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APPLIED MECHANICS

Some Problems of Hemitropic Micropolar Continuum

by

J. Paweł NOWACKI and Witold NOWACKI

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Summary. Static and dynamic problems for a linear thermoelastic hemitropic Cosserat's continuum are discussed. In the static case, particular solutions of the governing equations and generalized Maysel's formulae are obtained. Also, mean strain and mean stress values are given. The dynamic equations of the theory are reduced to that involving simple wave operators or integro-differential operators.

1. Introduction. The linear theory of asymmetric elasticity for a solid without central symmetry (hemitropic) was developed by Aero and Kuvshinskii in [1] and [2]. Next, it was generalized to coupled thermoelasticity by Nowacki in [3], and particular thermoelastic problems were solved by Lenz in [4] and [5]. In the present article we discuss a number of general static and dynamic problems for the theory given in [1–3] assuming that the body is isotropic and homogeneous. Therefore, we assume that external load and heat supply fields produce the thermoelastic deformation described by an ordered array $(\mathbf{u}, \boldsymbol{\varphi}, \theta)$ in which $\mathbf{u}$ and $\boldsymbol{\varphi}$ denote the displacement vector and the rotation vector, respectively, and θ stands for the temperature. The state of strain is represented by the two asymmetric tensors: the strain tensor γ_{ji} and the curvature-twist tensor κ_{ji} related to $\mathbf{u}$ and $\boldsymbol{\varphi}$ by

$$\gamma_{ji} = u_{i,j} - \varepsilon_{kji}\, \varphi_k, \qquad \kappa_{ji} = \varphi_{i,j}. \tag{1.1}$$

The state of stress is described by the two asymmetric tensors: the force stress tensor σ_{ji} and the couple stress tensor μ_{ji}. The state of stress and the entropy function S are related to the state of strain and the temperature θ through the constitutive equations (see [3])

$$\sigma_{ji} = (\mu+\alpha)\,\gamma_{ji} + (\mu-\alpha)\,\gamma_{ij} + (\lambda\gamma_{kk} - \eta\theta)\,\delta_{ij} + (\chi+\nu)\,\kappa_{ji} + (\chi-\nu)\,\kappa_{ij} + \kappa\kappa_{kk}\,\delta_{ij}, \tag{1.2}$$

$$\mu_{ji} = (\gamma+\varepsilon)\,\kappa_{ji} + (\gamma-\varepsilon)\,\kappa_{ij} + (\beta\kappa_{kk} - \zeta\theta)\,\delta_{ij} + (\chi+\nu)\,\gamma_{ji} + (\chi-\nu)\,\gamma_{ij} + \kappa\gamma_{kk}\,\delta_{ij}, \tag{1.3}$$

$$S = \eta\gamma_{kk} + \zeta\kappa_{kk} + \frac{c_\varepsilon}{T_0}\,\theta, \qquad \theta = T - T_0. \tag{1.4}$$

Here μ, λ, α, β, γ, ε, ν, χ, κ stand for material constants describing elastic properties of the solid, while η and ζ represent both elastic and thermal properties.

Moreover, c_ε is the specific heat, T denotes the absolute temperature and T_0 is the temperature of a natural state.

If we substitute σ_{ji} and μ_{ji} from (1.2) and (1.3) into the equations of motion

$$\sigma_{ji,j}+X_i=\rho\ddot{u}_i, \quad \varepsilon_{ijk}\,\sigma_{jk}+\mu_{ji,j}+Y_i=J\ddot{\varphi}_i \tag{1.5}$$

and S from (1.4) into the entropy balance law

$$T\dot{S}=-\operatorname{div}\mathbf{q}+W, \tag{1.6}$$

where $\mathbf{q}$ is given by the Fourier law

$$\mathbf{q}=-k\operatorname{grad}\theta \tag{1.7}$$

and if we take into account the definitions of the strain tensors (1.1) we arrive at a complete set of equations of the coupled thermoelastic hemitropic micropolar theory

