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The Helmholtz conditions for systems of second order homogeneous differential equations[1]

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Abstract. Variationality of systems of second order ordinary differential equations is studied within the class of positive homogeneous systems. The concept of a higher order positive homogeneous function, related to Finsler geometry, is represented by the well-known Zermelo conditions, and applied to the theory of variational equations. In particular, it is shown that every system of $m+1$ second order variational and positive homogeneous differential equations is linearly dependent and admits subsystems of m differential equations which are variational in sense of parameter-invariant variational problems, and vice versa. An example of a positive homogeneous variational system of second order differential equations is given.

Keywords. variational differential equation, Helmholtz conditions, Lagrangian, positive homogeneous function, Zermelo conditions, Finsler geometry

MSC. 34A26, 34A34, 58B20

1. Introduction

In this paper we study variationality of systems of second order ordinary differential equations given by positive homogeneous functions. Euler-Lagrange equations associated with systems of this class admit positive homogeneous Lagrangian, and they have solutions independent of parametrization which preserves orientation. From this point of view Lagrangians of the class of positive homogeneous and variational systems may represent fundamental functions for possible higher order generalizations of Finsler geometry.

Recently in [10], we have analysed by means of the geometric theory of jet differential groups (Grigore and Krupka [3], Krupka and Krupka [5], Krupka and Urban [6]), the concept of positive homogeneity for functions depending on curves and their derivatives up to an arbitrary finite order. It appeared that this *higher order positive homogeneity* is equivalent with the well-known Zermelo conditions (see e.g. Zermelo [11], McKiernan [8], Matsyuk [7]), generalizing the standard Euler formula for positive homogeneous functions depending on curves and their first derivatives only. On this basis, every solution of a system of differential equations with left-hand sides given by positive homogeneous functions is an orientation-preserving solution.

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In Section 2 we briefly recall basic concepts of the general theory of variational differential equations (see e.g. Havas [4]). In Section 3 we give second order version of our results contained in [10]; the Zermelo conditions for second order systems are given explicitly.

Our main results contained in Section 4 include: a) every positive homogeneous system of $m+1$ second order equations of $m+1$ dependent variables is linearly dependent, b) variationality of a system of $m+1$ second order differential equations, defined by positive homogenous functions, is equivalent with variationality of certain of its subsystem of m equations in sense of parametrized variational problems, c) explicit relationship between Lagrangians of both of these systems is given. Finally, we give an example of two second order equations whose solution is a unit circle in $\mathbf{R}^2$, with analysis of variationality and positive homogeneity.

The methods can be extended to the theory of differential equations on manifolds, as well as to higher order systems. Examples in higher order dimension can be constructed analogously.

Throughout the paper we denote by y^K, $K = 1, 2, \dots, m+1$, the *canonical coordinate functions* on the Euclidean space $\mathbf{R}^{m+1}$, and by $\dot{y}^K$, $\ddot{y}^K$ and $\dddot{y}^K$ their first, second and third order derivatives, respectively. If $\gamma : I \to \mathbf{R}^{m+1}$, $\gamma(t) = (\gamma^1(t), \gamma^2(t), \dots, \gamma^{m+1}(t))$, is a curve, then for every K, $y^K \circ \gamma(t) = \gamma^K(t)$, $\dot{y}^K \circ \gamma(t) = D(y^K\gamma)(t)$, $\ddot{y}^K \circ \gamma(t) = D^2(y^K\gamma)(t)$, and $\dddot{y}^K \circ \gamma(t) = D^3(y^K\gamma)(t)$.

2. The Helmholtz conditions

Suppose we are given a system of $m+1$ second order ordinary differential equations

$$\varepsilon_K(y^Q, \dot{y}^Q, \ddot{y}^Q) = 0, \tag{1}$$

where $K, Q = 1, 2, \dots, m+1$; the number of equations and the number of dependent variables are both equal $m+1$. Solutions of the system (1) are differentiable regular curves $\gamma : \mathrm{J} \to \mathbf{R}^{\mathrm{m}+1}$, $\gamma(t) = (y^1\gamma(t), y^2\gamma(t), \dots, y^{m+1}\gamma(t))$, in $\mathbf{R}^{m+1}$, defined on an open interval of the real line $\mathbf{R}$, which satisfy the system (1).

In accordance with the general theory of variational differential equations, we shall say that the system (1) is *variational,* if there exists a real-valued function $\mathscr{L} = \mathscr{L}(y^Q, \dot{y}^Q)$ for which (1) is the system of *Euler-Lagrange equations*; this means that for every K,

$$\varepsilon_K = \frac{\partial \mathscr{L}}{\partial y^K} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{y}^K} = \frac{\partial \mathscr{L}}{\partial y^K} - \frac{\partial^2 \mathscr{L}}{\partial y^Q \partial \dot{y}^K}\dot{y}^Q - \frac{\partial^2 \mathscr{L}}{\partial \dot{y}^Q \partial \dot{y}^K}\ddot{y}^Q. \tag{2}$$

If $\mathscr{L}$ exists, it is called the *Lagrange function* for the system (1) which coincide with the system of equations for extremals of a certain variational functional, associated with $\mathscr{L}$. We note that in the definition above the system of equations is supposed to be as it stands: the functions defining the left-hand sides are supposed to be fixed. All our assertions will be concerned with this system of functions; for example, no variational integrating factors are considered. It is the standard result that for a second order variational system of functions $\varepsilon_K = \varepsilon_K(y^Q, \dot{y}^Q, \ddot{y}^Q)$ there exists a second order Lagrangian $\mathscr{L} = \mathscr{L}(y^Q, \dot{y}^Q, \ddot{y}^Q)$, namely the *Vainberg-Tonti Lagrangian,*

$$\mathscr{L}(y^Q, \dot{y}^Q, \ddot{y}^Q) = y^K \int_0^1 \varepsilon_K(sy^Q, s\dot{y}^Q, s\ddot{y}^Q)ds \tag{3}$$

(see e.g. Tonti [9]).

