On the algebraic Bethe ansatz approach to correlation functions: the Heisenberg spin chain

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and more recently: J. S. Caux, K. Kozlowski, G. Niccoli. . .

ENIGMA School 07 – Lalonde les Maures

Outline

1. Introduction
   - The Heisenberg spin-1/2 chain
   - Exact computations of correlation functions
2. Basis of the method
   - Correlation functions in the finite chain
   - Elementary blocks in the thermodynamic limit
   - A simple example: the emptiness formation probability
3. Further analysis: the two-point function
   - Analytical + Numerical methods
   - Analytical resummations for the two-point function

The Heisenberg spin chain

- Model for magnetism in solids (Heisenberg, 1928)
  ⋆ Crystals with effective one-dimensional magnetic properties
  ⋆ Can be tested via inelastic neutron scattering experiments
- Archetype of quantum integrable models
  ⋆ Spectrum resolution via Bethe ansatz (1931) and its developments
  ⋆ Links to two-dimensional statistical mechanics (vertex models generalizing Ising)
- Very rich (non-commutative) algebraic structures
  ⋆ Yang-Baxter algebras, R-matrices, Quantum groups
  ⋆ They appear in different situations eventually far from magnetism (Gauge and String theories and AdS/CFT correspondence)
  ⋆ Link to combinatorics in special point (ice model)

The spin-1/2 XXZ Heisenberg chain

The *XXZ* spin-1/2 Heisenberg chain in a magnetic field is a quantum interacting model defined on a one-dimensional lattice with M sites, with Hamiltonian, $H = H^{(0)} - hS_z$,

$$H^{(0)} = \sum_{m=1}^{M} \left\{ \sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta(\sigma_m^z \sigma_{m+1}^z - 1) \right\},$$

$$S_z = \frac{1}{2} \sum_{m=1}^{M} \sigma_m^z, \qquad [H^{(0)}, S_z] = 0.$$

Quantum space of states : $\mathcal{H} = \otimes_{m=1}^{M} \mathcal{H}_m$, $\mathcal{H}_m \sim \mathbb{C}^2$, $\dim \mathcal{H} = 2^M$.

$\sigma_m^{x,y,z}$ are the local spin operators (in the spin-$\frac{1}{2}$ representation) at site m: they act as the corresponding Pauli matrices in the space $\mathcal{H}_m$ and as the identity operator elsewhere.

$+$ periodic boundary conditions

Correlation functions of Heisenberg chain

- Free fermion point $\Delta = 0$: Lieb, Shultz, Mattis, Wu, McCoy, Sato, Jimbo, Miwa, . . .
- From 1984: Izergin, Korepin,... (first attempts using Bethe ansatz for general Δ)
- General Δ: multiple integral representations
  $\star$ 1992-96 Jimbo and Miwa $\longrightarrow$ from q-vertex op. and qKZ eq.
  $\star$ 1999 Kitanine, Maillet, Terras $\longrightarrow$ from Algebraic Bethe Ansatz
- Several developments since 2000: Kitanine, Maillet, Slavnov, Terras; Boos, Korepin, Smirnov; Boos, Jimbo, Miwa, Smirnov, Takeyama; Göhmann, Klümper, Seel; Caux, Hagemans, Maillet . . .

Correlation functions

$$\langle \mathcal{O} \rangle = \frac{\mathrm{tr}_{\mathcal{H}} \left(\mathcal{O} \, e^{-\mathbf{H}/kT} \right)}{\mathrm{tr}_{\mathcal{H}} \left(e^{-\mathbf{H}/kT} \right)}$$

$$= \langle \psi_g | \, \mathcal{O} \, | \psi_g \rangle \qquad \text{at} \quad T = 0$$

where $|\psi_g\rangle$ is the state with lowest eigenvalue.

Why is it so difficult? (Bethe ansatz already 75 years old...!)

Main problems to be solved to achieve this :

- Compute exact eigenstates and energy levels of the Hamiltonian (Bethe ansatz)
- Obtain the action of local operators on the eigenstates: main problem since eigenstates are highly non-local!
- Compute the resulting scalar products with the eigenstates

The methods...

