices, and covariant differentiation, are done with the nongalilean metric $g^{(0)}_{ik}$). We find for the corrections to the curvature tensor:

$$R^{i(1)}_{klm} = \tfrac{1}{2}(h^i_{k;m;l} + h^i_{m;k;l} - h_{km}{}^{;i}{}_{;l} - h^i_{k;l;m} - h^i_{l;k;m} + h_{kl}{}^{;i}{}_{;m}). \tag{108.2}$$

The corrections to the Ricci tensor are then

$$R^{(1)}_{ik} = R^{l(1)}_{ilk} = \tfrac{1}{2}(h^l_{i;k;l} + h^l_{k;i;l} - h_{ik}{}^{;l}{}_{;l} - h_{;i;k}). \tag{108.3}$$

The corrections to the mixed components of the Ricci tensor are obtained from the relations

$$R^{k(0)}_i + R^{k(1)}_i = (R^{(0)}_{il} + R^{(1)}_{il})(g^{kl(0)} - h^{kl}),$$

so that

$$R^{k(1)}_i = g^{kl(0)} R^{(1)}_{il} - h^{kl} R^{(0)}_{il}. \tag{108.4}$$

The exact metric in vacuum must satisfy the exact Einstein equations $R_{ik} = 0$. Since the unperturbed metric $g^{(0)}_{ik}$ satisfies the equations $R^{(0)}_{ik} = 0$, we find for the perturbation, $R^{(1)}_{ik} = 0$, i.e.,

$$h^l_{i;k;l} + h^l_{k;i;l} - h_{ik}{}^{;l}{}_{;l} - h_{;i;k} = 0. \tag{108.5}$$

In the general case of arbitrary gravitational waves, simplification of this equation to a form like (107.8) is not possible. This can, however, be done in the important case of waves of high frequency: when the wavelength λ and the oscillation period λ/c are small compared to the characteristic distances L and times L/c over which the "background field" changes. Each differentiation of a component h_{ik} increases the order of the quantity by a factor L/λ relative to derivatives of the unperturbed metric $g^{(0)}_{ik}$. If we limit the accuracy to terms of the two highest orders $[(L/\lambda)^2$ and $(L/\lambda)]$ we can interchange the orders of differentiation; in fact, the difference

$$h^l_{i;k;l} - h^l_{i;l;k} \approx h^l_m R^{m(0)}_{ikl} - h^m_i R^{l(0)}_{mkl}$$

is of order $(L/\lambda)^0$, whereas each of the expressions $h^l_{i;k;l}$ and $h^l_{i;l;k}$ contains terms of both higher orders. Imposing on h_{ik} the supplementary conditions

$$\psi^k_{i;k} = 0 \tag{108.6}$$

[analogous to (107.5)], we get the equation

$$h_{ik}{}^{;l}{}_{;l} = 0 \tag{108.7}$$

which generalizes (107.8).

For the reasons given in § 107, the condition (108.6) does not fix a unique choice of coordinates. They can still be subjected to a transformation $x'^i = x^i + \xi^i$, where the small quantities ξ^i satisfy the equation $\xi^{i;k}{}_{;k} = 0$. These transformations can be used, in particular, to impose on the h_{ik} the condition $h \equiv h^i_i = 0$. Then $\psi^k_i = h^k_i$, so that the h^k_i are subjected to conditions

$$h^k_{i;k} = 0, \quad h = 0. \tag{108.8}$$

After this the set of admissible transformations is reduced to the requirement $\xi^i{}_{;i} = 0$.

$r > \lambda$, but $r < L$

The pseudotensor t^{ik} contains, in addition to the unperturbed part $t^{ik(0)}$, terms of various orders in the h_{ik}. We arrive at an expression analogous to (107.11) if we consider the quantities t^{ik} averaged over regions of four-space with dimensions large compared to λ but small compared to L. Such an averaging (which we denote by the angular brackets $\langle \ldots \rangle$) does not affect the $g_{ik}^{(0)}$ and annihilates all quantities that are linear in the rapidly oscillating quantities h_{ik}. Of the quadratic terms, we preserve only the terms of higher (second) order in $1/\lambda$; these are the terms quadratic in the derivatives $h_{ik,l} \equiv \partial h_{ik}/\partial x^l$.