$$\Box_2\,\mathbf{u}+(\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div}\mathbf{u}+2\alpha\operatorname{rot}\boldsymbol{\varphi}+ \\ +(\chi+\nu)\,\nabla^2\boldsymbol{\varphi}+(\chi+\kappa-\nu)\operatorname{grad}\operatorname{div}\boldsymbol{\varphi}+\mathbf{X}=\eta\operatorname{grad}\theta, \tag{1.8}$$

$$\Box_4\,\boldsymbol{\varphi}+(\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\boldsymbol{\varphi}+2\alpha\operatorname{rot}\mathbf{u}+4\nu\operatorname{rot}\boldsymbol{\varphi}+ \\ +(\chi+\nu)\,\nabla^2\mathbf{u}+(\chi+\kappa-\nu)\operatorname{grad}\operatorname{div}\mathbf{u}+\mathbf{Y}=\zeta\operatorname{grad}\theta, \tag{1.9}$$

$$D\theta-\eta\operatorname{div}\dot{\mathbf{u}}-\zeta\operatorname{div}\dot{\boldsymbol{\varphi}}=-W_0, \tag{1.10}$$

where the following notations are used

$$\Box_2=(\mu+\alpha)\,\nabla^2-\rho\partial_t^2, \qquad \Box_4=(\gamma+\varepsilon)\,\nabla^2-4\alpha-J\partial_t^2,$$

$$D=\frac{1}{T_0}(k\nabla^2-c_\varepsilon\,\partial_t), \qquad W_0=\frac{W}{T_0}.$$

Here W is the quantity of heat generated in a unit volume and unit time, k is the coefficient of heat conduction and $\mathbf{q}$ denotes the flux of heat vector. Moreover, $\mathbf{X}$ and $\mathbf{Y}$ denote the body force vector and the body couple vector, respectively. For $\nu=\chi=\kappa=\zeta=0$ the system of Eqs. (1.8)–(1.10) reduces to that of the isotropic micropolar thermoelasticity with central symmetry (see [6]).

2. Steady state problems of thermoelasticity. As a starting point we assume the equilibrium equations in terms of the displacements and rotations. Equating the time derivatives in Eqs. (1.8)–(1.9) to zero and letting $\mathbf{X}=\mathbf{Y}=\mathbf{0}$, we obtain

$$(\mu+\alpha)\,\nabla^2\mathbf{u}+(\lambda+\mu-\alpha)\operatorname{grad}\operatorname{div}\mathbf{u}+2\alpha\operatorname{rot}\boldsymbol{\varphi}+ \\ +(\chi+\nu)\,\nabla^2\boldsymbol{\varphi}+(\kappa+\chi-\nu)\operatorname{grad}\operatorname{div}\boldsymbol{\varphi}=\eta\operatorname{grad}\theta, \tag{2.1}$$

$$[(\gamma+\varepsilon)\,\nabla^2-4\alpha]\,\boldsymbol{\varphi}+(\beta+\gamma-\varepsilon)\operatorname{grad}\operatorname{div}\boldsymbol{\varphi}+2\alpha\operatorname{rot}\mathbf{u}+ \\ +(\chi+\nu)\,\nabla^2\mathbf{u}+(\kappa+\chi-\nu)\operatorname{grad}\operatorname{div}\mathbf{u}+4\nu\operatorname{rot}\boldsymbol{\varphi}=\zeta\operatorname{grad}\theta. \tag{2.2}$$

The temperature θ in (2.1) and (2.2) is to be treated as a known function that satisfies the heat conduction equation

$$k\nabla^2\theta=-W \tag{2.3}$$

subject to given boundary condition.