- q-KZ and q-vertex operators :
  ⋆ Valid (with some hypothesis) for infinite (and semi-infinite) chains, zero magnetic field and zero temperature
  ⋆ Elementary blocks of correlation functions (static) and form factors (massive case)
  ⋆ Multiple integrals and recently algebraic solutions of q-KZ
- Bethe ansatz
  ⋆ Valid for finite and infinite chains, with magnetic field and temperature, and with impurities or with integrable boundaries (open chain)
  ⋆ Determinant representation of form factors (finite chain), multiple integrals for correlation functions (infinite chain), master formula for spin-spin correlation functions.
  ⋆ Some results for a continuum model (NLS)

Algebraic Bethe ansatz and correlation functions

Compute $\langle \psi_g | \prod_j \sigma_j^{\alpha_j} | \psi_g \rangle$?

1. **Diagonalise the Hamiltonian using ABA**
   (Faddeev, Sklyanin, Takhtajan, 1979)
   $\longrightarrow$ key point : Yang-Baxter algebra $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$
   $\longrightarrow$ eigenstates: $B(\lambda_1) \dots B(\lambda_n)|0\rangle$

2. **Act with local operators on eigenstates**
   $\longrightarrow$ problem: relation between B (creation) and σ_j^{α} a priori very complicated !
   $\longrightarrow$ solve the quantum inverse problem (Kitanine, Maillet, V.T., 1999):
   $$\sigma_j^{\alpha_j} = f_j^{\alpha_j}(A, B, C, D) = \prod(A, B, C, D)$$
   $\longrightarrow$ use Yang-Baxter commutation relations

3. **Compute the resulting scalar products**
   (Slavnov; Kitanine, Maillet, V.T.)

4. **Thermodynamic limit**
   $\longrightarrow$ elementary building blocks of correlation functions as multiple integrals (2000)

5. **Two-point function : further analysis. . .** (since 2002)

Diagonalization of the Hamiltonian via ABA

$\sigma_n^\alpha \longrightarrow$ **monodromy matrix** $T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_{[a]}$

with $T(\lambda) \equiv T_{a,1\dots M}(\lambda) = L_{aM}(\lambda - \xi_N) \dots L_{a2}(\lambda - \xi_2) L_{a1}(\lambda - \xi_1)$

$$L_{an}(\lambda) = \begin{pmatrix} \sinh(\lambda + \eta\sigma_n^z) & \sinh\eta\,\sigma_n^- \\ \sinh\eta\,\sigma_n^+ & \sinh(\lambda - \eta\sigma_n^z) \end{pmatrix}_{[a]}$$

a →auxiliary space $\simeq \mathbb{C}^2$
n →local quantum space at site n

↪ **Yang-Baxter algebra**:
- ∘ generators A, B, C, D
- ∘ commutation relations given by the R-matrix of the model

$$R_{ab}(\lambda,\mu)\, T_a(\lambda) T_b(\mu) = T_b(\mu) T_a(\lambda)\, R_{ab}(\lambda,\mu)$$

→ commuting conserved charges: $t(\lambda) = A(\lambda) + D(\lambda)$ $\quad [t(\lambda), t(\mu)] = 0$

$H = 2\sinh\eta\, \frac{\partial}{\partial\lambda} \log t(\lambda)\big|_{\lambda=\frac{\eta}{2}} + c$ for all $\xi_j = 0$

→ construction of the space of states by action of B (creation) and C (annihilation) on a reference state $|0\rangle \equiv |\uparrow\uparrow\dots\uparrow\rangle$

eigenstates : $|\psi\rangle = \prod_k B(\lambda_k)|0\rangle$ with $\{\lambda_k\}$ solution of the Bethe equations.

Action of local operators on eigenstates

Solution of the quantum inverse scattering problem ($\sigma_n^\alpha \longleftarrow T(\lambda)$)

$$\sigma_n^- = \prod_{k=1}^{n-1} t(\xi_k) \cdot B(\xi_n) \cdot \prod_{k=1}^{n} t^{-1}(\xi_k)$$

$$\sigma_n^+ = \prod_{k=1}^{n-1} t(\xi_k) \cdot C(\xi_n) \cdot \prod_{k=1}^{n} t^{-1}(\xi_k)$$

$$\sigma_n^z = \prod_{k=1}^{n-1} t(\xi_k) \cdot (A - D)(\xi_n) \cdot \prod_{k=1}^{n} t^{-1}(\xi_k)$$

