MEASURABLE 3-COLORINGS OF ACYCLIC GRAPHS

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Abstract. This is the first of two lectures on measurable chromatic numbers given in June 2010 at the University of Barcelona. Our main result here is that acyclic locally finite analytic graphs on Polish spaces admit Baire measurable 3-colorings.

1. Introduction

Suppose that X and Y are sets. A *graph* on X is an irreflexive symmetric set $G \subseteq X \times X$. The *restriction* of G to a set $A \subseteq X$ is given by $G \restriction A = G \cap (A \times A)$. We say that a set $A \subseteq X$ is G-*discrete* if $G \restriction A = \emptyset$. A Y-*coloring* of G is a function $c \colon X \to Y$ for which each set of the form $c^{-1}(\{y\})$ is G-discrete.

A G-*path* is a sequence $(x_i)_{i \in n+1}$ such that $(x_i, x_{i+1}) \in G$ for all $i \in n$. We refer to n as the *length* of the G-path. A graph G is *acyclic* if there is at most one injective G-path between any two points. The axiom of choice trivially implies that every acyclic graph has a two-coloring.

Suppose now that Γ is a pointclass of subsets of Hausdorff spaces. The Γ-*measurable chromatic number* of a graph G on a Hausdorff space X is the least cardinality of a Hausdorff space Y for which there is a Γ-measurable Y-coloring of G. In [KST99], the following is noted:

Theorem 1 (Kechris-Solecki-Todorcevic)**.** *There is an acyclic Borel graph on a Polish space which does not have a Borel ω-coloring.*

We say that X is *analytic* if it is the continuous image of a closed subset of ω^ω. When X is non-empty, this implies that X is the continuous image of ω^ω itself. Here we consider general circumstances under which acyclic locally countable analytic graphs admit measurable 3-colorings.

Some examples are known. In [KST99], the following is established:

Theorem 2 (Kechris-Solecki-Todorcevic)**.** *The Borel chromatic number of the graph associated with any Borel function on an analytic Hausdorff space is* 1*,* 2*,* 3*, or* $\aleph_0$*.*

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Building on this, in [Mil08] it is noted that the Borel chromatic number of the graph associated with a fixed-point-free Borel function is infinite if and only if there is no Borel set such that both it and its complement intersect every forward orbit of f, and this is used to establish the following:

Theorem 3 (Miller). *Under* $\mathtt{ZFC} + \mathtt{add}(\mathtt{null}) = \mathfrak{c}$*, the universally measurable chromatic number and* ω*-universally Baire measurable chromatic number of the graph associated with a Borel function on a Hausdorff space coincide with value* 1, 2*, or* 3*.*

Recall that an equivalence relation is *hyperfinite* if it is the increasing union of finite Borel equivalence relations. We say that an equivalence relation is *measure hyperfinite* if for every Borel probability measure on the underlying space, its restriction to some set of full measure is hyperfinite. In [Mil08], the following is also established:

Theorem 4 (Miller). *Under* $\mathtt{ZFC}+\mathtt{add}(\mathtt{null}) = \mathfrak{c}$*, there is a universally measurable* 3*-coloring of every acyclic analytic graph on a Hausdorff space whose induced equivalence relation is measure hyperfinite.*

On the other hand, in [HK96] the following is established:

Theorem 5 (Hjorth-Kechris). *Every countable analytic equivalence relation on a Polish space is hyperfinite on a comeager set.*

Theorems 1 and 5 rule out the analog of Theorem 4 for ω-universally Baire measurable colorings. Here we establish a salvage for this, as well as versions for other natural notions of measurability.

2. Measurable colorings of locally finite graphs

Suppose that X is a Hausdorff space and E is a countable analytic equivalence relation on X. Let X_E denote the subspace of $X^{<\omega}$ consisting of all finite injective sequences x with the property that $x(i) E x(j)$ for all $i, j \in \mathrm{dom}(x)$. Let G_E denote the graph on X_E consisting of all pairs of distinct sequences whose images have non-empty intersection.