We look for a solution of Eqs. (2.1–(2.2) in the form

$$\mathbf{u}=\mathbf{u}'+\mathbf{u}'', \quad \boldsymbol{\varphi}=\boldsymbol{\varphi}'+\boldsymbol{\varphi}'', \tag{2.4}$$

where $(\mathbf{u}', \boldsymbol{\varphi}')$ given by

$$\mathbf{u}'=\operatorname{grad}\Phi, \quad \boldsymbol{\varphi}'=\operatorname{grad}\Gamma \tag{2.5}$$

is a particular solution of (2.1)—(2.2), while $(\mathbf{u}'', \boldsymbol{\varphi}'')$ stands for a general solution of these equations with $\theta=0$. Substituting (2.5) into (2.1)-(2.2) we conclude that $(\mathbf{u}', \boldsymbol{\varphi}')$ in a particular solution if Φ and Γ meet the following equations

$$(\lambda+2\mu)\,\nabla^2\Phi+(\kappa+2\chi)\,\nabla^2\Gamma=\eta\theta, \tag{2.6}$$

$$[(\beta+2\gamma)\,\nabla^2-4\alpha]\,\Gamma+(\kappa+2\chi)\,\nabla^2\Phi=\zeta\theta. \tag{2.7}$$

Elimination of $\nabla^2\Phi$ from (2.6), (2.7) leads to

$$(\nabla^2-\omega^2)\,\Gamma=\tau\theta, \tag{2.8}$$

where

$$\omega^2=\frac{4\alpha(\lambda+2\mu)}{\Delta}, \quad \tau=\frac{\zeta(\lambda+2\mu)+\eta(\kappa+2\chi)}{\Delta}, \quad \Delta=(\beta+2\gamma)(\lambda+2\mu)-(\kappa+2\chi)^2.$$

Therefore, the function Γ satisfies equation of the Helmholtz type. If we introduce the new function $\Phi^0=\Phi+\dfrac{2\chi+\kappa}{\lambda+2\mu}\Gamma$, then Eq. (2.6) takes the form

$$\nabla^2\Phi^0=\frac{\eta}{\lambda+2\mu}\,\theta, \tag{2.9}$$

which is the same as for a solid with central symmetry. Therefore, the thermoelastic potential Φ for a hemitropic solid is a sum of the potential Φ^0 for a body with central symmetry and the function $-\dfrac{2\chi+\kappa}{\lambda+2\mu}\Gamma$.

The solution of Eqs. (2.8)-(2.9) for an infinite space takes the form:

$$\begin{aligned}\Phi^0(\mathbf{x})&=-\frac{\eta}{4\pi(\lambda+2\mu)}\int\limits_B\frac{\theta(\mathbf{x}')}{R(\mathbf{x},\mathbf{x}')}\,dV(\mathbf{x}'),\\ \Gamma(\mathbf{x})&=-\frac{\tau}{4\pi}\int\limits_B\frac{e^{-\omega R}\,\theta(\mathbf{x}')}{R(\mathbf{x},\mathbf{x}')}\,dV(\mathbf{x}'),\end{aligned} \tag{2.10}$$

where $R=|\mathbf{x}-\mathbf{x}'|$.

The potential Φ^0 and Γ generate the force stresses σ'_{ij} and the couple stresses μ'_{ij} according to (1.1)—(1.3):

$$\sigma'_{ji}=2\mu(\Phi^0_{,ij}-\delta_{ij}\Phi^0_{,kk})+2(\chi-\mu\tau_1)(\Gamma_{,ij}-\delta_{ij}\Gamma_{,kk})+2\alpha\varepsilon_{kij}\Gamma_{,k}, \tag{2.11}$$

$$\mu'_{ji}=2\chi(\Phi^0_{,ij}-\delta_{ij}\Phi^0_{,kk})+2(\gamma-\chi\tau_1)(\Gamma_{,ij}-\delta_{ij}\Gamma_{,kk})+2\nu\varepsilon_{kij}\Gamma_{,k}+4\alpha\Gamma\delta_{ij}. \tag{2.12}$$

For a micropolar solid with central symmetry $\chi=\nu=\kappa=\zeta=0$, which implies $\Gamma=0$. Therefore, in this case (2.11) reduces to a formula involving only Φ^0, while (2.12) reads $\mu'_{ij}=0$.