To this accuracy, all terms in t^{ik} that are expressed as four-divergences can be dropped. In fact, the integrals of such quantities over a region of four-space (the region of averaging) are transformed by Gauss' theorem, as a result of which their order of magnitude in $1/\lambda$ is reduced by unity. In addition, those terms drop out which vanish because of (108.7) and (108.8) after integration by parts. Thus, integrating by parts and dropping integrals of four-divergences, we find:

$$\langle h^{ln}{}_{,p} h^{p}_{l,n} \rangle = -\langle h^{ln} h^{p}_{l,p,n} \rangle = 0,$$

$$\langle h^{il}{}_{,n} h^{k,n}_{l} \rangle = -\langle h^{il} h^{k,n}_{l,n} \rangle = 0.$$

As a result the only second-order terms that remain are

$$\langle t^{ik(2)} \rangle = \frac{c^4}{32\pi k} \langle h^{n,i}_{q} h^{q,k}_{n} \rangle. \tag{108.9}$$

We note that to this same accuracy, $\langle t_i^{i(2)} \rangle^0 = 0$.

Since it has a definite energy, the gravitational wave is itself the source of some additional gravitational field. Like the energy producing it, this field is a second-order effect in the h_{ik}. But in the case of high-frequency gravitational waves the effect is significantly strengthened: the fact that the pseudotensor t^{ik} is quadratic in the derivatives of the h_{ik} introduces the large factor λ^{-2}. In such a case we may say that the wave itself produces the background field on which it propagates. This field is conveniently treated by carrying out the averaging described above over regions of four-space with dimensions large compared to λ. Such an averaging smooths out the short-wave "ripple" and leaves the slowly varying background metric (R. A. Isaacson, 1968).

To derive the equation determining this metric, we must, in expanding the R_{ik}, keep not only linear terms but also quadratic terms in h_{ik}: $R_{ik} = R_{ik}^{(0)} + R_{ik}^{(1)} + R_{ik}^{(2)}$. As already pointed out, the averaging does not affect the zero-order terms. Thus, the averaged field equations $\langle R_{ik} \rangle = 0$ take the form

$$R_{ik}^{(0)} = -\langle R_{ik}^{(2)} \rangle, \tag{108.10}$$

where we should keep only terms of second-order in $1/\lambda$ in $R_{ik}^{(2)}$. They are easily found from the identity (96.7). The terms quadratic in h_{ik} that arise on the right side of this identity, and have the form of a four-divergence, vanish (to the accuracy considered) when the averaging is done, and there remains

$$\langle (R^{ik} - \tfrac{1}{2} g^{ik} R)^{(2)} \rangle = -\frac{8\pi k}{c^4} \langle t^{ik(2)} \rangle,$$

or, since $\langle t^{ik(2)} \rangle = 0$, to this same accuracy:

$$\langle R_{ik}^{(2)} \rangle = -\frac{8\pi k}{c^4} \langle t_{ik}^{(2)} \rangle.$$

Finally, using (108.9), we get eq. (108.10) in the final form

$$R_{ik}^{(0)} = \tfrac{1}{4}\langle h^n_{q,i} h^q_{n,k}\rangle. \tag{108.11}$$

If the "background" is produced entirely by the waves themselves, (108.11) and (108.7) must be solved simultaneously. An estimate of the expressions on both sides of (108.11) shows that in this case the radius of curvature of the background metric, which is of order L, is related to the wavelength λ and the order of magnitude of its field h by $L^{-2} \sim h^2/\lambda^2$, i.e. $\lambda/L \sim h$.

§ 109. Strong gravitational waves

In this section we shall consider the solution of the Einstein equations which is a generalization of the weak, plane gravitational wave in a flat space–time (I. Robinson and H. Bondi, 1957).

We shall look for a solution in which, in a suitable reference frame, all the components of the metric tensor are functions of a single variable, which we call x^0 (without, however, prejudging its character). This condition still permits coordinate transformations of the form

$$x^\alpha \to x^\alpha + \phi^\alpha(x^0), \tag{109.1}$$

$$x^0 \to \phi^0(x^0), \tag{109.2}$$

where ϕ^0, ϕ^α are arbitrary functions.