Proposition 6. *Suppose that* X *is a Hausdorff space and* E *is a countable analytic equivalence relation on* X*. Then there is a Borel* ω*-coloring of* G_E*.*

Proof. We will show that there are countably many G_E-discrete Borel subsets of X_E whose union is X_E. Clearly we can assume that $X \neq \emptyset$, so there is a continuous surjection $\varphi \colon \omega^\omega \to X$. For each $l \in \omega$, set

$$S_l = \{s \in (\omega^{<\omega})^l \mid (\varphi(\mathcal{N}_{s(k)}))_{k \in l} \text{ is pairwise disjoint})\}.$$

The first separation theorem ensures that for each $s \in S_l$, there is a pairwise disjoint sequence $(B_{k,s})_{k \in l}$ of Borel subsets of X such that $\varphi(\mathcal{N}_{s(k)}) \subseteq B_{k,s}$ for all $k \in l$. By the Lusin-Novikov uniformization theorem, there are partial functions $f_n \colon X \rightharpoonup X$, whose graphs are Borel subsets of E, such that $E = \bigcup_{n \in \omega} \operatorname{graph}(f_n)$. Let P denote the family of triples of the form $p = (l, n, s)$, where $l \in \omega$, $n \in \omega^{l \times l}$, and $s \in S_l$. For every such p, define $B_p \subseteq X^l$ by

$$x \in B_p \iff \forall j, k \in l \ (x(k) \in B_{k,s} \text{ and } f_{n(j,k)} \circ x(j) = x(k)).$$

Clearly B_p is a Borel subset of X_E.

Lemma 7. *Suppose that $p = (l, n, s)$ is in P. Then B_p is G_E-discrete.*

Proof of lemma. Suppose that $x, y \in B_p$. If $x(j) \neq y(k)$ for all $j, k \in l$, then $(x, y) \notin G_E$. Otherwise, the fact that $(B_{k,s})_{k \in l}$ is pairwise disjoint ensures the existence of $j \in l$ with $x(j) = y(j)$. The definition of B_p then implies that $x(k) = f_{n(j,k)} \circ x(j) = f_{n(j,k)} \circ y(j) = y(k)$ for all $k \in n$, so $x = y$, thus $(x, y) \notin G_E$, and the lemma follows. ⊠

As $|P| = \aleph_0$ and $X_E = \bigcup_{p \in P} B_p$, the proposition follows. ⊠

We say that a σ-ideal $\mathcal{I}$ on X *approximates analytic sets* if for all analytic sets $A \subseteq X$, there are Borel sets $B, C \subseteq X$ with the property that $B \subseteq A \subseteq C$, $C \setminus B \in \mathcal{I}$, and both B and $X \setminus C$ are standard Borel. This trivially implies the more general fact for sets in the smallest σ-algebra containing the analytic sets.

Proposition 8. *Suppose that X is a standard Borel space, Y is a Hausdorff space, $\mathcal{I}$ is a σ-ideal on X which approximates analytic sets, and $\varphi \colon X \to Y$ is Borel. Then the family $\varphi_* \mathcal{I} = \{B \subseteq Y \mid \varphi^{-1}(B) \in \mathcal{I}\}$ is a σ-ideal on Y which approximates analytic sets.*

Proof. Clearly $\varphi_* \mathcal{I}$ is a σ-ideal on Y, so it is enough to show that if $A_Y \subseteq Y$ is in the smallest σ-algebra containing the analytic sets, then there is a standard Borel set $B_Y \subseteq Y$ such that $B_Y \subseteq A_Y$ and $A_Y \setminus B_Y \in \varphi_* \mathcal{I}$. Towards this end, set $A_X = \varphi^{-1}(A_Y)$, fix a Borel set $B_X \subseteq X$ such that $B_X \subseteq A_X$ and $A_X \setminus B_X \in \mathcal{I}$, and let E denote the equivalence relation on X given by $xEy \iff \varphi(x) = \varphi(y)$.