Now we are to present a method of solving Eqs. (2.1)–(2.2) for a bounded simply connected body B when the portion of its boundary (say ∂B_1) is clamped and the remaining part (say ∂B_2) is free from stresses: $\partial B=\partial B_1\cup\partial B_2$. Assume also that a steady temperature field is the only cause producing deformation of the solid. Thus we have

$$\begin{cases} X_i=Y_i=0, \quad \theta=0 \quad \text{on} \quad B, \\ u_i=\varphi_i=0 \quad \text{on} \quad \partial B_1; \quad p_i=m_i=0 \quad \text{on} \quad \partial B_2. \end{cases} \tag{2.13}$$

To find the displacement $\mathbf{u}(\mathbf{x})$ and rotation $\boldsymbol{\varphi}(\mathbf{x})$ we use the reciprocity theorem (see [3])

$$\int_B (X_i u'_i+Y_i\varphi'_i)\,dv+\int_{\partial B}(p_i u'_i+m_i\varphi'_i)\,da+\int_B(\eta\operatorname{div}\mathbf{u}'+\zeta\operatorname{div}\varphi')\,\theta dv= \tag{2.14}$$
$$=\int_B (X'_i u_i+Y'_i\varphi_i)\,dv+\int_{\partial B}(p'_i u_i+m'_i\varphi_i)\,da+\int_B(\eta\operatorname{div}\mathbf{u}+\zeta\operatorname{div}\varphi)\,\theta'\,dv,$$

in which (u_i, φ_i, θ) is to comply with (2.13), while $(u'_i, \varphi'_i, \theta')$ meets the conditions

$$\begin{cases} X'_i=\delta_{ij}\delta(\mathbf{x}-\mathbf{x}'), \quad Y'_i=0, \quad \theta'=0 \quad \text{on} \quad B, \\ u'_i=\varphi'_i=0 \quad \text{on} \quad \partial B_1, \quad p'_i=m'_i=0 \quad \text{on} \quad \partial B_n. \end{cases} \tag{2.15}$$

It follows from (2.15) that the state with primes is produced by a unit concentrated force applied at the point $\mathbf{x}'$ and directed along the axis x_j when the body is subject to isothermal conditions and the same boundary data as for $(\mathbf{u}, \boldsymbol{\varphi}, \theta)$.

Substituting (2.13) and (2.15) into (2.14) we obtain

$$u_j(\mathbf{x}')=\int_B(\eta\gamma^{(j)}_{kk}+\zeta\kappa^{(j)}_{kk})\,\theta(\mathbf{x})\,dV(\mathbf{x}). \tag{2.16}$$

Here $u'_i=U^{(j)}_i(\mathbf{x},\mathbf{x}')$, $\varphi'_i=\Phi^{(j)}_i(\mathbf{x},\mathbf{x}')$ are the Green functions obeying (2.15) and $\gamma^{(j)}_{kk}=U^{(j)}_{k,k}$, $\kappa^{(j)}_{kk}=\Phi^{(j)}_{k,k}$.

If the state $(u'_i, \varphi'_i, \theta')$ in (2.14) is to satisfy the conditions

$$\begin{cases} X'_i=0, \quad Y'_i=\delta_{ij}\delta_3(\mathbf{x}-\mathbf{x}'), \quad \theta'=0 \quad \text{on} \quad B, \\ u'_i=\varphi'_i=0 \quad \text{on} \quad \partial B_1, \quad p'_i=m'_i=0 \quad \text{on} \quad \partial B_2 \end{cases} \tag{2,17}$$

a combination of (2.13) and (2.17) leads to

$$\varphi_j(\mathbf{x}')=\int_B(\eta\hat{\gamma}^{(j)}_{kk}+\zeta\hat{\kappa}^{(j)}_{kk})\,\theta\,dV(\mathbf{x}). \tag{2.18}$$

Here $u_i' = \hat{U}_i^{(j)}(\mathbf{x}, \mathbf{x}')$, $\varphi_i' = \hat{\Phi}_i^{(j)}(\mathbf{x}, \mathbf{x}')$ are the Green functions subject to (2.17) and $\hat{\gamma}_{kk}^{(j)} = \hat{U}_{k,k}^{(j)}$, $\hat{\kappa}_{kk}^{(j)} = \hat{\Phi}_{k,k}^{(j)}$.

The formulae (2.16) and (2.18) constitute generalization of Maysel's theorem, valid for Hooke's solid, to the hemitropic micropolar theory.