$\longrightarrow$ use the Yang-Baxter commutation relations for A, B, C, D to get the action on arbitrary states:

$$\langle 0| \prod_{k=1}^{N} C(\lambda_k) \cdot \prod_{j=1}^{m} T_{\varepsilon_j, \varepsilon'_j}(\lambda_{N+j}) = \sum_{\mathcal{P} \subset \{\lambda\}} \Omega_{\mathcal{P}}(\{\lambda\}, \{\epsilon_j, \epsilon'_j\}) \, \langle 0| \prod_{b \in \mathcal{P}} C(\lambda_b)$$

$\longrightarrow$ correlation functions = sums over scalar products

Computation of scalar products

Scalar product

$$\underbrace{\langle 0| \prod_{l=1}^{N} C(\mu_l)}_{\text{arbitrary state}} \cdot \underbrace{\prod_{k=1}^{N} B(\lambda_k)|0\rangle}_{\text{eigenstate}} = \frac{\det\ U(\{\mu_l\},\{\lambda_k\})}{\det\ V(\{\mu_l\},\{\lambda_k\})},$$

with $\quad U_{ab} = \partial_{\lambda_a}\tau(\mu_b,\{\lambda_k\}), \quad V_{ab} = \dfrac{1}{\sinh(\mu_b - \lambda_a)}, \quad 1 \leqslant a, b \leqslant N,$

where $\tau(\mu_b,\{\lambda_k\})$ is the eigenvalue of the transfer matrix $t(\mu_b)$.

$\longrightarrow$ **"m-point" elementary blocks for the correlation functions** in the finite chain:

$$\langle \psi_g | \prod_{j=1}^{m} E_j^{\epsilon_j',\epsilon_j} |\psi_g\rangle = \underbrace{\sum\sum\ldots\sum}_{m \text{ sums}} \Omega_m(\{\lambda\},\{\epsilon_j,\epsilon_j'\})\ \det{}_m \tilde{M}$$

with $(E^{\epsilon',\epsilon})_{lk} = \delta_{l,\epsilon'}\delta_{k,\epsilon}$

Matrix elements of local operators

For example :

$$\langle 0| \prod_{j=1}^{N} C(\mu_j)\ \sigma_n^z \prod_{k=1}^{N} B(\lambda_k)\ |0\rangle$$

$$= \langle 0| \prod_{j=1}^{N} C(\mu_j) \prod_{k=1}^{n-1} t(\xi_k) \cdot (A-D)(\xi_n) \cdot \prod_{k=1}^{n} t^{-1}(\xi_k) \prod_{k=1}^{N} B(\lambda_k)\ |0\rangle$$

Here the sets $\{\lambda_k\}$ and $\{\mu_j\}$ are both solutions of Bethe equations $\longrightarrow$

$$\langle 0| \prod_{j=1}^{N} C(\mu_j)\ \sigma_n^z \prod_{k=1}^{N} B(\lambda_k)\ |0\rangle = \Phi_n\ \langle 0| \prod_{j=1}^{N} C(\mu_j)\ (A-D)(\xi_n) \prod_{k=1}^{N} B(\lambda_k)\ |0\rangle$$

$$= \Phi_n\ \langle \tilde{\psi}| \prod_{k=1}^{N} B(\lambda_k)\ |0\rangle$$

$\rightsquigarrow$ determinant representations of matrix elements (using the scalar product formula)

Elementary blocks in the thermodynamic limit

Sums become integrals:

$$\frac{1}{M}\sum_{j=1}^{N} f(\lambda_j) \underset{M\to\infty}{\longrightarrow} \int_{C_h} f(\lambda)\rho(\lambda)d\lambda$$

$\{\lambda_j\}\rightarrow$ solution of Bethe eq. for the ground state
$\rho(\lambda)\rightarrow$ density of the ground state
solution of a linear integral eq.

$\longrightarrow$ **multiple integral representation for the "m-point" elementary building blocks of the correlation functions**

$$\langle\psi_g|\prod_{j=1}^{m} E_j^{\epsilon_j',\epsilon_j}|\psi_g\rangle = \int_{C_h} d^m\lambda \;\; \Omega_m(\{\lambda_k\},\{\epsilon_j,\epsilon_j'\}) \; \det_m S_h(\{\lambda_k\})$$

where $\Omega_m(\{\lambda_k\},\{\epsilon_j,\epsilon_j'\})$ is purely algebraic and $S_h(\{\lambda_k\})$, C_h depend on the regime and on the magnetic field h.