The necessary and sufficient conditions for variationality of systems of differential equations are the well-known *Helmholtz conditions*. We formulate the Helmholtz conditions for second order systems.

Theorem 2.1. *[Helmholtz conditions] Suppose that we have a system of functions $\varepsilon_K = \varepsilon_K(y^Q, \dot{y}^Q, \ddot{y}^Q)$. The following two conditions* are equivalent:

(a) *The equation* (2) *has a solution.*

(b) *The functions ε_K satisfy the system*

$$\frac{\partial \varepsilon_K}{\partial \ddot{y}^M} - \frac{\partial \varepsilon_M}{\partial \ddot{y}^K} = 0, \tag{4}$$

$$\frac{\partial \varepsilon_K}{\partial \dot{y}^M} + \frac{\partial \varepsilon_M}{\partial \dot{y}^K} - \frac{d}{dt}\left(\frac{\partial \varepsilon_K}{\partial \ddot{y}^M} + \frac{\partial \varepsilon_M}{\partial \ddot{y}^K}\right) = 0, \tag{5}$$

$$\frac{\partial \varepsilon_K}{\partial y^M} - \frac{\partial \varepsilon_M}{\partial y^K} - \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \varepsilon_K}{\partial \dot{y}^M} - \frac{\partial \varepsilon_M}{\partial \dot{y}^K}\right) = 0. \tag{6}$$

Proof. This result is standard; see e.g. Havas [4] and references therein.

We note that (4) and (5) immediately implies that the functions ε_K must be linear in second derivative variables, i.e. $\varepsilon_K = A_K + B_{KQ}\ddot{y}^Q$, where $B_{KQ} = \partial C_K/\partial \dot{y}^Q = \partial C_Q/\partial \dot{y}^K = B_{QK}$. The second order Lagrangian (3) can be then reduced to a first order Lagrangian by deleting a total derivative term; namely

$$\begin{aligned}\mathscr{L}_0(y^Q, \dot{y}^Q) = y^K \int_0^1 A_K(sy^Q, s\dot{y}^Q)ds - \dot{y}^K \int_0^1 C_K(sy^Q, s\dot{y}^Q)ds \\ - y^K \int_0^1 \left(\frac{\partial C_K}{\partial y^P}\dot{y}^P\right)_{(sy^Q, s\dot{y}^Q)} ds.\end{aligned} \tag{7}$$

Clearly, the system of Helmholtz conditions can be rewritten to an equivalent system for the first order functions A_K, B_{KQ}; however, we use in this paper the conditions given by Theorem 2.1.

3. Second order positive homogeneous systems

We wish to study in this work variationality of systems of second order ordinary differential equations, which are given by *second order positive homogeneous functions* (in the Zermelo sense). We studied the class of higher order positive homogeneous functions in [10]. Let us briefly recall the basic facts. The concept of a positive homogeneous function we use, extends the classical positive homogeneity for functions depending on curves $t \to y^K(t)$ and their first derivatives $t \to \dot{y}^K(t)$, expressed by the standard *Euler formula*,

$$\frac{\partial F}{\partial \dot{y}^M}\dot{y}^M = F,$$

to functions depending also on second derivatives $t \to \ddot{y}^K(t)$. We shall say that a function $F = F(y^K, \dot{y}^K, \ddot{y}^K)$ is *positive homogeneous in the variables $\dot{y}^K$ and $\ddot{y}^K$*, or simply *positive homogeneous*, if

$$F(y^K, a_1\dot{y}^K, a_1^2\ddot{y}^K + a_2\dot{y}^K) = a_1 F(y^K, \dot{y}^K, \ddot{y}^K) \tag{8}$$

for all regular curves $\gamma : \mathrm{J} \to \mathbf{R}^{\mathrm{m}+1}$, $\gamma(t) = (y^1\gamma(t), y^2\gamma(t), \dots, y^{m+1}\gamma(t))$, in $\mathbf{R}^{m+1}$, defined on an open interval of the real line $\mathbf{R}$, and for all numbers $a_1 > 0$, $a_2 \in \mathbf{R}$. The condition (8) has, however, a geometric meaning: an integral (variational) functional, associated with F, does *not* depend on parametrization.

In [10], we proved the following two results on positive homogeneous functions of arbitrary finite order; here we give their second order versions.

The next theorem shows that the Zermelo conditions are necessary and sufficient conditions for a function $F = F(y^K, \dot{y}^K, \ddot{y}^K)$ to be positive homogeneous.

Theorem 3.1. Let $F = F(y^K, \dot{y}^K, \ddot{y}^K)$ *be a function. The following conditions are equivalent:*

(a) *F is positive-homogeneous in the variables $\dot{y}^K$ and $\ddot{y}^K$.*
(b) *F satisfies the Zermelo conditions*

$$\frac{\partial F}{\partial \dot{y}^M}\dot{y}^M + 2\frac{\partial F}{\partial \ddot{y}^M}\ddot{y}^M = F, \tag{9}$$

$$\frac{\partial F}{\partial \ddot{y}^M}\dot{y}^M = 0. \tag{10}$$

Proof. The proof, for arbitrary finite order, can be found in [10].

The following result concerns solutions of systems of second order ordinary differential equations. Consider system (1), $\varepsilon_K(y^Q, \dot{y}^Q, \ddot{y}^Q) = 0$. We shall say that the system (1) is *positive homogeneous*, if all functions ε_K are positive homogeneous in the sense of previous definition (8). Let γ be a solution of the system (1), defined on an open interval $I \subset \mathbf{R}$. Then γ is called an *orientation-preserving solution*, if for every diffeomorphism $\tau : J \to I$ of open intervals such that $D\tau > 0$ on J, the regular curve $\gamma \circ \tau$ is again a solution of (1).