By the Jankov-von Neumann uniformization theorem, there is a left inverse ψ for φ which is measurable with respect to the smallest σ-algebra containing the analytic sets. Then so too is $\psi \circ \varphi$, thus there is an $\mathcal{I}$-conull Borel set $C_X \subseteq X$ such that $(\psi \circ \varphi) \restriction C_X$ is Borel. Then $(\psi \circ \varphi)(B_X \cap C_X)$ is an analytic partial transversal of E, so by the first separation theorem it is contained in a Borel partial transversal $D_X \subseteq B_X \cap C_X$ of E, thus the set $B_Y = \varphi(D_X)$ is as desired. ⊠

Let E_G denote the minimal equivalence relation on X containing G. The *connected component* of a point is its E_G-class.

Proposition 9. *Suppose that X is a Hausdorff space, G is a locally countable analytic graph on X, and $\mathcal{I}$ is a σ-ideal on X which approximates analytic sets. Then there is an E_G-invariant $\mathcal{I}$-conull standard Borel set $B \subseteq X$ on which G is Borel.*

Proof. Clearly we can assume that $X = \operatorname{dom}(G)$. Set $E = E_G$. By Proposition 6, there are involutions $\iota_n \colon X \to X$ with analytic graphs such that $E = \bigcup_{n \in \omega} \operatorname{graph}(\iota_n)$. Set

$$A_{m,n} = \{x \in X \mid (\iota_m(x), \iota_n(x)) \in G\}$$

for all $m, n \in \omega$. Then there is an $\mathcal{I}$-conull standard Borel set $C \subseteq X$ with the property that each of the sets $C_{m,n} = A_{m,n} \cap C$ is standard Borel and each of the functions $\iota_n \restriction C \colon C \to X$ is Borel. Noting that $[C]_E = \bigcup_{n \in \omega} \iota_n(C)$ and $G \restriction [C]_E = \bigcup_{m,n \in \omega} (\iota_m \times \iota_n)(C_{m,n})$, it follows that these sets are also standard Borel. ⊠

A *G-barrier* is a set $Y \subseteq X$ for which every connected component of $G \restriction (X \setminus Y)$ is finite.

Proposition 10. *Suppose that X is a Polish space, Y is a Hausdorff space, G is an acyclic locally finite analytic graph on Y, and $\varphi \colon X \to Y$ is continuous. Then there is an E_G-invariant Borel set $C \subseteq Y$ such that $\varphi^{-1}(C)$ is comeager and $G \restriction C$ admits a discrete Borel barrier.*

Proof. By Proposition 9, we can assume that Y is standard Borel and G is Borel. The *G-boundary* of a set $B \subseteq Y$ is given by

$$\partial_G(B) = \{x \in B \mid \exists y \in Y \setminus B \ ((x, y) \in G)\}.$$

Let $\mathscr{Y}$ denote the standard Borel space of all triples (y, S, T) with the property that $\{y\} \cup S \cup \partial_G(Y \setminus S) \subseteq T$, $T \times T \subseteq E$, and $|T| < \aleph_0$. Let $\mathscr{G}$ denote the locally countable Borel graph on $\mathscr{Y}$ consisting of all pairs of distinct triples whose final entries are not disjoint. Proposition 6 easily implies that there is a Borel ω-coloring c of $\mathscr{G}$.

We say that a sequence is a *(G, B)-comb* if its image is contained in $\partial_G(Y \setminus B)$. We will now define an increasing sequence $(B_s)_{s \in \omega^{<\omega}}$ of G-discrete Borel subsets of Y such that for no $s \in \omega^{<\omega}$ is there an injective G-ray which is a (G, B_s)-comb.

We begin by setting $B_\emptyset = \emptyset$. Suppose now that $s \in \omega^{<\omega}$ and we have already found B_s. Let $\mathscr{B}_s$ denote the Borel set of all $(y, S, T) \in \mathscr{Y}$ which satisfy the following conditions:

(1) The set $B_s \cup S$ is G-discrete.

