Exercise: Calculation of Dirac determinant $\det(\gamma_\mu p_\mu - m^*)$, p_0 = $i(\omega_n + i\mu)$

Exercise 2: Show that $2\sum_{n=-\infty}^{+\infty} \ln \beta^2[\omega^2 + (\omega_n + i\mu)^2] = \sum_{n=-\infty}^{+\infty} \left\{\ln \beta^2[\omega_n^2 + (\omega - \mu)^2] + \ln \beta^2[\omega_n^2 + (\omega + \mu)^2]\right\}$

Partition Function for Fermionic Fields

Since both positive and negative frequencies have to be summed over, the latter expression can be put in a form analogous to the above expression in the bosonic case,

$$\ln Z = \sum_n \sum_{\vec{p}} \left\{ \ln\left[\beta^2\left(\omega_n^2 + (\omega-\mu)^2\right)\right] + \ln\left[\beta^2\left(\omega_n^2 + (\omega+\mu)^2\right)\right]\right\} \ .$$

In the further evaluation we can go similar steps as in the bosonic case, with two exceptions: (1) the presence of a chemical potential, splitting the contributions of particles and antiparticles; (2) the Matsubara frequencies are now odd multiples of πT, so that the infinite sum to be exploited reads

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^2\pi^2 + \Theta^2} = \frac{1}{\Theta}\left(\frac{1}{2} - \frac{1}{e^{\Theta}+1}\right) \ .$$

Integrating over the auxiliary variable Θ, and dropping terms independent of β and μ, we finally obtain

$$\ln Z = 2V \int \frac{d^3p}{(2\pi)^3} \left[\beta\omega + \ln(1 + e^{-\beta(\omega-\mu)}) + \ln(1 + e^{-\beta(\omega+\mu)})\right] \ .$$

Notice that the factor 2 corresponding to the spin-$\frac{1}{2}$ nature of the fermions comes out automatically. Separate contributions from particles (μ) and antiparticles (-μ) are evident. Finally, the zero-point energy of the vacuum also appears in this formula.

Interactions: Hubbard-Stratonovich Trick

A general class of interactions for which the Hubbard-Stratonovich (HS) transformation is immediately applicable, are four-fermion couplings of the current-current type

$$\mathcal{L}_{int} = G(\bar{\psi}\psi)^2 \; . \tag{3}$$

A Fermi gas with this typ of interaction serves as a model for electronic superconductivity (Bardeen-Cooper-Schrieffer (BCS) model, 1957) or for chiral symmetry breaking in quark matter (Nambu–Jona-Lasinio (NJL) model, 1961).

The HS-transformation for (3) reads

$$\exp\left[G(\bar{\psi}\psi)^2\right] = \mathcal{N}\int \mathcal{D}\sigma \exp\left[\frac{\sigma^2}{4G} + \bar{\psi}\psi\sigma\right]$$

and allows to bring the functional integral over fermionic fields into a quadratic (Gaussian) form so that fermions can be integrated out.

This is also called *Bosonization* procedure.

Bose-Einstein Condensation: Charged Scalar Field

Consider a complex scalar field (two real components ϕ_1, ϕ_2):
$\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$, $\Phi^* = (\phi_1 - i\phi_2)/\sqrt{2}$

$$\mathcal{L} = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi - \lambda (\Phi^* \Phi)^2 ,$$

with U(1) symmetry: $\Phi \longrightarrow \Phi e^{-i\alpha}$, where α is a real constant.
Noether theorem: continuous symmetry $\rightarrow$ conserved current

$$\begin{aligned} \mathcal{L} \rightarrow \mathcal{L}' &= \partial_\mu (\Phi^* e^{i\alpha(x)})(\partial^\mu \Phi e^{-i\alpha(x)}) - m^2 \Phi^* \Phi - \lambda (\Phi^* \Phi)^2 , \\ &= \mathcal{L} + \Phi^* \Phi \partial_\mu \alpha \partial^\mu \alpha + i \partial_\mu \alpha (\Phi^* \partial^\mu \Phi - \Phi \partial^\mu \Phi^*) \end{aligned}$$

Equation of motion for the "field" $\alpha(x)$

$$\partial^\mu \frac{\partial \mathcal{L}'}{\partial(\partial^\mu \alpha)} = \frac{\partial \mathcal{L}'}{\partial \alpha}$$