Let us discuss some simple states of deformation.

A. Suppose B is clamped over its entire boundary ∂B and the deformation is due to a constant temperature field. Therefore, we have

$$\begin{cases} X_i = Y_i = 0, \quad \theta = \text{const} \quad \text{on} \quad B, \\ u_i = \varphi_i = \boldsymbol{\varphi} \quad \text{on} \quad \partial B, \end{cases} \tag{2.19}$$

which implies that the system (2.1)–(2.2) reduces to a homogeneous system of equations. The homogeneous equations together with the zero boundary conditions lead to

$$u_i = \varphi_i = 0 \quad \text{on} \quad B, \tag{2.20}$$

which, in turn, implies

$$\gamma_{ji} = \kappa_{ji} = 0 \quad \text{on} \quad B. \tag{2.21}$$

Therefore, the constitutive relations and (2.21) yield

$$\sigma_{ji} = -\eta\delta_{ij}\,\theta, \quad \mu_{ij} = -\zeta\delta_{ij}\,\theta, \tag{2.22}$$

i.e. for a clamped solid subject to a constant temperature the stress tensors are symmetric and take constant values.

B. Let B be simply connected and free from stresses over its entire boundary ∂B. We look for such a temperature field $\theta(x)$ that produces no stresses inside B.

To find an answer, we write the constitutive Eqs. (1.2)–(1.3) in an alternative form

$$\gamma_{ji} = \gamma_{ji}^0 + \gamma_{ji}', \quad \kappa_{ji} = \kappa_{ji}^0 + \kappa_{ji}', \tag{2.23}$$

where

$$\gamma_{ji}^0 = a_t\,\theta\delta_{ij}, \quad \kappa_{ji}^0 = b_t\,\theta\delta_{ij}, \tag{2.23$'$}$$

a_t, b_t are constants, and γ_{ji}' and κ_{ji}' are linear functions of the stress tensors. Since γ_{ji} and κ_{ji} meet the compatibility conditions [8]

$$\begin{aligned} &\varepsilon_{jhl}\,\gamma_{li,h} - \kappa_{ij} + \delta_{ij}\,\kappa_{kk} = 0, \\ &\varepsilon_{jhl}\,\kappa_{li,h} = 0 \end{aligned} \tag{2.24}$$

and by the hypothesis $\gamma_{ji}' = \kappa_{ji}' = 0$ on B, Eqs. (2.23)–(2.24) imply

$$\begin{aligned} &\varepsilon_{jhl}\,\gamma_{li,h} - \kappa_{ij}^0 + \delta_{ij}\,\kappa_{kk}^0 = 0, \\ &\varepsilon_{jhl}\,\kappa_{li,h}^0 = 0 \end{aligned} \tag{2.25}$$

which can be satisfied by a given $\theta(x)$ if and only if $\theta = 0$. Therefore, for a body which is free from stresses on the boundary the only free stress state is a natural state. This implies that a hemitropic solid is subject to a more stringent condition than corresponding Hooke's model or Cosserat's medium with central symmetry.

3. Mean strain and mean stress. The mean values of the strains and stresses on B are defined by the formulae

$$(3.1)\qquad (\bar{\gamma}_{ji}, \bar{\kappa}_{ji}, \bar{\sigma}_{ji}, \bar{\mu}_{ji}) = \frac{1}{V}\int_B (\gamma_{ji}, \kappa_{ji}, \sigma_{ji}, \mu_{ji})\, dv, \qquad V=\int_B dv.$$

Taking into account the definitions of γ_{ji} and κ_{ji} (see (1.1)), we obtain

$$(3.2)\qquad \bar{\gamma}_{ji} = \frac{1}{V}\int_{\partial B} u_i n_j\, da - \varepsilon_{kji}\bar{\varphi}_k, \qquad \bar{\kappa}_{ji} = \frac{1}{V}\int_{\partial B} \varphi_i n_j\, da,$$

which implies that

$$(3.3)\qquad \bar{\gamma}_{kk} = \frac{1}{V}\int_{\partial B} u_k n_k\, da = \frac{1}{V}\int_B u_{k,k}\, dv = \frac{\delta v(B)}{V},$$

$$(3.4)\qquad \bar{\kappa}_{kk} = \frac{1}{V}\int_{\partial B} \varphi_k n_k\, da = \frac{1}{V}\int_B \varphi_{k,k}\, dv,$$

where $\delta v(B)$ is the volume change.