The character of the solution depends essentially on whether we can make all the $g_{0\alpha}$ vanish by using the three transformations (109.1). This can be done if the determinant $|g_{\alpha\beta}| \neq 0$. In fact, under the transformation (109.1), $g_{0\alpha} \to g_{0\alpha} + g_{\alpha\beta}\dot{\phi}^\beta$ (where the dot denotes differentiation with respect to x^0); if $|g_{\alpha\beta}| \neq 0$, the system of equations

$$g_{0\alpha} + g_{\alpha\beta}\dot{\phi}^\beta = 0$$

determines the $\phi^\beta(x^0)$ that accomplish the required transformation. Such a case will be treated in § 117; here we shall be interested in the solution in which

$$|g_{\alpha\beta}| = 0. \tag{109.3}$$

In this case there is no reference system in which all the $g_{0\alpha} = 0$. Instead, however, the four transformations (109.1–2) can be used to make

$$g_{01} = 1, \quad g_{00} = g_{02} = g_{03} = 0. \tag{109.4}$$

Here the variable x^0 has "lightlike" character: for $dx^\alpha = 0$, $dx^0 \neq 0$, the interval $ds = 0$; we shall denote the variable x^0 chosen in this way by $x^0 = \eta$. Under the conditions (109.4) the line element can be written in the form

$$ds^2 = 2dx^1\, d\eta + g_{ab}(dx^a + g^a dx^1)(dx^b + g^b dx^1). \tag{109.5}$$

Throughout this section, the indices $a, b, c, \ldots$ take on values 2, 3; $g_{ab}(\eta)$ can be regarded as a two-dimensional tensor. Calculation of the quantities R_{ab} leads to the following field equations:

$$R_{ab} = -\tfrac{1}{2} g_{ac}\dot{g}^c g_{bd}\dot{g}^d = 0.$$

It then follows that $g_{ac}\dot{g}^c = 0$, or $\dot{g}^c = 0$, i.e. $g^c = \text{const}$. We can, therefore, by a transformation $x^a + g^a x^1 \to x^a$, bring the metric to the form

$$ds^2 = 2dx^1\, d\eta + g_{ab}(\eta)dx^a dx^b. \tag{109.6}$$

The determinant $-g$ of this metric tensor coincides with the determinant $|g_{ab}|$, while the only nonzero Christoffel symbols are the following:

$$\Gamma^a_{b0} = \tfrac{1}{2}\kappa^a_b, \quad \Gamma^1_{ab} = -\tfrac{1}{2}\kappa_{ab},$$

where we have introduced the two-dimensional tensor $\kappa_{ab} = \dot{g}_{ab}$, $\kappa^b_a = g^{bc}\kappa_{ac}$. Of all the components of the Ricci tensor, the only one that does not vanish identically is R_{00}, so that we have the equation

$$R_{00} = -\tfrac{1}{2}\dot{\kappa}^a_a - \tfrac{1}{4}\kappa^b_a\kappa^a_b = 0. \tag{109.7}$$

Thus, the three functions $g_{22}(\eta)$, $g_{23}(\eta)$, $g_{33}(\eta)$ must satisfy just one equation. Therefore two of them can be chosen arbitrarily. It is convenient to write (109.7) in another form writing the g_{ab} in the form

$$g_{ab} = -\chi^2\gamma_{ab}, \quad |\gamma_{ab}| = 1. \tag{109.8}$$

Then the determinant $-g = |g_{ab}| = \chi^4$, and substitution in (109.7) gives, after simple transformations,

$$\ddot{\chi} + \tfrac{1}{8}(\dot{\gamma}_{ac}\gamma^{bc})(\dot{\gamma}_{bd}\gamma^{ad})\chi = 0 \tag{109.9}$$

(γ^{ab} is the two-dimensional tensor reciprocal to γ_{ab}). If we assign arbitrary functions $\gamma_{ab}(\eta)$ (related to one another through the relation $|\gamma_{ab}| = 1$) these equations determine the function $\chi(\eta)$.

We thus arrive at a solution containing two arbitrary functions. It is easy to see that it is a generalization of the case considered in § 107 of a weak plane gravitational wave propagating in one direction.† The latter is obtained if we make the transformation

$$\eta = \frac{t+x}{\sqrt{2}}, \quad x^1 = \frac{t-x}{\sqrt{2}}$$

and set $\gamma_{ab} = \delta_{ab} + h_{ab}(\eta)$ (where the h_{ab} are small quantities, subject to the condition $h_{22} + h_{33} = 0$) and $\chi = 1$; a constant value of χ satisfies (109.9) if we neglect small second-order terms.