If, moreover, $\gamma \circ \tau$ is a solution of (1) for arbitrary reparametrization τ, we say γ is a *set-solution*. In order that γ be a set-solution, it is sufficient that γ is orientation-preserving and the curve $t \to \gamma(-t)$ is also a solution. This observation explains, in particular, the role of the Zermelo conditions.

For second order equations, we have the following general result: the class of positive homogeneous systems of differential equations has solutions which do *not* depend on orientation-preserving parametrization.

Theorem 3.2. *Let $\varepsilon_K(y^Q, \dot{y}^Q, \ddot{y}^Q) = 0$ be a positive-homogeneous system of second order differential equations. Then every solution of this system is an orientation-preserving solution.*

Proof. This result is valid for arbitrary higher order positive homogeneous systems; see [10]. Nevertheless, we prove this proposition for second order systems explicitly. Suppose that the functions ε_K, defining the system (1), are positive homogeneous, i.e. functions satisfying the conditions (9) and (10). Let $\gamma : I \to \mathbf{R}^{m+1}$ be a curve in $\mathbf{R}^{m+1}$, and let $\tau : J \to I$ be a diffeomorphism of open intervals in $\mathbf{R}$. Choose $t_0 \in J$, and we may suppose that $D\tau(t_0) > 0$. From positive homogeneity condition (8) we get for every K,

$$\begin{aligned}
&\varepsilon_K(y^Q(\gamma\circ\tau)(t_0), \dot{y}^Q(\gamma\circ\tau)(t_0), \ddot{y}^Q(\gamma\circ\tau)(t_0)) \\
&= \varepsilon_K(y^Q\gamma(\tau(t_0)), D(y^Q\gamma)(\tau(t_0))D\tau(t_0), \\
&\quad D^2(y^Q\gamma)(\tau(t_0))(D\tau(t_0))^2 + D(y^Q\gamma)(\tau(t_0))D^2\tau(t_0)) \\
&= \varepsilon_K(y^Q\gamma(\tau(t_0)), \dot{y}^Q\gamma(\tau(t_0))D\tau(t_0), \ddot{y}^Q\gamma(\tau(t_0))(D\tau(t_0))^2 + \dot{y}^Q\gamma(\tau(t_0))D^2\tau(t_0)) \\
&= D\tau(t_0)\cdot\varepsilon_K(y^Q\gamma(\tau(t_0)), \dot{y}^Q\gamma(\tau(t_0)), \ddot{y}^Q\gamma(\tau(t_0))).
\end{aligned}$$

Hence $\gamma \circ \tau$ is a solution if and only if γ is a solution, which completes the proof.

4. The Helmholtz conditions for second order positive homogeneous systems

In this section we study variationality of positive homogeneous systems of second order equations. Let us first recall a result we proved in [10]: the necessary and sufficient condition for a variational system of second order equations to be positive homogeneous.

Theorem 4.1. *Suppose that the system* (1), $\varepsilon_K(y^Q, \dot{y}^Q, \ddot{y}^Q) = 0$*, is variational. The following two conditions are equivalent:*
(a) *The system* (1) *is positive homogeneous.*

(b) *The system* (1) *admits a positive homogeneous Lagrangian* $\mathscr{L} = \mathscr{L}(y^Q, \dot{y}^Q)$.

Proof. The proof can be found in [10].

For purpose of formulating and proving our main theorem let us comment on coordinate charts in $\mathbf{R}^{m+1}$ we shall use. Throughout, we consider *regular* curves in $\mathbf{R}^{m+1}$ or, in other words, curves with non-vanishing tangent vector at every point of a curve. In this case, there exists an index L such that $1 \leq L \leq m+1$ and $\dot{y}^L \neq 0$ at every point of a curve. We introduce another coordinates of regular curves in $\mathbf{R}^{m+1}$, namely the *adapted coordinates* which arise from the canonical coordinates and their derivatives in the following way:

$$\begin{aligned} &w^L = y^L,\ \dot{w}^L = \dot{y}^L,\ \ddot{w}^L = \ddot{y}^L, \\ &w^\nu = y^\nu,\ w_1^\nu = \frac{1}{\dot{y}^L}\dot{y}^\nu,\ w_2^\nu = \frac{1}{(\dot{y}^L)^2}\left(\ddot{y}^\nu - \frac{\ddot{y}^L}{\dot{y}^L}\dot{y}^\nu\right), \end{aligned} \tag{11}$$

and conversely we have

$$\begin{aligned} &y^L = w^L,\ \dot{y}^L = \dot{w}^L,\ \ddot{y}^L = \ddot{w}^L, \\ &y^\nu = w^\nu,\ \dot{y}^\nu = w_1^\nu \dot{w}^L,\ \ddot{y}^\nu = w_2^\nu (\dot{w}^L)^2 + w_1^\nu \ddot{w}^L. \end{aligned} \tag{12}$$

Remark 4.2. [Invariant coordinates] It is not difficult to see that the coordinates $w^L, w^\nu, w_1^\nu, w_2^\nu$, defined by (11), are invariant under the composition of diffeomorphisms τ of a neighbourhood of the origin 0 in $\mathbf{R}$ such that $\tau(0) = 0$; i.e. we have $w^K\gamma = w^K(\gamma \circ \tau)$, $w_1^\sigma \gamma = w_1^\sigma(\gamma \circ \tau)$, and $w_2^\sigma \gamma = w_2^\sigma(\gamma \circ \tau)$. On the other hand, the coordinates $\dot{w}^L$ and $\ddot{w}^L$ are *not* invariant. We remark that in the geometric theory of jet differential invariants coordinates of this kind arise when we study quotient spaces of regular velocities with respect to a differential group action; for futher details we refer to Grigore and Krupka [3], M. Krupka and D. Krupka [5], Krupka and Urban [6].