A. Assume that the boundary of B is clamped. Since $u_i=\varphi_i=0$ on ∂B, (3.2) and (3.3) imply

$$(3.5)\qquad \bar{\gamma}_{ji} = -\varepsilon_{kji}\bar{\varphi}_k, \quad \bar{\kappa}_{ji}=0, \quad \delta v(B)=0.$$

The mean values of stresses corresponding to the case are obtained from the constitutive relations (1.2)–(1.3):

$$(3.6)\qquad \bar{\sigma}_{ji} = -2\alpha\varepsilon_{kji}\bar{\varphi}_k - \eta\delta_{ij}\bar{\theta}, \quad \bar{\mu}_{ji} = -2\nu\varepsilon_{kji}\bar{\varphi}_k - \zeta\delta_{ij}\bar{\theta}.$$

B. Assume that the body is free in the sense that the body forces and the body couples vanish ($X_i=Y_i=0$) and the boundary ∂B is free from stresses ($p_i=m_i=0$). If we multiply the equilibrium equations

$$(3.7)\qquad \sigma_{ji,j}+X_i=0, \quad \varepsilon_{ijk}\sigma_{jk}+\mu_{ji,j}+Y_i=0$$

by x_j and integrate the result over B, we obtain

$$(3.8)\qquad \int_{\partial B} p_i x_j\, da + \int_B X_i x_j\, dv = \int_B \sigma_{ji}\, dv,$$

$$(3.9)\qquad \int_{\partial B} m_i x_j\, da + \int_B Y_i x_j\, dv + \varepsilon_{ipk}\int_B \sigma_{pk} x_j\, dv = \int_B \mu_{ji}\, dv.$$

Since the body is free, (3.8) and (3.9) imply

$$(3.10)\qquad \bar{\sigma}_{ji}=0, \quad \bar{\mu}_{ji} = \frac{1}{V}\varepsilon_{ipk}\int_B \sigma_{pk} x_j\, dv,$$

and

$$\bar{\sigma}_{jj}=0, \quad \bar{\mu}_{,j}\neq 0. \tag{3.11}$$

Contraction of the constitutive Eqs. (1.2)–(1.3) together with the mean value operation lead to

$$\begin{cases} \bar{\sigma}_{jj}=3K\bar{\gamma}_{jj}+3G\bar{\kappa}_{jj}-3\eta\bar{\theta}, \\ \bar{\mu}_{jj}=3G\bar{\gamma}_{jj}+3L\bar{\kappa}_{jj}-3\zeta\theta, \end{cases} \tag{3.12}$$

where

$$K=\lambda+\frac{2}{3}\mu, \qquad G=\kappa+\frac{2}{3}\chi, \qquad L=\beta+\frac{2}{3}\gamma.$$

solving Eqs. (3.12) with respect to $\bar{\gamma}_{jj}$ and $\bar{\kappa}_{jj}$, and taking into account (3.11), we obtain

$$\bar{\gamma}_{jj}=3(\eta K'-\zeta G')\,\bar{\theta}-G'\,\bar{\mu}_{jj}, \qquad \bar{\kappa}_{jj}=3(\zeta L'-\eta G')\,\bar{\theta}+L'\,\bar{\mu}_{jj}, \tag{3.13}$$

or

$$\delta v(B)=3(\eta K'-\zeta G')\int_B \theta dv-\bar{\mu}_{jj}\,G'\,V,$$

$$\int_B \varphi_{k,k}\,dv=3(\zeta L'-\eta G')\int_B \theta dv+L'\,\bar{\mu}_{jj}\,V,$$

where the following notations are used

$$K'=\frac{L}{3(KL-G^2)}, \qquad G'=\frac{G}{3(KL-G^2)}, \qquad L'=\frac{K}{3(KL-G^2)}.$$

Therefore, for a free solid the volume change depends not only on the mean temperature but also on the mean value of the couple stresses.