Suppose that a weak gravitational wave of finite extent (a "wave packet") is passing some point x. At the beginning of the passage we have $h_{ab} = 0$, $\chi = 1$; at the end of the passage we again have $h_{ab} = 0$, $\partial^2\chi/\partial t^2 = 0$, but the inclusion of second order terms in (109.9) leads to the appearance of a nonzero negative value of $\partial\chi/\partial t$:

$$\partial\chi/\partial t \approx -\frac{1}{8}\int\left(\frac{\partial h_{ab}}{\partial t}\right)^2 dt < 0$$

(the integral is taken over the time of passage of the wave). Thus, after the wave has passed, $\chi = 1 - \text{const}\cdot t$, and after a finite time interval, χ changes its sign. But vanishing of χ means vanishing of the metric determinant g, i.e. a singularity in the metric. This singularity, however, is not physical in character; it is related only to the unsatisfactory nature of the

† A solution of similar character in a large number of variables is given in I. Robinson and A. Trautman, *Phys. Rev. Lett.* **4**, 431 (1960); *Proc. Roy. Soc.* A **265**, 463 (1962).

reference frame, "spoiled" by the passing gravitational wave, and can be eliminated by a suitable transformation; after passage of the gravitational wave, the space–time does actually become flat again.

This can be shown directly. If we measure the variable η from its value corresponding to the singular point, $\chi = \eta$, so that

$$ds^2 = 2d\eta\, dx^1 - \eta^2[(dx^2)^2 + (dx^3)^2].$$

After the transformation

$$\eta x^2 = y, \quad \eta x^3 = z, \quad x^1 = \xi - \frac{y^2+z^2}{2\eta},$$

we get

$$ds^2 = 2d\eta\, d\xi - dy^2 - dz^2,$$

and the substitution $\eta = (t+x)/\sqrt{2}$, $\xi = (t-x)/\sqrt{2}$ finally brings the metric to galilean form.

This property of the gravitational wave—the creation of a fictitious singularity, is, of course, not related to the fact that the wave is weak; it also applies to the general solution of (109.7); just as in the example considered, near the singularity $\chi \sim \eta$, i.e. $-g \sim \eta^4$.†

PROBLEM

Find the condition for a metric of the form

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 + f(t-x, y, z)(dt-dx)^2$$

to be an exact solution of the Einstein equations for a field in vacuum (A. Peres, 1960).

Solution: The Ricci tensor is calculated most simply in the coordinates $u = (t-x)/\sqrt{2}$, $v = (t+x)/\sqrt{2}$, y, z, in which

$$ds^2 = -dy^2 - dz^2 + 2du\,dv + 2f(u, y, z)du^2.$$

Aside from $g_{22} = g_{33} = -1$, the only nonzero components of the metric tensor are $g_{uu} = 2f$, $g_{uv} = 1$; then $g^{vv} = -2f$, $g^{uv} = 1$, while the determinant $g = -1$. A direct calculation with (92.1) gives for the nonzero components of the curvature tensor:

$$R_{yuyu} = \frac{\partial^2 f}{\partial y^2}, \quad R_{zuzu} = -\frac{\partial^2 f}{\partial z^2}, \quad R_{yuzu} = -\frac{\partial^2 f}{\partial y \partial z}.$$

The only nonzero component of the Ricci tensor is $R_{uu} = \Delta f$, where Δ is the Laplacian in the coordinates y, z. Thus the Einstein equation is $\Delta f = 0$, i.e. the function $f(t-x, y, z)$ must be harmonic in the variables y, z.

If f is independent of y and z, or linear in them, there is no field—the space time is flat (the curvature tensor vanishes). The function $f(u, y, z) = yzf_1(u) + \frac{1}{2}(y^2-z^2)f_2(u)$, which is quadratic in y and z, corresponds to a plane wave propagating in the positive x direction; the curvature tensor in such a field depends only on t-x:

$$R_{yuzu} = -f_1(u), \quad R_{yuyu} = -R_{zuzu} = -f_2(u).$$

Corresponding to the two possible polarizations of the wave, the metric contains two arbitrary functions $f_1(u)$ and $f_2(u)$.

† This can be shown using (109.7) in precisely the same way as in § 97 for the analogous three-dimensional equation in the synchronous reference frame. Just as there, the appearance of a fictitious singularity is related to the crossing of coordinate curves.

v << c

§ 110. **Radiation of gravitational waves**

Let us consider next a weak gravitational field, produced by arbitrary bodies, moving with velocities small compared with the velocity of light.