Since $\partial \mathcal{L}' / \partial \alpha = 0$ follows a conserved "current": $\partial \mathcal{L}' / \partial(\partial^\mu \alpha) = \Phi^* \Phi \partial_\mu \alpha - i \Phi \partial_\mu \Phi^* + i \Phi^* \partial_\mu \Phi$.
Recover original field theory by setting $\alpha =$constant. Then

$$j_\mu = i(\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*) , \quad \partial^\mu j_\mu = 0$$

Full current: $J_\mu = \int d^3x j_\mu(x)$; conserved charge: $Q = \int d^3x j_0(x)$

Bose-Einstein Condensation: Charged Scalar Field (2)

Decompose the complex $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$ into real and imaginary parts: ϕ_1, ϕ_2.
Conjugate momenta: $\pi_1 = \partial\phi_1/\partial t$, $\pi_2 = \partial\phi_2/\partial t$
Hamiltonian density and charge:

$$\mathcal{H} = \frac{1}{2}\left[\pi_1^2 + \pi_2^2 + (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m^2\phi_1^2 + m^2\phi_2^2\right] + \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2 \,, \quad Q = \int d^3x(\phi_2\pi_1 - \phi_1\pi_2)$$

The partition function is

$$Z = \int [d\pi_1][d\pi_2] \int_{\text{periodic}} [d\phi_1][d\phi_2] \exp\left[\int^\beta d^4x \left(i\pi_1\frac{\partial\phi_1}{\partial\tau} + i\pi_2\frac{\partial\phi_2}{\partial\tau} - \mathcal{H}(\pi_1,\pi_2,\phi_1,\phi_2) + \mu(\phi_2\pi_1 - \phi_1\pi_2)\right)\right]$$

Integration over conjugate field momenta can be done with the result:

$$\begin{aligned} Z \;=\; (N')^2 \int_{\text{periodic}} [d\phi_1][d\phi_2] \exp\Big\{ \int^\beta d^4x\Big[-\frac{1}{2}\left(\frac{\partial\phi_1}{\partial\tau} - i\mu\phi_2\right)^2 - \frac{1}{2}\left(\frac{\partial\phi_2}{\partial\tau} - i\mu\phi_1\right)^2 \\ -\frac{1}{2}(\nabla\phi_1)^2 - \frac{1}{2}(\nabla\phi_2)^2 - \frac{1}{2}m^2\phi_1^2 - \frac{1}{2}m^2\phi_2^2 - \frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2\Big]\Big\} \,. \end{aligned}$$

Differs from naïve expectation $\mathcal{L}(\phi_1, \phi_2, \partial_\mu\phi_1, \partial_\mu\phi_2; \mu = 0) + \mu j_0(\phi_1, \phi_2, \partial\phi_1/\partial\tau, \partial\phi_2/\partial\tau)$ by $\mu^2\Phi^*\Phi$

Bose-Einstein Condensation: Charged Scalar Field (3)

In the following: ideal gas ($\lambda = 0$). For $\lambda \neq 0$, perform HS-transformation! (Exercise)
Expand components of $\Phi = (\phi_1 + i\phi_2)/\sqrt{2}$ in Fourier series:

$$\phi_1(\vec{x},\tau) = \sqrt{2}\zeta\cos\theta + \left(\frac{\beta}{V}\right)^{1/2} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} \mathrm{e}^{i(\vec{p}\vec{x}+\omega_n\tau)} \phi_{1;n}(\vec{p}) \,,$$

$$\phi_2(\vec{x},\tau) = \sqrt{2}\zeta\sin\theta + \left(\frac{\beta}{V}\right)^{1/2} \sum_{n=-\infty}^{\infty} \sum_{\vec{p}} \mathrm{e}^{i(\vec{p}\vec{x}+\omega_n\tau)} \phi_{2;n}(\vec{p}) \,.$$

Infrared character of Φ carried by ζ and θ, independent of $(\vec{x},\tau)$, so $\phi_{1;0}(\vec{0}) = \phi_{2;0}(\vec{0}) = 0$.
Possibility of condensation of bosons into the zero-momentum state: finite fraction of particles in $n = 0, \vec{p} = \vec{0}$ state.