4. Dynamic problems of thermoelasticity. The dynamic equations of our theory (see (1.8)–(1.10)) are mutually coupled and not easy for an analytical treatment. To discuss general properties of these equations we assume the following decomposition formulae for $\mathbf{u}$, $\boldsymbol{\varphi}$, $\mathbf{X}$ and $\mathbf{Y}$:

$$\mathbf{u}=\operatorname{grad}\Phi+\operatorname{rot}\boldsymbol{\Psi}, \qquad \boldsymbol{\varphi}=\operatorname{grad}\Gamma+\operatorname{rot}\mathbf{H}, \qquad \operatorname{div}\boldsymbol{\Psi}=\operatorname{div}\mathbf{H}=0, \tag{4.1}$$

$$\mathbf{X}=\rho(\operatorname{grad}\vartheta+\operatorname{rot}\chi), \qquad \mathbf{Y}=J(\operatorname{grad}\sigma+\operatorname{rot}\eta), \qquad \operatorname{div}\mathbf{X}=\operatorname{div}\boldsymbol{\eta}=0. \tag{4.2}$$

Substituting (4.1) and (4.2) into (1.8)–(1.10), we conclude that (1.8)–(1.10) are satisfied if the scalar functions Φ and Γ meet

$$\Box_1\,\Phi+(\kappa+2\chi)\,\nabla^2\,\Gamma-\eta\theta+\rho\vartheta=0, \tag{4.3}$$

$$(\kappa+2\chi)\,\nabla^2\,\Phi+\Box_3\,\Gamma-\zeta\theta+J\sigma=0, \tag{4.4}$$

$$D\theta-\eta\nabla^2\,\dot{\Phi}-\zeta\nabla^2\,\dot{\Gamma}=-W_0 \tag{4.5}$$

and the vector functions $\boldsymbol{\Psi}$ and $\mathbf{H}$ satisfy

$$\Box_2\,\boldsymbol{\Psi}+[(\chi+\nu)\,\nabla^2+2\alpha\operatorname{rot}]\,\mathbf{H}+\rho\boldsymbol{\chi}=0, \tag{4.6}$$

$$[(\chi+\nu)\,\nabla^2+2\alpha\operatorname{rot}]\,\boldsymbol{\Psi}+(\Box_4+4\nu\operatorname{rot})\,\mathbf{H}+J\eta=0, \tag{4.7}$$

where

$$\square_1 = (\lambda+2\mu)\,\nabla^2 - \rho\partial_t^2, \qquad \square_3 = (\beta+2\gamma)\,\nabla^2 - 4\alpha - J\partial_t^2.$$

Note that Eqs. (4.6)–(4.7) are independent of the temperature field. They represent a motion of transverse waves: the function $\mathbf{\Psi}$ characterizes the transverse displacements and $\mathbf{H}$ describes the transverse microrotation.

On the other hand, Eqs. (4.3)–(4.5) do depend on the temperature. From now on we concentrate on a discussion of Eqs. (4.3)–(4.5) when the solid is unbounded and waves are produced by the source functions ϑ, σ, W_0 and initial disturbances of Φ, Γ, θ.

Eqs. (4.3), (4.4) and (4.5) represent, in this order, the longitudinal displacement wave, the longitudinal microrotation wave and the thermal wave. Since these three types of waves are mutually coupled they propagate with the same phase velocity.