Because of the presence of matter, the equations of the gravitational field will differ from the simple wave equation of the form $\Box h_i^k = 0$ (107.8) by having, on the right side of the equality, terms coming from the energy-momentum tensor of the matter. We write these equations in the form

$$\frac{1}{2}\Box\psi_i^k = \frac{8\pi k}{c^4}\tau_i^k, \tag{110.1}$$

where we have introduced in place of the h_i^k the more convenient quantities

$$\psi_i^k = h_i^k - \tfrac{1}{2}\delta_i^k h,$$

and where τ_i^k denotes the auxiliary quantities which are obtained upon going over from the exact equations of gravitation to the case of a weak field in the approximation we are considering. It is easy to verify that the components τ_0^0 and τ_α^0 are obtained directly from the corresponding components T_i^k by taking out from them the terms of the order of magnitude in which we are interested; as for the components τ_β^α, they contain along with terms obtained from the T_β^α, also terms of second order from $R_i^k - \frac{1}{2}\delta_i^k R$.†

The quantities ψ_i^k satisfy the condition (107.5) $\partial\psi_i^k/\partial x^k = 0$. From (110.1) it follows that this same equation holds for the τ_i^k:

$$\frac{\partial \tau_i^k}{\partial x^k} = 0. \tag{110.2}$$

This equation here replaces the general relation $T_{i;k}^k = 0$.

Using the equations which we have obtained, let us consider the problem of the energy radiated by moving bodies in the form of gravitational waves. The solution of this problem requires the determination of the gravitational field in the "wave zone", i.e. at distances large compared with the wavelength of the radiated waves. *

In principle, all the calculations are completely analogous to those which we carried out for electromagnetic waves. Equation (110.1) for a weak gravitational field coincides in form with the equation of the retarded potentials (§ 62). Therefore we can immediately write its general solution in the form

$$\psi_i^k = -\frac{4k}{c^4}\int (\tau_i^k)_{t-\frac{R}{c}}\frac{dV}{R}. \tag{110.3}$$

Since the velocities of all the bodies in the system are small, we can write, for the field at large distances from the system (see §§ 66 and 67),

$$\psi_i^k = -\frac{4k}{c^4 R_0}\int (\tau_i^k)_{t-\frac{R_0}{c}}\, dV, \tag{110.4}$$

† From eqs. (110.1) we can again obtain the formulas (106.1–2) that were used in § 106 for the weak constant field far from bodies. In the first approximation we neglect terms with second time derivatives (containing $1/c^2$), and of all the components of τ_i^k, only $\tau_0^0 = \mu c^2$ remains. The solution of the equations $\Delta\psi_\alpha^\beta = 0$, $\Delta\psi_0^\alpha = 0$, $\Delta\psi_0^0 = 16\pi k\,\mu/c^2$ that vanishes at infinity is $\psi_\alpha^\beta = 0$, $\psi_0^\alpha = 0$, $\psi_0^0 = 4\phi/c$, where ϕ is the Newtonian gravitational potential; cf. (99.2). One then finds for the tensor $h_i^k = \psi_i^k - \frac{1}{2}\psi\delta_i^k$ the values (106.1–2).

nothing about $r \ll \lambda_{GW}$? In fact on P. 348-349 (dimension) $r \gg \lambda$ and $r \ll L$ (distance) where R_0 is the distance from the origin, chosen anywhere in the interior of the system. From now on we shall, for brevity, omit the index $t-(R_0/c)$ in the integrand.

For the evaluation of these integrals we use equation (110.2). Dropping the index on the τ_i^k and separating space and time components, we write (110.2) in the form

$$\frac{\partial \tau_{\alpha\gamma}}{\partial x^\gamma} - \frac{\partial \tau_{\alpha 0}}{\partial x^0} = 0, \qquad \frac{\partial \tau_{0\gamma}}{\partial x^\gamma} - \frac{\partial \tau_{00}}{\partial x^0} = 0. \tag{110.5}$$

Multiplying the first equation by x^β, we integrate over all space,

$$\frac{\partial}{\partial x^0}\int \tau_{\alpha 0} x^\beta \, dV = \int \frac{\partial \tau_{\alpha\gamma}}{\partial x^\gamma} x^\beta \, dV = \int \frac{\partial(\tau_{\alpha\gamma} x^\beta)}{\partial x^\gamma} dV - \int \tau_{\alpha\beta} \, dV.$$

Since at infinity $\tau_{ik} = 0$, the first integral on the right, after transformation by Gauss' theorem, vanishes. Taking half the sum of the remaining equation and the same equation with transposed indices, we find

$$\int \tau_{\alpha\beta} \, dV = -\frac{1}{2}\frac{\partial}{\partial x^0}\int (\tau_{\alpha 0} x^\beta + \tau_{\beta 0} x^\alpha) \, dV.$$