$$Z = (N')^2 \left[\Pi_n \Pi_{\vec{p}} \int i d\phi_{1;n}(\vec{p}) d\phi_{2;n}(\vec{p})\right] \mathrm{e}^{S} \,,$$

$$S = \beta V(\mu^2 - m^2)\zeta^2 - \frac{1}{2}\sum_n \sum_{\vec{p}} (\phi_{1;-n}(-\vec{p}), \phi_{2;-n}(-\vec{p})) D \begin{pmatrix} \phi_{1;n}(\vec{p}) \\ \phi_{2;n}(\vec{p}) \end{pmatrix} \,,$$

$$D = \beta^2 \begin{pmatrix} \omega_n^2 + \omega^2 - \mu^2 & -2\mu\omega_n \\ 2\mu\omega_n & \omega_n^2 + \omega^2 - \mu^2 \end{pmatrix} .$$

Bose-Einstein Condensation: Charged Scalar Field (4)

Carrying out integrations yields:

$$\ln Z = \beta V(\mu^2 - m^2)\zeta^2 + \ln(\det D)^{-1}$$

Second term can be handled as:

$$\begin{aligned} \ln\det D &= \ln\left\{\Pi_n \Pi_{\vec{p}} \beta^4 \left[(\omega_n^2 + \omega^2 - \mu^2)^2 + 4\mu^2\omega_n^2\right]\right\} \\ &= \ln\left\{\Pi_n \Pi_{\vec{p}} \beta^2 \left[\omega_n^2 + (\omega - \mu)^2\right]\right\} + \ln\left\{\Pi_n \Pi_{\vec{p}} \beta^2 \left[\omega_n^2 + (\omega - \mu)^2\right]\right\} \end{aligned}$$

Putting all together and evaluating the Matsubara sums, we obtain

$$\ln Z = \beta V(\mu^2 - m^2)\zeta^2 - V\int \frac{d^3p}{(2\pi)^3}[\beta\omega + \ln(1 - \mathrm{e}^{-\beta(\omega-\mu)}) + \ln(1 - \mathrm{e}^{-\beta(\omega+\mu)})]$$

Result independent of θ because of $U(1)$ symmetry. Parameter ζ from variation of $\ln Z$.

$$\frac{\partial \ln Z}{\partial \zeta} = 2\beta V(\mu^2 - m^2)\zeta = 0$$

implies that $\zeta = 0$ unless $|\mu| = m$, when ζ is determined from $\rho = Q/V$

$$\rho = \frac{T}{V}\left(\frac{\partial \ln Z}{\partial \mu}\right)_{\mu=m} = 2m\zeta^2 + \rho^*(\beta, \mu = m)\ ; \quad \rho^* = \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{\mathrm{e}^{\beta(\omega-\mu)} - 1} - \frac{1}{\mathrm{e}^{\beta(\omega+\mu)} - 1}\right)$$

Critical temperature T_c for condensation from $\rho = \rho^*(\beta_c, \mu = m)$.

Bose-Einstein Condensation: Exercises

1. Find the critical temperature T_c for condensation from $\rho = \rho^*(\beta_c, \mu = m)$.
   Show that the nonrelativistic (NR, $\rho \ll m^3$) and ultrarelativistic (UR, $\rho \gg m^3$) limits are given by

$$T_{c,\mathrm{NR}} = \frac{2\pi}{m}\left(\frac{\rho}{\zeta(3/2)}\right)^{2/3} \;\; ; \;\; T_{c,\mathrm{UR}} = \left(\frac{3\rho}{m}\right)^{1/2}$$

2. For $T < T_c$, the value of ζ is the order parameter of the 2^{nd} order condensation phase transition

$$\zeta^2 = \left[\rho - \rho^*(\beta, \mu = m)\right]/(2m) \;\; , \;\; T < T_c \, .$$

   At the critical temperature, the expansion $\zeta \sim t^\nu$ for small $t = T - T_c$ determines a critical exponent ν. Find the value of ν in both cases.

3. Consider the interacting case $\lambda \neq 0$. Perform the Hubbard-Stratonovich transformation by introducing a collective scalar field σ. Discuss the effect of $\lambda \neq 0$ on the thermodynamic potential in the mean-field approximation for σ, i.e. by neglecting the path integral over the σ-fields and determining σ in the thermodynamical equilibrium from a gap equation $\partial \ln Z/\partial\sigma = 0$.