$\longrightarrow$ Proof of the results and conjectures of Jimbo, Miwa et al. + extension to non-zero magnetic field; more recently, extension to time dependent (KMST) and non zero temperature (Göhmann, Klümper, Seel)

What about this result ?

→ A priori, the problem is solved:

- expression of all elementary blocks $\langle\psi_g|\,E_1^{\epsilon_1',\epsilon_1}\ldots E_m^{\epsilon_m',\epsilon_m}\,|\psi_g\rangle$
- any correlation function $=\sum$ (elementary blocks)

→ From a practical point of view, there are two main problems:

(1) physical correlation function = HUGE sum of elementary blocks at large distances

Example: two-point function

$$\langle\psi_g|\sigma_1^z\,\sigma_m^z|\psi_g\rangle \equiv \langle\psi_g|(E_1^{11}-E_1^{22})\underbrace{\prod_{j=2}^{m-1}(E_j^{11}+E_j^{22})}_{\text{propagator}}(E_m^{11}-E_m^{22})|\psi_g\rangle$$

$$=\sum_{2^m\ \text{terms}}\text{(elementary blocks)}\quad\underset{m\to\infty}{\sim}\ ?$$

⇝ **re-summation ?**

(2) each block has a complicated expression

Example: emptiness formation probability for $h = 0$ in the massless regime $(-1 < \Delta = \cosh\zeta < 1)$

$$\tau(m) \equiv \langle\psi_g| \prod_{k=1}^{m} \frac{1-\sigma_k^z}{2} |\psi_g\rangle$$

$$= (-1)^m \Big(-\frac{\pi}{\zeta}\Big)^{\frac{m(m-1)}{2}} \int_{-\infty}^{\infty} \frac{d^m\lambda}{2\pi} \prod_{a>b}^{m} \frac{\sinh\frac{\pi}{\zeta}(\lambda_a-\lambda_b)}{\sinh(\lambda_a-\lambda_b-i\zeta)}$$

$$\times \prod_{j=1}^{m} \frac{\sinh^{j-1}(\lambda_j - i\zeta/2)\sinh^{m-j}(\lambda_j+i\zeta/2)}{\cosh^m \frac{\pi}{\zeta}\lambda_j}$$

$\rightsquigarrow$ **dependence on m ?**

(1)+(2) $\Rightarrow$ need further analysis!

A simple example: the emptiness formation probability

Integral representation as a single elementary block but previous expression not symmetric

$\longrightarrow$ **symmetrisation** of the integrand:

$$\tau(m) = \lim_{\xi_1,\dots\xi_m \to -\frac{i\zeta}{2}} \frac{1}{m!} \int_{-\infty}^{\infty} d^m\lambda \prod_{a,b=1}^{m} \frac{1}{\sinh(\lambda_a - \lambda_b - i\zeta)}$$

$$\times \prod_{a<b}^{m} \frac{\sinh(\lambda_a - \lambda_b)}{\sinh(\xi_a - \xi_b)} \cdot Z_m(\{\lambda\},\{\xi\}) \cdot \det_m[\rho(\lambda_j,\xi_k)]$$

where $Z_m(\{\lambda\},\{\xi\})$ is the partition function of the 6-vertex model with domain wall boundary conditions and $\rho(\lambda,\xi) = [-2i\zeta \sinh \frac{\pi}{\zeta}(\lambda_j - \xi_k)]^{-1}$ is the inhomogeneous version of the density for the ground state (massless regime $\Delta = \cos\zeta$, $h = 0$).