Next, we multiply the second equation of (110.5) by $x^\alpha x^\beta$, and again integrate over all space. An analogous transformation leads to

$$\frac{\partial}{\partial x^0}\int \tau_{00} x^\alpha x^\beta \, dV = -\int (\tau_{\alpha 0} x^\beta + \tau_{\beta 0} x^\alpha) \, dV.$$

Comparing the two results, we find

$$\int \tau_{\alpha\beta} \, dV = \frac{1}{2}\frac{\partial^2}{\partial x_0^2}\int \tau_{00} x^\alpha x^\beta \, dV. \tag{110.6}$$

Thus the integrals of all the $\tau_{\alpha\beta}$ appear as expressions in terms of integrals containing only the component τ_{00}. But this component, as was shown earlier, is simply equal to the corresponding component T_{00} of the energy-momentum tensor and can be written to sufficient accuracy [see (99.1)] as:

$$\tau_{00} = \mu c^2. \tag{110.7}$$

Substituting this in (110.6) and introducing the time $t = x^0/c$, we find for (110.4)

$$\psi_{\alpha\beta} = -\frac{2k}{c^4 R_0}\frac{\partial^2}{\partial t^2}\int \mu x^\alpha x^\beta \, dV. \tag{110.8}$$

At large distances from the bodies, we can consider the waves as plane (over not too large regions of space). Therefore we can calculate the flux of energy radiated by the system, say along the direction of the x^1 axis, by using formula (107.12). In this formula there enter the components $h_{23} = \psi_{23}$ and $h_{22} - h_{33} = \psi_{22} - \psi_{33}$. From (110.8), we find for them the expressions†

$$h_{23} = -\frac{2k}{3c^4 R_0}\dddot{D}_{23}, \qquad h_{22} - h_{33} = -\frac{2k}{3c^4 R_0}(\dddot{D}_{22} - \dddot{D}_{33}) \tag{110.9}$$

† The tensor (110.8) does not satisfy the conditions under which formula (107.12) was derived. However, the transformation of reference frame that brings the h_{ik} to the required gauge does not affect the values of the components of (110.9) that are used here.

(the dot denotes time differentiation), where we have introduced the mass quadrupole tensor (99.8):

$$D_{\alpha\beta} = \int \mu(3x^{\alpha}x^{\beta} - \delta_{\alpha\beta}r^2)\, dV. \tag{110.10}$$

As a result, we obtain the energy flux along the x^1 axis in the form

$$ct^{10} = \frac{k}{36\pi c^5 R_0^2}\left[\left(\frac{\dddot{D}_{22} - \dddot{D}_{33}}{2}\right)^2 + \dddot{D}_{23}^2\right]. \tag{110.11}$$

The flux of energy into an element of solid angle in the given direction is then obtained by multiplying by $R_0^2\, do$.

The two terms in this expression correspond to the radiation of waves of two independent polarizations. To write them in invariant form (independent of the choice of the direction of radiation) we introduce the three-dimensional unit polarization tensor $e_{\alpha\beta}$ of the plane gravitational wave, which determines the nonzero components of $h_{\alpha\beta}$ (in the gauge for the h_{ik} in which $h_{0\alpha} = h_{00} = h = 0$). The polarization tensor is symmetric and satisfies the conditions

$$e_{\alpha\alpha} = 0, \quad e_{\alpha\beta}n_{\beta} = 0, \quad e_{\alpha\beta}e_{\alpha\beta} = 1, \tag{110.12}$$

where **n** is a unit vector in the direction of propagation of the wave.

Using this tensor we can write the intensity of radiation of a given polarization into solid angle do in the form

$$dI = \frac{k}{72\pi c^5}(\dddot{D}_{\alpha\beta}e_{\alpha\beta})^2\, do. \tag{110.13}$$

This expression depends implicitly on the direction of **n** through the transversality condition $e_{\alpha\beta}n_{\beta} = 0$. The total angular distribution for all polarizations is gotten by summing (110.13) over polarizations, or, what is equivalent, averaging over polarization and multiplying by 2 (the number of independent polarizations). The averaging is done using the formula

$$\begin{aligned}\overline{e_{\alpha\beta}e_{\gamma\delta}} = \tfrac{1}{4}\{n_{\alpha}n_{\beta}n_{\gamma}n_{\delta} + (n_{\alpha}n_{\beta}\delta_{\gamma\delta} + n_{\gamma}n_{\delta}\delta_{\alpha\beta}) - \\ -(n_{\alpha}n_{\gamma}\delta_{\beta\delta} + n_{\beta}n_{\gamma}\delta_{\alpha\delta} + n_{\alpha}n_{\delta}\delta_{\beta\gamma} + n_{\beta}n_{\delta}\delta_{\alpha\gamma}) - \\ - \delta_{\alpha\beta}\delta_{\gamma\delta} + (\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\beta\gamma}\delta_{\alpha\delta})\end{aligned} \tag{110.14}$$

(the expression on the right is a tensor formed from the unit tensor and the components of the vector **n**; it has the required symmetry in its indices, it gives unity on contraction on pairs of indices α, γ and β, δ, and vanishes after scalar multiplication with **n**).