Remark 4.3. [Total derivative operator] For further need in proofs we find the transformation of the total derivative operator into the adapted coordinates. Suppose $f = f(y^K, \dot{y}^K, \ddot{y}^K)$ to be a function given in the canonical coordinates, and denote by $\tilde{f}$ a function in the adapted coordinates defined by

$$\tilde{f}(w^L, \dot{w}^L, \ddot{w}^L, w^\nu, w_1^\nu, w_2^\nu) = f(y^L, \dot{y}^L, \ddot{y}^L, y^\nu, \dot{y}^\nu, \ddot{y}^\nu).$$

Then we obtain

$$\frac{df}{dt} = \frac{\partial f}{\partial y^L}\dot{y}^L + \frac{\partial f}{\partial \dot{y}^L}\ddot{y}^L + \frac{\partial f}{\partial \ddot{y}^L}\dddot{y}^L + \frac{\partial f}{\partial y^\nu}\dot{y}^\nu + \frac{\partial f}{\partial \dot{y}^\nu}\ddot{y}^\nu + \frac{\partial f}{\partial \ddot{y}^\nu}\dddot{y}^\nu$$

$$
\begin{aligned}
&= \frac{\partial \tilde{f}}{\partial w^L}\dot{w}^L + \left(\frac{\partial \tilde{f}}{\partial \dot{w}^L} - \frac{\partial \tilde{f}}{\partial w_1^\nu}\frac{w_1^\nu}{\dot{w}^L} + \frac{\partial \tilde{f}}{\partial w_2^\nu}\left(\frac{w_1^\nu \ddot{w}^L}{\left(\dot{w}^L\right)^3} - 2\frac{w_2^\nu}{\dot{w}^L}\right)\right)\ddot{w}^L \\
&+ \left(\frac{\partial \tilde{f}}{\partial \ddot{w}^L} - \frac{\partial \tilde{f}}{\partial w_2^\nu}\frac{w_1^\nu}{\left(\dot{w}^L\right)^2}\right)\dddot{w}^L + \frac{\partial \tilde{f}}{\partial w^\nu} w_1^\nu \dot{w}^L \\
&+ \left(\frac{\partial \tilde{f}}{\partial w_1^\nu}\frac{1}{\dot{w}^L} - \frac{\partial \tilde{f}}{\partial w_2^\nu}\frac{\ddot{w}^L}{\left(\dot{w}^L\right)^3}\right)\left(w_2^\nu\left(\dot{w}^L\right)^2 + w_1^\nu \ddot{w}^L\right) \\
&+ \frac{1}{\left(\dot{w}^L\right)^2}\frac{\partial \tilde{f}}{\partial w_2^\nu}\left(w_3^\nu\left(\dot{w}^L\right)^3 + 3w_2^\nu \dot{w}^L \ddot{w}^L + w_1^\nu \dddot{w}^L\right) \\
&= \dot{w}^L \frac{d\tilde{f}}{dw^L},
\end{aligned}
$$

where

$$
\frac{d\tilde{f}}{dw^L} = \frac{\partial \tilde{f}}{\partial w^L} + \frac{\partial \tilde{f}}{\partial \dot{w}^L}\frac{\ddot{w}^L}{\dot{w}^L} + \frac{\partial \tilde{f}}{\partial \ddot{w}^L}\frac{\dddot{w}^L}{\dot{w}^L} + \frac{\partial \tilde{f}}{\partial w^\nu}w_1^\nu + \frac{\partial \tilde{f}}{\partial w_1^\nu}w_2^\nu + \frac{\partial \tilde{f}}{\partial w_2^\nu}w_3^\nu
$$

is the formal derivative of a function $\tilde{f} = \tilde{f}(w^L, \dot{w}^L, \ddot{w}^L, w^\nu, w_1^\nu, w_2^\nu)$.

Now we formulate the main result of this paper.

Theorem 4.4. *The following two conditions for the system* (1) *are equivalent:*
(a) *The system* (1) *is variational and positive homogeneous.*
(b) *For every index L, $1 \le L \le m+1$, there exists a coordinate transformation, represented by adapted coordinates $w^L, \dot{w}^L, \ddot{w}^L, w^\nu, w_1^\nu, w_2^\nu$* (11)*, such that the functions $\tilde{\varepsilon}_K$, defined by*

$$
\tilde{\varepsilon}_K(w^L, \dot{w}^L, \ddot{w}^L, w^\nu, w_1^\nu, w_2^\nu) = \varepsilon_K(y^L, \dot{y}^L, \ddot{y}^L, y^\nu, \dot{y}^\nu, \ddot{y}^\nu), \tag{13}
$$

are of the form $\tilde{\varepsilon}_K = \mu_K \dot{w}^L$, where $\mu_K = \mu_K(w^L, w^\nu, w_1^\nu, w_2^\nu)$, $\mu_L = -\mu_\sigma w_1^\sigma$, and the system of m ordinary differential equations

$$
\mu_\sigma(w^L, w^\nu, w_1^\nu, w_2^\nu) = 0, \tag{14}
$$

$\sigma = 1, 2, \ldots, m+1$, $\sigma \ne L$, is variational.

Proof. 1. We consider the system (1) satisfying conditions (4), (5), (6) (Helmholtz), and conditions (9), (10) (Zermelo) for second order systems. Let L, $1 \le L \le m+1$, be a fixed index, and let $w^L, \dot{w}^L, \ddot{w}^L, w^\nu, w_1^\nu, w_2^\nu$ be the L-adapted coordinates, defined by (11).