$\longrightarrow$ (1) Exact computation for $\Delta = 1/2$
(2) Asymptotic behaviour for $m \longrightarrow \infty$

Exact computation for $\Delta = 1/2$

The determinant structure combined with the periodicity properties at $\Delta = 1/2$ enable us to separate and compute the multiple integral :

$$\tau_{inh}(m, \{\xi_j\}) = \frac{(-1)^{\frac{m^2-m}{2}}}{2^{m^2}} \prod_{a>b}^{m} \frac{\sinh 3(\xi_b - \xi_a)}{\sinh(\xi_b - \xi_a)} \times \prod_{\substack{a,b=1 \\ a \neq b}}^{m} \frac{1}{\sinh(\xi_a - \xi_b)} \cdot \det_m \left(\frac{3 \sinh \frac{\xi_j - \xi_k}{2}}{\sinh \frac{3(\xi_j - \xi_k)}{2}} \right).$$

In the homogeneous limit:

$$\tau(m) = \left(\frac{1}{2}\right)^{m^2} \prod_{k=0}^{m-1} \frac{(3k+1)!}{(m+k)!} = \left(\frac{1}{2}\right)^{m^2} A_m$$

with A_m - number of alternating sign matrices

$\longrightarrow$ first exact result for $\Delta \neq 0$ (and proof of a conjecture of Razumov and Stroganov)

Asymptotic Results: (saddle-point)

⋆ massless case $(-1 < \Delta = \cos\zeta \leqslant 1)$

$$\lim_{m\to\infty} \frac{\log\tau(m)}{m^2} = \log\frac{\pi}{\zeta} + \frac{1}{2}\int_{\mathbb{R}-i0} \frac{d\omega}{\omega}\,\frac{\sinh\frac{\omega}{2}(\pi-\zeta)\cosh^2\frac{\omega\zeta}{2}}{\sinh\frac{\pi\omega}{2}\sinh\frac{\omega\zeta}{2}\cosh\omega\zeta}$$

$$= \begin{cases} -\frac{1}{2}\log 2 & \text{for } \Delta = 0 \\ \frac{3}{2}\log 3 - 3\log 2 & \text{for } \Delta = \frac{1}{2} \\ \log\left[\frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4})}\right] & \text{for } \Delta = 1 \text{ (XXX chain)} \end{cases}$$

⋆ massive case $(\Delta = \cosh\zeta > 1)$

$$\lim_{m\to\infty} \frac{\log\tau(m)}{m^2} = -\frac{\zeta}{2} - \sum_{n=1}^{\infty} \frac{e^{-n\zeta}}{n}\,\frac{\sinh(n\zeta)}{\cosh(2n\zeta)}$$

$$\underset{\zeta\to 0}{\longrightarrow} \log\left[\frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4})}\right] \qquad \text{(XXX)}$$

$$\underset{\zeta\to +\infty}{\longrightarrow} -\infty \qquad \text{(Ising)}$$

Further analysis: the two-point function

Consider the correlation function of the product of two local operators at zero temperature :

$$g_{12} = \langle \psi_g | \theta_1 \theta_2 | \psi_g \rangle$$

Two main strategies to evaluate such a function:

(i) compute the action of local operators on the ground state $\theta_1 \theta_2 |\psi_g\rangle = |\tilde{\psi}\rangle$ and then calculate the resulting scalar product:

$$g_{12} = \langle \psi_g | \tilde{\psi} \rangle$$

(ii) insert a sum over a complete set of eigenstates $|\psi_i\rangle$ to obtain a sum over one-point matrix elements (form factor type expansion) :

$$g_{12} = \sum_i \langle \psi_g | \theta_1 | \psi_i \rangle \cdot \langle \psi_i | \theta_2 | \psi_g \rangle$$

Analytical + Numerical methods for dynamical correlation functions in a field (Biegel, Karbach, Müller; Caux, Hagemans, Maillet)

Use *(ii)* form factor expansion over a complete set of intermediate eigenstates $|\psi_i\rangle$:

$$\langle S_j^\alpha(t)\, S_{j'}^\beta(0)\rangle = \sum_i \langle\psi_g|S_j^\alpha(t)|\psi_i\rangle \cdot \langle\psi_i|S_{j'}^\beta(0)|\psi_g\rangle$$

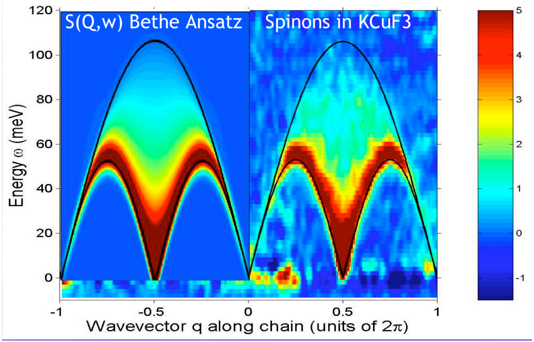
for a finite chain of length M even, and a ground state $|\psi_g\rangle$ depending on the magnetic field with a fixed number of reversed spins N, and $2N \leq M$.