The result is

$$dI = \frac{k}{36\pi c^5}\left[\frac{1}{4}(\dddot{D}_{\alpha\beta}n_{\alpha}n_{\beta})^2 + \frac{1}{2}\dddot{D}_{\alpha\beta}^2 - \dddot{D}_{\alpha\beta}\dddot{D}_{\alpha\gamma}n_{\beta}n_{\gamma}\right] do. \tag{110.15}$$

The total radiation in all directions, i.e., the energy loss of the system per unit time $(-d\mathscr{E}/dt)$, can be found by averaging dI/do over all directions and multiplying the result by 4π. The averaging is easily performed using the formulas given in the footnote on p. 189, and gives

$$-\frac{d\mathscr{E}}{dt} = \frac{k}{45c^5}\dddot{D}_{\alpha\beta}^2. \tag{110.16}$$

We note that the radiation of gravitational waves is a fifth order effect in $1/c$. This fact, together with the smallness of the gravitational constant k, makes the usual effects extremely small.

PROBLEMS

1. Two bodies, attracting each other according to Newton's law, move in circular orbits (around their common center of inertia). Determine the average (over a rotation period) of the intensity of radiation of gravitational waves and its distribution in polarization and direction.

Solution: Choosing the coordinate origin at the center of inertia, we have for the radius vectors of the two bodies:

$$\mathbf{r}_1 = \frac{m_2}{m_1+m_2}\mathbf{r}, \quad \mathbf{r}_2 = -\frac{m_1}{m_1+m_2}\mathbf{r}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2.$$

The components of the tensor $D_{\alpha\beta}$ are (if the xy plane coincides with the plane of motion):

$$D_{xx} = \mu r^2(3\cos^2\psi - 1), \quad D_{yy} = \mu r^2(3\sin^2\psi - 1),$$

$$D_{xy} = 3\mu r^2 \cos\psi \sin\psi, \quad D_{zz} = -\mu r^2,$$

where $\mu = m_1 m_2/(m_1+m_2)$, ψ is the polar angle of the vector $\mathbf{r}$ in the xy plane. For circular motion r = const, and $\dot{\psi} = r^{-\frac{3}{2}}\sqrt{k(m_1+m_2)} \equiv \omega$.

We assign the direction of $\mathbf{n}$ by the polar angle θ and azimuth ϕ, with the polar axis z perpendicular to the plane of the motion. Let us consider the two polarizations for which: (1) $e_{\theta\phi} = 1/\sqrt{2}$; (2) $e_{\theta\theta} = -e_{\phi\phi} = 1/\sqrt{2}$. Projecting the tensor $D_{\alpha\beta}$ on the directions of the spherical unit vectors $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$, calculating with formula (110.13) and averaging over the time, we find the result for these two cases and for the sum $I = I_1 + I_2$:

$$\overline{\frac{dI_1}{do}} = \frac{k\mu^2\omega^6 r^4}{2\pi c^5} 4\cos^2\theta, \quad \overline{\frac{dI_2}{do}} = \frac{k\mu^2\omega^6 r^4}{2\pi c^5}(1+\cos^2\theta)^2,$$

$$\overline{\frac{dI}{do}} = \frac{k\mu^2\omega^6 r^4}{2\pi c^5}(1 + 6\cos^2\theta + \cos^4\theta),$$

and after integrating over all directions:

$$-\frac{d\mathscr{E}}{dt} = I = \frac{32k\mu^2\omega^6 r^4}{5c^5} = \frac{32k^4 m_1^2 m_2^2(m_1+m_2)}{5c^5 r^5}, \quad \frac{\bar{I}_1}{\bar{I}_2} = \frac{5}{7}$$

[for calculating the total intensity I alone, we should, of course, have used (110.16)].