First we wish to show that the transformed system $\tilde{\varepsilon}_K$ in adapted coordinates is of the form $\tilde{\varepsilon}_K = \mu_K \dot{w}^L$, for some functions $\mu_K = \mu_K(w^L, w^\nu, w_1^\nu, w_2^\nu)$. Using the transformation equations (11), (12), between the canonical and the adapted chart we get

$$
\begin{aligned}
&\frac{\partial \varepsilon_K}{\partial y^\nu} = \frac{\partial \tilde{\varepsilon}_K}{\partial w^\nu}, \quad \frac{\partial \varepsilon_K}{\partial y^L} = \frac{\partial \tilde{\varepsilon}_K}{\partial w^L}, \\
&\frac{\partial \varepsilon_K}{\partial \dot{y}^\nu} = \frac{1}{\dot{w}^L}\frac{\partial \tilde{\varepsilon}_K}{\partial w_1^\nu} - \frac{\ddot{w}^L}{(\dot{w}^L)^3}\frac{\partial \tilde{\varepsilon}_K}{\partial w_2^\nu}, \\
&\frac{\partial \varepsilon_K}{\partial \dot{y}^L} = \frac{\partial \tilde{\varepsilon}_K}{\partial \dot{w}^L} - \frac{w_1^\lambda}{\dot{w}^L}\frac{\partial \tilde{\varepsilon}_K}{\partial w_1^\lambda} + \left(\frac{w_1^\lambda \ddot{w}^L}{(\dot{w}^L)^3} - 2\frac{w_2^\lambda}{\dot{w}^L}\right)\frac{\partial \tilde{\varepsilon}_K}{\partial w_2^\lambda}, \\
&\frac{\partial \varepsilon_K}{\partial \ddot{y}^\nu} = \frac{1}{(\dot{w}^L)^2}\frac{\partial \tilde{\varepsilon}_K}{\partial w_2^\nu}, \quad \frac{\partial \varepsilon_K}{\partial \ddot{y}^L} = \frac{\partial \tilde{\varepsilon}_K}{\partial \ddot{w}^L} - \frac{w_1^\lambda}{(\dot{w}^L)^2}\frac{\partial \tilde{\varepsilon}_K}{\partial w_2^\lambda}.
\end{aligned}
\tag{15}
$$

Applying (15), we can directly transform the Zermelo conditions into the adapted coordinates.

From (9) we get

$$\frac{\partial \tilde{\varepsilon}_K}{\partial \dot{w}^L}\dot{w}^L + 2\frac{\partial \tilde{\varepsilon}_K}{\partial \ddot{w}^L}\ddot{w}^L = \tilde{\varepsilon}_K, \tag{16}$$

and from (10)

$$\frac{\partial \tilde{\varepsilon}_K}{\partial \ddot{w}^L}\dot{w}^L = 0, \tag{17}$$

(no summation through L). Since the coordinate function $\dot{w}^L$ is non-vanishing, from (16) and (17) we immediately get

$$\frac{\partial \tilde{\varepsilon}_K}{\partial \dot{w}^L}\dot{w}^L = \tilde{\varepsilon}_K, \quad \frac{\partial \tilde{\varepsilon}_K}{\partial \ddot{w}^L} = 0; \tag{18}$$

the Zermelo conditions for the system of functions $\tilde{\varepsilon}_K$ in adapted coordinates. These conditions for $\tilde{\varepsilon}_K$, however, can be solved, and we get

$$\tilde{\varepsilon}_K = \mu_K \dot{w}^L, \tag{19}$$

where $\mu_K = \mu_K(w^L, w^\nu, w_1^\nu, w_2^\nu)$.

Now we apply the variationality of the system (1). Rewriting the Helmholtz conditions for functions $\tilde{\varepsilon}_K$ in adapted coordinates, we obtain from (4)

$$\frac{\partial \tilde{\varepsilon}_\sigma}{\partial w_2^\nu} - \frac{\partial \tilde{\varepsilon}_\nu}{\partial w_2^\sigma} = 0, \quad \frac{\partial \tilde{\varepsilon}_L}{\partial w_2^\sigma} + \frac{\partial \tilde{\varepsilon}_\sigma}{\partial w_2^\lambda}w_1^\lambda = 0, \tag{20}$$

from (5) we get

$$\begin{aligned}
&\frac{1}{\dot{w}^L}\left(\frac{\partial \tilde{\varepsilon}_\sigma}{\partial w_1^\nu} + \frac{\partial \tilde{\varepsilon}_\nu}{\partial w_1^\sigma}\right) + \frac{\ddot{w}^L}{(\dot{w}^L)^3}\left(\frac{\partial \tilde{\varepsilon}_\sigma}{\partial w_2^\nu} + \frac{\partial \tilde{\varepsilon}_\nu}{\partial w_2^\sigma}\right) - \frac{1}{\dot{w}^L}\frac{d}{dw^L}\left(\frac{\partial \tilde{\varepsilon}_\sigma}{\partial w_2^\nu} + \frac{\partial \tilde{\varepsilon}_\nu}{\partial w_2^\sigma}\right) = 0,\\
&\frac{1}{\dot{w}^L}\frac{\partial \tilde{\varepsilon}_L}{\partial w_1^\sigma} + 2\frac{\ddot{w}^L}{(\dot{w}^L)^3}\frac{\partial \tilde{\varepsilon}_L}{\partial w_2^\sigma} - 2\frac{1}{\dot{w}^L}\frac{d}{dw^L}\left(\frac{\partial \tilde{\varepsilon}_L}{\partial w_2^\sigma}\right)\\
&+ \frac{1}{\dot{w}^L}\tilde{\varepsilon}_\sigma - \frac{w_1^\lambda}{\dot{w}^L}\frac{\partial \tilde{\varepsilon}_\sigma}{\partial w_1^\lambda} - 2\frac{w_2^\lambda}{\dot{w}^L}\frac{\partial \tilde{\varepsilon}_\sigma}{\partial w_2^\lambda} = 0,\\
&\frac{\partial \tilde{\varepsilon}_L}{\partial \dot{w}^L} - \frac{w_1^\lambda}{\dot{w}^L}\frac{\partial \tilde{\varepsilon}_L}{\partial w_1^\lambda} - \left(\frac{w_2^\lambda}{\dot{w}^L} + \frac{w_1^\lambda \ddot{w}^L}{(\dot{w}^L)^3}\right)\frac{\partial \tilde{\varepsilon}_L}{\partial w_2^\lambda} + \frac{w_1^\lambda}{\dot{w}^L}\frac{d}{dw^L}\left(\frac{\partial \tilde{\varepsilon}_L}{\partial w_2^\lambda}\right) = 0,
\end{aligned} \tag{21}$$