Exercise: Calculation of Dirac determinant $\det(\gamma_\mu p_\mu - m^*)$, p_0 = $i(\omega_n + i\mu)$

Solution: 1. Use explicit form of gamma matrices (and Pauli matrices)

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2. Write down the determinant

$$\|\gamma_\mu p_\mu - m^*\| = \begin{vmatrix} (p_0 - m^*) & 0 & p_3 & (p_1 - ip_2) \\ 0 & (p_0 - m^*) & (p_1 + ip_2) & -p_3 \\ -p_3 & (-p_1 + ip_2) & (-p_0 - m^*) & 0 \\ (-p_1 - ip_2) & p_3 & 0 & (-p_0 - m^*) \end{vmatrix}$$

3. Determine the subdeterminants

$$\begin{aligned} D_{11} &= -(p_0 + m^*) \left(\vec{p}^2 + m^2 - p_0^2\right) \\ D_{13} &= p_3 \left(\vec{p}^2 + m^2 - p_0^2\right) \\ D_{14} &= -(p_1 + ip_2) \left(\vec{p}^2 + m^{*2} - p_0^2\right) \end{aligned}$$

4. Calculate the determinant according to standard rules

$$\begin{aligned} \|\gamma_\mu p_\mu - m^*\| &= (p_0 - m^*) D_{11} + p_3 D_{13} - (p_1 - ip_2) D_{14} \\ &= \left(-p_0^2 + p_1^2 + p_2^2 + p_3^2 + m^{*2}\right) \left(\vec{p}^2 + m^{*2} - p_0^2\right) \\ &= \left(\vec{p}^2 + m^{*2} - p_0^2\right)^2 \end{aligned}$$

5. Result:

$$= \left[\omega^2 + (\omega_n + i\mu)^2\right]^2, \qquad \omega^2 = \vec{p}^2 + m^{*2}$$

Exercise 2: Show that $2\sum_{n=-\infty}^{+\infty}\ln\beta^2[\omega^2+(\omega_n+i\mu)^2] = \sum_{n=-\infty}^{+\infty}\left\{\ln\beta^2[\omega_n^2+(\omega-\mu)^2]+\ln\beta^2[\omega_n^2+(\omega+\mu)^2]\right\}$

Solution: 1. Consider an analytic function $F(z_n)$ where $z_n = (i\omega_n - \mu)$, with $\omega_n = (2n+1)\pi T$

$$\sum_{n=-\infty}^{+\infty} F\left((\omega_n+i\mu)^2\right) = \sum_{n=0}^{+\infty} F\left((\omega_n+i\mu)^2\right) + \sum_{n=-\infty}^{-1} F\left((\omega_n+i\mu)^2\right)$$

$$\sum_{n=-\infty}^{-1} F\left((\omega_n+i\mu)^2\right) = \sum_{n=1}^{\infty} F\left((\omega_{-n}+i\mu)^2\right) = \sum_{n=0}^{\infty} F\left((\omega_{-n-1}+i\mu)^2\right) = \sum_{n=0}^{\infty} F\left((-\omega_{-n-1}-i\mu)^2\right)$$

2. For the fermionic Matsubara frequencies holds $-\omega_{-n-1} = -\pi T(2(-n-1)+1) = \pi T(2n+1) = \omega_n$

$$\sum_{n=-\infty}^{+\infty} F\left((\omega_n+i\mu)^2\right) = \sum_{n=0}^{+\infty} F\left((\omega_n+i\mu)^2\right) + \sum_{n=0}^{+\infty} F\left((\omega_n-i\mu)^2\right) = \sum_{n=0}^{+\infty} F\left((\omega_n+i\mu)^2\right) + \sum_{n=0}^{+\infty} F^*\left((\omega_n+i\mu)^2\right)$$

$$= 2\sum_{n=0}^{+\infty} Re\ F\left((\omega_n+i\mu)^2\right)$$

3. Using this relationship based on the symmetry of the Matsubara frequencies, transform:

$$\begin{aligned}
2\sum_{n=-\infty}^{+\infty}\ln\beta^2[\omega^2+(\omega_n+i\mu)^2] &= 4\sum_{n=0}^{+\infty} Re\ \ln\beta^2[(\omega^2+\omega_n^2-\mu^2)+i(2\omega_n\mu)] \\
&= 2\sum_{n=0}^{+\infty}\ln\beta^2[(\omega^2+\omega_n^2-\mu^2)^2+(2\omega_n\mu)^2] \\
&= 2\sum_{n=0}^{+\infty}\left\{\ln\beta^2[\omega_n^2+(\omega-\mu)^2]+\ln\beta^2[\omega_n^2+(\omega+\mu)^2]\right\} \\
&= \sum_{n=-\infty}^{+\infty}\left\{\ln\beta^2[\omega_n^2+(\omega-\mu)^2]+\ln\beta^2[\omega_n^2+(\omega+\mu)^2]\right\}
\end{aligned}$$