The loss of energy from the radiating system leads to a gradual (secular) approach of the two bodies. Since $\mathscr{E} = -km_1m_2/2r$, the velocity of approach is

$$\dot{r} = \frac{2r^2}{km_1m_2}\frac{d\mathscr{E}}{dt} = -\frac{64k^3 m_1 m_2(m_1+m_2)}{5c^5 r^3}.$$

2. Find the average (over a rotation period) of the energy radiated in the form of gravitational waves by a system of two bodies moving in elliptical orbits (P. C. Peters and J. Mathews).†

Solution: In contrast to the case of circular motion, the distance r and the angular velocity vary along the orbit according to the laws

$$\frac{a(1-e^2)}{r} = 1 + e\cos\psi, \quad \frac{d\psi}{dt} = \frac{1}{r^2}[k(m_1+m_2)a(1-e^2)]^{\frac{1}{2}},$$

where e is the eccentricity and a is the semimajor axis of the orbit (*cf. Mechanics*, § 15). A quite lengthy calculation using (110.16) gives:

$$-\frac{d\mathscr{E}}{dt} = \frac{8k^4 m_1^2 m_2^2(m_1+m_2)}{15a^5c^5(1-e^2)^5}(1+e\cos\psi)^4[12(1+e\cos\psi)^2 + e^2\sin^2\psi].$$

In averaging over the period of rotation, the integration over t is replaced by integration over ψ,

† For the angular, polarization, and spectral distributions of this radiation, cf. *Phys. Rev.* **131**, 435 (1963).

and gives the result:

$$-\frac{\overline{d\mathcal{E}}}{dt} = \frac{32k^4 m_1^2 m_2^2 (m_1+m_2)}{5c^5 a^5} \frac{1}{(1-e^2)^{\frac{7}{2}}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right).$$

We note the rapid increase in intensity of radiation with increasing eccentricity of the orbit.

3. Determine the time-averaged rate of loss of angular momentum from a system of bodies in stationary motion and emitting gravitational waves.

Solution: For convenience of writing formulas, we temporarily regard the body as consisting of discrete particles. We represent the average rate of loss of energy of the system as the work of the "frictional forces" **f** acting on the particles:

$$\frac{\overline{d\mathcal{E}}}{dt} = \Sigma\,\overline{\mathbf{f}\cdot\mathbf{v}} \tag{1}$$

(we omit the index labeling the particles). Then the average rate of loss of angular momentum is given by

$$\frac{\overline{dM_\alpha}}{dt} = \Sigma\overline{(\mathbf{r}\times\mathbf{f})_\alpha} = \Sigma e_{\alpha\beta\gamma}\overline{x_\beta f_\gamma} \tag{2}$$

(cf. the derivation of formula (75.7)). To determine **f**, we write

$$\frac{\overline{d\mathcal{E}}}{dt} = -\frac{k}{45c^5}\overline{\dddot{D}_{\alpha\beta}\dddot{D}_{\alpha\beta}} = -\frac{k}{45c^5}\overline{\dot{D}_{\alpha\beta}D^{(V)}_{\alpha\beta}}$$

(where we have used the fact that the average values of total time derivatives vanish). Substituting $\dot{D}_{\alpha\beta} = \Sigma m(3x_\alpha v_\beta + 3x_\beta v_\alpha - 2\mathbf{r}\cdot\mathbf{v}\delta_{\alpha\beta})$ and comparing with (1), we find:

$$f_\alpha = -\frac{2k}{15c^5}D^{(V)}_{\alpha\beta}mx_\beta.$$

Substitution of this expression in (2) gives the result:

$$\frac{\overline{dM_\alpha}}{dt} = \frac{2k}{45c^5}e_{\alpha\beta\gamma}D^{(V)}_{\beta\delta}D_{\gamma\delta} = \frac{2k}{45c^5}e_{\alpha\beta\gamma}\overline{\dddot{D}_{\beta\delta}\ddot{D}_{\delta\gamma}}. \tag{3}$$

4. For a system of two bodies moving in elliptical orbits, find the average loss of angular momentum per unit time.

Solution: A calculation with formula (3) of the preceding problem, analogous to that done in problem 2, gives the result:

$$-\frac{\overline{dM_z}}{dt} = \frac{32k^{\frac{7}{2}} m_1^2 m_2^2 \sqrt{m_1+m_2}}{5c^5 a^{\frac{7}{2}}} \frac{1}{(1-e^2)^2}\left(1+\frac{7}{4}e^2\right).$$

For circular motion ($e = 0$) the values of $\dot{\mathcal{E}}$ and $\dot{M}$ are, as they should be, related by $\dot{\mathcal{E}} = \dot{M}\omega$.