and from (6)

$$\begin{aligned}
&\frac{\partial \tilde{\varepsilon}_\sigma}{\partial w^\nu} - \frac{\partial \tilde{\varepsilon}_\nu}{\partial w^\sigma} + \frac{1}{2}\frac{\ddot{w}^L}{(\dot{w}^L)^2}\left(\frac{\partial \tilde{\varepsilon}_\sigma}{\partial w_1^\nu} - \frac{\partial \tilde{\varepsilon}_\nu}{\partial w_1^\sigma}\right) - \frac{1}{2}\frac{d}{dw^L}\left(\frac{\partial \tilde{\varepsilon}_\sigma}{\partial w_1^\nu} - \frac{\partial \tilde{\varepsilon}_\nu}{\partial w_1^\sigma}\right) = 0,\\
&\frac{\partial \tilde{\varepsilon}_L}{\partial w^\sigma} - \frac{\partial \tilde{\varepsilon}_\sigma}{\partial w^L} - \frac{1}{2}\dot{w}^L\frac{d}{dw^L}\left(\frac{1}{\dot{w}^L}\frac{\partial \tilde{\varepsilon}_L}{\partial w_1^\sigma} - \frac{\partial \tilde{\varepsilon}_\sigma}{\partial w_1^L} + \frac{w_1^\lambda}{\dot{w}^L}\frac{\partial \tilde{\varepsilon}_\sigma}{\partial w_1^\lambda} + 2\frac{w_2^\lambda}{\dot{w}^L}\frac{\partial \tilde{\varepsilon}_\sigma}{\partial w_2^\lambda}\right) = 0.
\end{aligned} \tag{22}$$

Substituting for $\tilde{\varepsilon}_K$ from (19) in the variationality conditions (20), (21), and (22), then the conditions for system of functions $\{\mu_\sigma, \mu_L\}$ read

$$\frac{\partial \mu_\sigma}{\partial w_2^\nu} - \frac{\partial \mu_\nu}{\partial w_2^\sigma} = 0, \tag{23}$$

$$\frac{\partial \mu_\sigma}{\partial w_1^\nu} + \frac{\partial \mu_\nu}{\partial w_1^\sigma} - \frac{d}{dw^L}\left(\frac{\partial \mu_\sigma}{\partial w_2^\nu} + \frac{\partial \mu_\nu}{\partial w_2^\sigma}\right) = 0, \tag{24}$$

$$\frac{\partial \mu_\sigma}{\partial w^\nu} - \frac{\partial \mu_\nu}{\partial w^\sigma} - \frac{1}{2}\frac{d}{dw^L}\left(\frac{\partial \mu_\sigma}{\partial w_1^\nu} - \frac{\partial \mu_\nu}{\partial w_1^\sigma}\right) = 0, \tag{25}$$

and

$$\begin{aligned}
&\frac{\partial \mu_L}{\partial w_2^\sigma} + \frac{\partial \mu_\sigma}{\partial w_2^\lambda} w_1^\lambda = 0, \\
&\frac{\partial \mu_L}{\partial w_1^\sigma} - 2\frac{d}{dw^L}\left(\frac{\partial \mu_L}{\partial w_2^\sigma}\right) + \mu_\sigma - \frac{\partial \mu_\sigma}{\partial w_1^\lambda} w_1^\lambda - 2\frac{\partial \mu_\sigma}{\partial w_2^\lambda} w_2^\lambda = 0, \\
&\mu_L - \frac{\partial \mu_L}{\partial w_1^\lambda} w_1^\lambda - \frac{\partial \mu_L}{\partial w_2^\lambda} w_2^\lambda + w_1^\lambda \frac{d}{dw^L}\left(\frac{\partial \mu_L}{\partial w_2^\lambda}\right) = 0, \\
&\frac{\partial \mu_L}{\partial w^\sigma} - \frac{\partial \mu_\sigma}{\partial w^L} - \frac{1}{2}\frac{d}{dw^L}\left(\frac{\partial \mu_L}{\partial w_1^\sigma} - \mu_\sigma + \frac{\partial \mu_\sigma}{\partial w_1^\lambda} w_1^\lambda + 2\frac{\partial \mu_\sigma}{\partial w_2^\lambda} w_2^\lambda\right) = 0.
\end{aligned} \tag{26}$$

To find $\mu_L = \mu_L(w^L, w^\nu, w_1^\nu, w_2^\nu)$ (L fixed) satisfying the previous conditions, we note that it is possible to solve the conditions (26) directly. However, here we apply again the positive homogeneity of the system $\varepsilon_K(y^Q, \dot{y}^Q, \ddot{y}^Q)$. The Theorem 4.1 allow us to choose a positive homogeneous Lagrangian $\mathscr{L} = \mathscr{L}(y^Q, \dot{y}^Q)$ for the system of functions $\varepsilon_K(y^Q, \dot{y}^Q, \ddot{y}^Q)$. We have

$$\varepsilon_K \dot{y}^K = \left(\frac{\partial \mathscr{L}}{\partial y^K} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{y}^K}\right)\dot{y}^K = \frac{\partial \mathscr{L}}{\partial y^K}\dot{y}^K - \frac{\partial^2 \mathscr{L}}{\partial y^Q \partial \dot{y}^K}\dot{y}^Q \dot{y}^K - \frac{\partial^2 \mathscr{L}}{\partial \dot{y}^Q \partial \dot{y}^K}\ddot{y}^Q \dot{y}^K.$$