Elimination of θ from Eqs. (4.3)–(4.5) yields

$$(\square_1 D - \eta^2\,\partial_t\,\nabla^2)\,\Phi + \nabla^2\,\hat{D}\Gamma = -\eta W_0 - \rho D\vartheta, \tag{4.8}$$

$$\nabla^2\,\hat{D}\Phi + (\square_3 D - \zeta^2\,\partial_t\,\nabla^2)\,\Gamma = -\zeta W_0 - JD\sigma, \tag{4.9}$$

where

$$T_0\,\hat{D} = \omega_0\,k\nabla^2 - (c_\varepsilon\,\omega_0 + T_0\,\eta\zeta)\,\partial_t, \qquad \omega_0 = \kappa + 2\chi.$$

Eqs. (4.8)—(4.9) constitute a complex hyperbolic-parabolic system of equations. For $\chi=\kappa=\zeta=0$ this system reduces to that of the classical thermoelasticity [9]

$$(\square_1 D - \eta^2\,\partial_t\,\nabla^2)\,\Phi = -\eta W_0 - \rho D\vartheta. \tag{4.10}$$

Consider now the homogeneous system associated with (4.3)–(4.5). The heat conduction equation (4.5) takes the form

$$D\theta = \nabla^2(\eta\dot{\Phi} + \zeta\dot{\Gamma}). \tag{4.11}$$

If the RHS of (4.11) is treated as a known source function, and if $G(\mathbf{x}, \mathbf{x}', t)$ is the Green function satisfying the parabolic equation

$$DG(\mathbf{x}, \mathbf{x}', t) = \delta(\mathbf{x}-\mathbf{x}')\,\delta(t) \tag{4.12}$$

in an infinite space subject to homogeneous initial conditions and the regularity condition: $G \to 0$ as $R \to \infty$, a solution of (4.11) is represented by

$$\theta(\mathbf{x}', t) = \int_0^t d\tau \int_B G(\mathbf{x}, \mathbf{x}', t-\tau)\,\nabla^2\left(\eta\frac{\partial\Phi}{\partial\tau} + \zeta\frac{\partial\Gamma}{\partial\tau}\right) dv(\mathbf{x}), \tag{4.13}$$

where

$$G(\mathbf{x}, \mathbf{x}', t) = -\frac{T_0}{8c_\varepsilon(\pi\hat{\kappa}t)^{3/2}}\,e^{-\frac{R^2}{4\hat{\kappa}t}}, \qquad R = |\mathbf{x}-\mathbf{x}'|, \qquad \hat{\kappa} = \frac{c_\varepsilon}{k}. \tag{4.14}$$

Substituting θ from (4.13) into (4.3) and (4.4), we obtain two integro-differential equations for the functions Φ and Γ.

It is shown in the classical thermoelasticity [9] that the term $\eta \operatorname{div} \dot{\mathbf{u}}$ has small influence on $\mathbf{u}$ and θ. If we neglect the terms $\eta \operatorname{div} \dot{\mathbf{u}}$ and $\zeta \operatorname{div} \dot{\boldsymbol{\varphi}}$ in the present theory, (4.5) reduces to

(4.15) $$D\theta = -W_0.$$

Let θ be a solution of (4.15). Substituting this solution into (4.4) and (4.5) and eliminating Γ or Φ, we obtain ($\vartheta = \sigma = 0$):

(4.16) $$(\square_1 \square_3 - \omega^2 \nabla^2 \nabla^2)\, \Phi = (\eta \square_3 - \zeta \nabla^2)\, \theta,$$

or

(4.17) $$(\square_1 \square_3 - \omega^2 \nabla^2 \nabla^2)\, \Gamma = (\zeta \square_1 - \eta \nabla^2)\, \theta,$$

respectively.

A discussion of monochromatic plane waves satisfying (4.16) or (4.17) shows that in this case both waves Φ and Γ are dispersed but not attenuated.

INSTITUTE OF MECHANICS, UNIVERSITY, PAŁAC KULTURY I NAUKI, 9th Floor, 00-901 WARSAW
(INSTYTUT MECHANIKI, UNIWERSYTET WARSZAWSKI, PKiN)

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Я. П. Новацки, В. Новацки, **Некоторые проблемы гемитропного микрополярного континуума**

Содержание. В работе обсуждено несколько стационарных и динамических вопросов, исходя из уравнений поля для термоупругого, линейного и гемитропного континуума Коссератов. Даны особые интегралы стационарных уравнений, уравнений термоупругости, а также обобщенные взоры Майселя. Даны средние значения деформаций и напряжений. Уравнения динамической термоупругости сведены к простым волновым уравнениям и к интегрально--дифференциальным уравнениям.