(2) There is no injective G-path from $\partial_G(Y \setminus S)$ to $\partial_G(T)$ which is a $(G, B_s \cup S)$-comb.
(3) If $y \notin B_s \cup S$, then $[y]_{E_{G \upharpoonright (Y \setminus (B_s \cup S))}}$ is finite.

For each $i \in \omega$, define

$$B_{s \frown i} = B_s \cup \bigcup \{S \mid \exists y \exists T \ ((y, S, T) \in \mathscr{B}_s \text{ and } c(y, S, T) = i)\}.$$

Lemma 11. *Suppose that $i \in \omega$ and $s \in \omega^{<\omega}$. Then $B_{s \frown i}$ is G-discrete.*

Proof of lemma. Suppose that $x, y \in B_{s \frown i}$. If both x and y are in B_s, then the G-discreteness of B_s ensures that $(x, y) \notin G$. If exactly one of x and y is in B_s, then there is a finite set S such that $B_s \cup S$ is a G-discrete set containing both x and y, thus $(x, y) \notin G$. If neither x nor y is in B_s, then there are finite sets S_x and S_y containing x and y such that $S_x \cap S_y = \partial_G(Y \setminus S_x) \cap S_y = \emptyset$, thus $(x, y) \notin G$. ⊠

Lemma 12. *Suppose that $i \in \omega$ and $s \in \omega^{<\omega}$. Then no injective G-ray is a $(G, B_{s \frown i})$-comb.*

Proof of lemma. Observe that any injective G-ray which is a $(G, B_{s \frown i})$-comb would necessarily be a (G, B_s)-comb. ⊠

For each $s \in \omega^{<\omega}$, define $E_s = E_{G \upharpoonright (Y \setminus B_s)}$. For each $p \in \omega^\omega$, set $B_p = \bigcup_{n \in \omega} B_{p \upharpoonright n}$ and $E_p = \bigcap_{n \in \omega} E_{p \upharpoonright n}$. Clearly B_p is G-discrete.

Lemma 13. *Suppose that $s \in \omega^{<\omega}$, $x \in X$, and $y \in [\varphi(x)]_E$. Then there exists $i \in \omega$ such that $y \in B_{s \frown i}$ or $[y]_{E_{s \frown i}}$ is finite.*

Proof of lemma. Clearly we can assume that $y \notin B_s$. Let S denote the set of points of terminal points of maximal injective $G \upharpoonright (Y \setminus B_s)$-paths $(y_i)_{i \in n+1}$ originating at y for which $(y_i)_{i \in n}$ is a (G, B_s)-comb. König's lemma easily implies that S is finite. Fix a finite set T such that $\{y\} \cup S \cup \partial_G(S) \subseteq T$ and $T \times T \subseteq E$. As $(y, S, T) \in \mathscr{B}_s$, it follows that $i = c(y, S, T)$ is as desired. ⊠

Lemma 13 implies that

$$\forall x \in X \forall y \in [\varphi(x)]_E \forall^* p \in \omega^\omega \ (y \in B_p \text{ or } [y]_{E_p} \text{ is finite}).$$

As E is countable, it follows that

$$\forall x \in X \forall^* p \in \omega^\omega \forall y \in [\varphi(x)]_E \ (y \in B_p \text{ or } [y]_{E_p} \text{ is finite}).$$

The Kuratowski-Ulam theorem therefore implies that

$$\forall^* p \in \omega^\omega \forall^* x \in X \forall y \in [\varphi(x)]_E \ (y \in B_p \text{ or } [y]_{E_p} \text{ is finite}),$$

which completes the proof of the theorem. ⊠

Proposition 14. *Suppose that X is a Polish space, Y is a Hausdorff space, G is an acyclic locally finite analytic graph on Y, and $\varphi\colon X \to Y$ is continuous. Then there is an E_G-invariant Borel set $C \subseteq Y$ such that $\varphi^{-1}(C)$ is comeager and $G \restriction C$ admits a Borel 3-coloring.*