But differentiating the positive homogeneity condition

$$\mathscr{L} = \frac{\partial \mathscr{L}}{\partial \dot{y}^K}\dot{y}^K,$$

we get

$$\frac{\partial \mathscr{L}}{\partial \dot{y}^Q} = \frac{\partial^2 \mathscr{L}}{\partial \dot{y}^Q \partial \dot{y}^K}\dot{y}^K + \frac{\partial \mathscr{L}}{\partial \dot{y}^Q}$$

hence

$$\varepsilon_K \dot{y}^K = \frac{\partial \mathscr{L}}{\partial y^K}\dot{y}^K - \frac{\partial^2 \mathscr{L}}{\partial y^Q \partial \dot{y}^K}\dot{y}^Q \dot{y}^K = \frac{\partial \mathscr{L}}{\partial y^K}\dot{y}^K - \frac{\partial}{\partial y^Q}\left(\frac{\partial \mathscr{L}}{\partial \dot{y}^K}\dot{y}^K\right)\dot{y}^Q = 0. \tag{27}$$

From (27) and (19) we now obtain

$$0 = \varepsilon_K \dot{y}^K = \tilde{\varepsilon}_\sigma w_1^\sigma \dot{w}^L + \tilde{\varepsilon}_L \dot{w}^L = (\dot{w}^L)^2(\mu_\sigma w_1^\sigma + \mu_L),$$

and thus μ_L is a linear combination of μ_σ of the form $\mu_L = -\mu_\sigma w_1^\sigma$.

Finally, from Theorem 2.1 we see that the conditions (23), (24) and (25) are the necessary and sufficient conditions for the system $\mu_\sigma(w^L, w^\nu, w_1^\nu, w_2^\nu) = 0$ to be variational.

2. Conversely, suppose that in some L-adapted coordinates the system of functions $\tilde{\varepsilon}_K$, defined by (13), is of the form $\tilde{\varepsilon}_K = \mu_K \dot{w}^L$, where $\mu_K = \mu_K(w^L, w^\nu, w_1^\nu, w_2^\nu)$, $\mu_L = -\mu_\sigma w_1^\sigma$, and let the system of functions μ_σ satisfies conditions of variationality (23), (24) and (25). It is now sufficient to verify the Helmholtz conditions (20), (21), (22), and the Zermelo conditions (18) for the system $\tilde{\varepsilon}_K$ and apply the transformation equations into the canonical coordinates, or equivalently, it is sufficient to verify conditions (26) for the function μ_L. This can be, however, proceed by a direct calculation. This completes the proof.

Remark 4.5. Clearly, Theorem 4.4 shows that every positive homogeneous system of $m+1$ second order differential equations, $\varepsilon_K(y^Q, \dot{y}^Q, \ddot{y}^Q) = 0$, is linearly dependent. Moreover, if this system is variational, then its subsystem of m equations is also variational in sense of parameter-invariant variational problems (cf. Theorem 3.2, Theorem 4.1), and vice-versa. The result (b) means that there exists a function $\mathfrak{L} = \mathfrak{L}(w^L, w^\nu, w_1^\nu)$ such that

$$\mu_\sigma = \frac{\partial \mathfrak{L}}{\partial w^\sigma} - \frac{d}{dw^L}\frac{\partial \mathfrak{L}}{\partial w_1^\sigma}. \tag{28}$$

It is worth to note that the adapted coordinates play a crucial role in variational analysis of positive homogeneous systems.

Corollary 4.6. *Suppose that system $\varepsilon_K(y^Q, \dot{y}^Q, \ddot{y}^Q) = 0$ is variational and positive homogeneous, with a positive homogeneous Lagrangian $\mathscr{L} = \mathscr{L}(y^Q, \dot{y}^Q)$ (cf.* Theorem 4.1*), and let $\widetilde{\mathscr{L}}$ be a function defined in the adapted coordinates by $\widetilde{\mathscr{L}}(w^L, \dot{w}^L, w^\nu, w_1^\nu) = \mathscr{L}(y^Q, \dot{y}^Q)$. Then variational system $\mu_\sigma(w^L, w^\nu, w_1^\nu, w_2^\nu) = 0$ (14) has a Lagrangian $\mathfrak{L} = \mathfrak{L}(w^L, w^\nu, w_1^\nu)$ given by*

$$\mathfrak{L} = \frac{\partial \widetilde{\mathscr{L}}}{\partial \dot{w}^L} = \frac{\partial \mathscr{L}}{\partial \dot{y}^L} + \frac{\partial \mathscr{L}}{\partial \dot{y}^\sigma}\frac{\dot{y}^\sigma}{\dot{y}^L}.$$

Proof. Applying the transformation (11) from the canonical to adapted coordinates, we obtain the system of Euler-Lagrange equations of the form

$$\begin{aligned}\mu_\sigma &= \frac{1}{\dot{w}^L}\frac{\partial \widetilde{\mathscr{L}}}{\partial w^\sigma} + \frac{\ddot{w}^L}{(\dot{w}^L)^3}\frac{\partial \widetilde{\mathscr{L}}}{\partial w_1^\sigma} - \frac{1}{\dot{w}^L}\frac{d}{dw^L}\left(\frac{\partial \widetilde{\mathscr{L}}}{\partial w_1^\sigma}\right)\\ &= \frac{\partial\left(\frac{1}{\dot{w}^L}\widetilde{\mathscr{L}}\right)}{\partial w^\sigma} - \frac{d}{dw^L}\left(\frac{\partial\left(\frac{1}{\dot{w}^L}\widetilde{\mathscr{L}}\right)}{\partial w_1^\sigma}\right).\end{aligned}$$

We put $\mathfrak{L} = (1/\dot{w}^L)\widetilde{\mathscr{L}}$, and because of the positive homogeneity of $\widetilde{\mathscr{L}}$ (cf. (18)), we get

$$\mathfrak{L} = \frac{1}{\dot{w}^L}\widetilde{\mathscr{L}} = \frac{1}{\dot{w}^L}\frac{\partial \widetilde{\mathscr{L}}}{\partial \dot{w}^L}\dot{w}^L = \frac{\partial \widetilde{\mathscr{L}}}{\partial \dot{w}^L} = \frac{\partial \mathscr{L}}{\partial \dot{y}^L} + \frac{\partial \mathscr{L}}{\partial \dot{y}^\sigma}\frac{\dot{y}^\sigma}{\dot{y}^L},$$

the first order Lagrangian of the system (14).

5. Example: Second order positive homogeneous variational equations

We consider an example of a system of two second order differential equations of two dependent variables,

$$\begin{aligned}\dot{x}y + \frac{1}{\dot{x}}y(\dot{y}^2 + y\ddot{y}) - \frac{1}{\dot{x}^2}y^2\dot{y}\ddot{x} &= 0,\\ -y\dot{y} - \frac{1}{\dot{x}^2}y\dot{y}(\dot{y}^2 + y\ddot{y}) + \frac{1}{\dot{x}^3}y^2\dot{y}^2\ddot{x} &= 0,\end{aligned} \tag{29}$$

where $x = x(t)$ and $y = y(t)$ are the canonical coordinate functions in $\mathbf{R}^2$. A solution of this system is a regular curve $t \to (x(t), y(t))$ in $\mathbf{R}^2$, satisfying (29); we suppose that $\dot{x}$ is a non-vanishing function at every point of a solution. It can be easily checked that this system is positive homogeneous; this means that the left-hand sides of (29) satisfy the Zermelo conditions (9), (10) from Theorem 3.1. However, Theorem 4.4 shows that equations (29) must be linearly dependent which can be, indeed, observed apparently. On the other hand, system (29) is variational; its left-hand sides satisfy the Helmholtz conditions (4), (5) and (6) from Theorem 2.1. One can directly compute the Vainberg-Tonti Lagrangian (3) for system (29),

$$\mathscr{L} = \frac{1}{3}y(y\dot{x} - x\dot{y}) + \frac{1}{3}\frac{1}{\dot{x}}y^2(\dot{y}^2 + y\ddot{y}) - \frac{1}{3}\frac{1}{\dot{x}^2}y\dot{y}\left(x(\dot{y}^2 + y\ddot{y}) + y^2\ddot{x}\right) + \frac{1}{3}\frac{1}{\dot{x}^3}xy^2\dot{y}^2\ddot{x},$$

and its first order reduction (7),

$$\mathscr{L}_0 = \frac{1}{3}y(y\dot{x} - x\dot{y}) - \frac{1}{2}\frac{1}{\dot{x}}y^2\dot{y}^2. \tag{30}$$

In accordance with Theorem 4.4, we find now an equivalent system (14) with (29): the one differential equation of one dependent variable in adapted coordinates which is variational in sense of (28). We put $w^L = x$, $w^1 = y$, and using coordinate transformation we get from the first equation of (29),

$$\mu = \left(1 + (w^1_1)^2 + w^1 w^1_2\right) w^1 = 0. \tag{31}$$

Indeed, (31) satisfies the Helmholtz condition for one second order equation,

$$\frac{\partial \mu}{\partial w^1_1} - \frac{d}{dw^L}\left(\frac{\partial \mu}{\partial w^1_2}\right) = 0,$$

(cf. Theorem 2.1). The first order Lagrangian for (31), described in Corollary 4.6, is of the form

$$\mathfrak{L} = \frac{1}{3}w^1(w^1 - w^L w^1_1) - \frac{1}{2}(w^1)^2(w^1_1)^2.$$

We note that there exists another first order Lagrangian of equation (31) which does not depend on w^L, namely

$$\mathfrak{L}_0 = \frac{1}{2}(w^1)^2\left(1 - (w^1_1)^2\right).$$

The solution of second order differential equation (31) is the unit circle in $\mathbf{R}^2$.

References

[1] I. Bucataru, A setting for higher order differential equations fields and higher order Lagrange and Finsler spaces, arXiv:1205.0920 (2012)

[2] M. Crampin, D. J. Saunders, Homogeneity and projective equivalence of differential equation fields, J. Geom. Mech. 4 (2012) 27–47.

[3] D. R. Grigore, D. Krupka, Invariants of velocities and higher-order Grassmann bundles, J. Geom. Phys. 24 (1998) 244–264.

[4] P. Havas, The range of application of the Lagrange formalism I, Nuovo Cimento 5 (1957) 363-388.

[5] D. Krupka, M. Krupka, Jets and contact elements, in: D. Krupka (Ed.), Proc. Sem. Diff. Geom., Math. Publ. 2, Silesian Univ. Opava, Czech Republic, 2000, 39–85.

[6] D. Krupka, Z. Urban, Differential invariants of velocities and higher order Grassmann bundles, in: O. Kowalski, D. Krupka, O. Krupková, J. Slovák (Eds.), Diff. Geom. Appl., Proc. Conf., in Honour of L. Euler, Olomouc, August 2007, World Scientific, Singapore, 2008, 463–473.

[7] R. Matsyuk, Autoparallel variational description of the free relativistic top third order dynamics, in: Diff. Geom. Appl., Proc. Conf., Opava, Czech Republic, August 2001, Silesian University in Opava, Czech Republic, 2001, 447–459.

[8] M. A. McKiernan, Sufficiency of parameter invariance conditions in areal and higher order Kawaguchi spaces, Publ. Math. Debrecen 13 (1966) 77–85.

[9] E. Tonti, Variational formulation of nonlinear differential equations I, II, Bull. Classe Sciences Acad. Roy. de Belgiques 55 (1969) 137–165, 262–278.

[10] Z. Urban, D. Krupka, The Zermelo conditions and higher order homogeneous functions, Publ. Math. Debrecen, to appear.

[11] E. Zermelo, Untersuchungen zur Variationsrechnung, Dissertation, Friedrich-Wilhelms-Universität, Berlin, 1894.

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