Proof. By Proposition 9, we can assume that Y is standard Borel and G is Borel. By Proposition 10, there is a E_G-invariant Borel set $C \subseteq X$ such that $\varphi^{-1}(C)$ is comeager and $G \restriction C$ admits a discrete Borel barrier $B \subseteq C$. Then $E \restriction (C \setminus B)$ is a finite Borel equivalence relation, and therefore admits a Borel transversal $D \subseteq C \setminus B$. For each $x \in C \setminus B$, let $d(x, D)$ denote the distance from x to D with respect to the usual graph metric on $G \restriction (C \setminus B)$, and define $c\colon C \to 3$ by

$$c(x) = \begin{cases} d(x, D) \pmod 2 & \text{if } x \in C \setminus B, \\ 2 & \text{if } x \in B. \end{cases}$$

It is clear that c is the desired 3-coloring. ⊠

Theorem 15 ($\mathtt{ZFC}$)**.** *Suppose that X is a Polish space and G is an acyclic locally finite analytic graph on X. Then there is a Baire measurable 3-coloring of G.*

Proof. By Proposition 14, there is a comeager E_G-invariant Borel set $C \subseteq X$ on which G admits a Borel 3-coloring. The axiom of choice ensures that any such function extends to a 3-coloring of G, and any such extension is clearly Baire measurable. ⊠

A set $B \subseteq X$ is *ω-universally Baire* if for every continuous function $\varphi\colon \omega^\omega \to X$ the set $\varphi^{-1}(B)$ has the Baire property.

Theorem 16 ($\mathtt{ZFC} + \mathtt{add}(\mathtt{meager}) = \mathfrak{c}$)**.** *Suppose that X is a Hausdorff space and G is an acyclic locally finite analytic graph on X. Then there is an ω-universally Baire measurable 3-coloring of G.*

Proof. Clearly we can assume that $X = \mathrm{dom}(G)$. Let $C(X, Y)$ denote the set of all continuous functions from X to Y. Fix enumerations $(\varphi_\alpha)_{\alpha \in \mathfrak{c}}$ of $C(\omega^\omega, X)$ and $(x_\alpha)_{\alpha \in \mathfrak{c}}$ of X. Proposition 14 ensures that for all $\alpha \in \mathfrak{c}$ there is an E_G-invariant Borel set $C_\alpha \subseteq X$ such that $x_\alpha \in C_\alpha$, $\varphi_\alpha^{-1}(C_\alpha)$ is comeager, and there is a Borel 3-coloring c_α of $G \restriction C_\alpha$. Set $D_\alpha = C_\alpha \setminus \bigcup_{\beta \in \alpha} C_\beta$ for all $\alpha \in \mathfrak{c}$, and observe that the function $c = \bigcup_{\alpha \in \mathfrak{c}} c_\alpha \restriction D_\alpha$ is as desired. ⊠

Suppose that X is a Hausdorff space and $\mathcal{I}$ is an ideal on X. We say that a set $B \subseteq X$ is *$\mathcal{I}$-measurable* if for every $\mathcal{I}$-positive analytic set $A \subseteq X$, there is an $\mathcal{I}$-positive analytic set contained in $A \cap B$ or $A \setminus B$.

Suppose that X and Y are Hausdorff spaces and $\mathcal{I}$ and $\mathcal{J}$ are ideals on X and Y. A *cohomomorphism* from X to Y is a function $\varphi\colon X \to Y$ which sends $\mathcal{I}$-positive analytic subsets of X to $\mathcal{J}$-positive analytic subsets of Y.

We say that $\mathcal{I}$ is a *Baire ideal* if for every $\mathcal{I}$-positive analytic set $A \subseteq X$, there is a continuous cohomomorphism from the ideal of meager subsets of ω^ω to $\mathcal{I} \restriction A$. Obviously the ideal of meager subsets of a Polish space is a Baire ideal, as is the ideal of Silver null subsets of 2^ω. As far as we know, every structural dichotomy theorem of descriptive set theory can be rephrased as asserting that a natural auxiliary ideal is a Baire ideal.

It is easy to see that if X is a Hausdorff space and $\mathcal{I}$ is a Baire ideal on X, then every ω-universally Baire subset of X is $\mathcal{I}$-measurable, thus every ω-universally Baire measurable function on X is $\mathcal{I}$-measurable. In particular, it follows that under $\mathtt{add}(\mathtt{meager}) = \mathfrak{c}$, every acyclic locally finite analytic graph on a Hausdorff space admits a 3-coloring which is simultaneously measurable with respect to all ideals arising from structural dichotomy theorems.

3. Measurable colorings of locally countable graphs

It is well known that there is an acyclic locally countable Borel graph on 2^ω which does not have a Baire measurable ω-coloring. As we shall see in this section, there is nevertheless a natural family of ideals $\mathcal{I}$ for which such graphs admit $\mathcal{I}$-measurable 3-colorings.

Let m denote the usual product measure on 2^ω. We say that $\mathcal{I}$ is a *Lebesgue ideal* if for every $\mathcal{I}$-positive analytic set $A \subseteq X$, there is a continuous cohomomorphism from the ideal of m-null subsets of 2^ω to $\mathcal{I}$. Given a probability measure μ on a Polish space X, the ideal of μ-null subsets of X is a Baire ideal, as is the ideal of Silver null subsets of 2^ω. Many of the ideals associated with descriptive set-theoretic dichotomy theorems are also Lebesgue ideals.

Proposition 17. *Suppose that X is a Hausdorff space, G is an acyclic locally countable analytic graph on 2^I, and $\mathcal{I}$ is an ideal on X which is both a Baire ideal and a Lebesgue ideal. Then every $\mathcal{I}$-positive analytic set $A \subseteq X$ contains an $\mathcal{I}$-positive Borel set $B \subseteq X$ such that $G \restriction [B]_E$ admits a Borel 3-coloring.*

Proof. By Proposition 9, we can assume that X is standard Borel and G is Borel. Set $E = E_G$. As $\mathcal{I}$ is a Baire ideal, there is a continuous cohomomorphism $\varphi\colon \omega^\omega \to X$ from the ideal of meager subsets of ω^ω to $\mathcal{I}$. By Theorem 5, there is a Borel set $C \subseteq A$ such that $\varphi^{-1}(C)$ is comeager and $E \restriction C$ is hyperfinite.

As C is $\mathcal{I}$-positive and $\mathcal{I}$ is a Lebesgue ideal, there is a continuous cohomomorphism $\psi\colon 2^\omega \to C$ from the ideal of m-null subsets of 2^ω to $\mathcal{I} \restriction C$. By Theorem 4, there is a Borel set $B \subseteq C$ such that $\varphi^{-1}(B)$ is m-conull and $G \restriction [B]_E$ admits a Borel 3-coloring. $\boxtimes$

Theorem 18 ($\mathtt{ZFC} + \mathtt{add}(\mathtt{meager}) = \mathfrak{c}$)**.** *Suppose that X is a Hausdorff space, G is an acyclic locally countable analytic graph on X, and $\mathcal{I}$ is an ideal on X which is a Baire ideal and a Lebesgue ideal. Then there is an $\mathcal{I}$-measurable 3-coloring of G.*

Proof. Clearly we can assume that $X = \mathrm{dom}(G)$. Fix enumerations $(A_\alpha)_{\alpha\in\mathfrak{c}}$ of all $\mathcal{I}$-positive analytic subsets of X and $(x_\alpha)_{\alpha\in\mathfrak{c}}$ of X. Proposition 17 ensures that for all $\alpha \in \mathfrak{c}$ there is an $\mathcal{I}$-positive Borel set $B_\alpha \subseteq A_\alpha$ such that $x_\alpha \in B_\alpha$ and there is a Borel 3-coloring c_α of $G \restriction B_\alpha$. Set $C_\alpha = [B_\alpha]_E \setminus \bigcup_{\beta\in\alpha}[B_\beta]_E$ for all $\alpha \in \mathfrak{c}$, and observe that the function $c = \bigcup_{\alpha\in\mathfrak{c}} c_\alpha \restriction C_\alpha$ is as desired. $\boxtimes$

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