- each form factor = explicit determinant of size N, depending on two sets of parameters solutions of Bethe equations and characterizing the states $\langle\psi_g|$ and $|\psi_i\rangle$ respectively
- Numerics are then used to compute the determinants and the (finite) sum (control of the results via sum rules)

$\hookrightarrow$ **numerical result for the dynamical spin-spin correlation functions**

$\hookrightarrow$ **successful comparison to neutron scattering experiments** for the structure factor (Fourier transform of the dynamical correlation function)

$$S^{\alpha\beta}(q,\omega) = \frac{1}{N}\sum_{j,j'=1}^{N} e^{iq(j-j')}\int_{-\infty}^{\infty} dt e^{i\omega t}\langle S_j^\alpha(t) S_{j'}^\beta(0)\rangle$$



- Left: Bethe ansatz data computed for a chain of 500 sites
- Right: Experimental data for KCuF3 (D.A. Tennant et al)

Analytical resummations for the two-point function

$$\langle \sigma_1^z \, \sigma_m^z \rangle = \phi_m \, \langle \psi_g | \, (A-D)(\xi_1) \cdot \underbrace{\prod_{i=2}^{m-1} (A+D)(\xi_i)}_{\text{propagator } (1 \to m)} \cdot (A-D)(\xi_m) \, |\psi_g\rangle$$

Use (i): compute resummed action of the "propagator" from site 1 to m on an arbitrary state:

$$\langle \psi | \prod_{a=1}^{m} t_\kappa(x_a) = \sum_{n=0}^{m} \langle \psi_n(\kappa) |$$

with $t_\kappa(x) = (A + \kappa D)(x)$ twisted transfer matrix

↪ partial resummation in the thermodynamic limit:

$$\langle \sigma_1^z \, \sigma_m^z \rangle = \sum_{\text{m+1 terms}} \text{(multiple integrals)} \qquad \text{(instead of } 2^m \text{ terms)}$$

↪ master formula for the finite chain

Example

Generating function $\langle Q^{\kappa}_{1,m} \rangle$ for σ^z correlation functions

$$\frac{1}{2}\langle (1-\sigma^z_1)(1-\sigma^z_{m+1})\rangle = \frac{\partial^2}{\partial\kappa^2}\langle \left(Q^{\kappa}_{1,m+1} - Q^{\kappa}_{1,m} - Q^{\kappa}_{2,m+1} + Q^{\kappa}_{2,m}\right)\rangle\Big|_{\kappa=1}$$

with

$$Q^{\kappa}_{1,m} = \prod_{n=1}^{m}\left(\frac{1+\kappa}{2} + \frac{1-\kappa}{2}\cdot\sigma^z_n\right) = \prod_{a=1}^{m}(A+\kappa D)(\xi_a)\prod_{b=1}^{m}(A+D)^{-1}(\xi_b)$$

$\rightsquigarrow$ to compute:

$$\langle Q^{\kappa}_{1,m}\rangle = \phi_m\ \langle \psi_g|\prod_{a=1}^{m} t_{\kappa}(\xi_a)|\psi_g\rangle \quad \text{with } t_{\kappa}(x) = (A+\kappa D)(x)$$

Master equation for σ^z correlation functions

Let the inhomogeneities $\{\xi\}$ be generic and the set $\{\lambda\}$ be an admissible off-diagonal solution of the Bethe equations (cf.Tarasov - Varchenko). Then there exists $\kappa_0 > 0$ such, that for $|\kappa| < \kappa_0$:

$$\langle Q^{\kappa}_{1,m} \rangle = \frac{1}{N!} \oint_{\Gamma\{\xi\}\cup\Gamma\{\lambda\}} \prod_{j=1}^{N} \frac{dz_j}{2\pi i} \cdot \prod_{a,b=1}^{N} \sinh^2(\lambda_a - z_b) \cdot \prod_{a=1}^{m} \frac{\tau_\kappa(\xi_a|\{z\})}{\tau(\xi_a|\{\lambda\})}$$

$$\times \frac{\det_N\left(\frac{\partial\tau_\kappa(\lambda_j|\{z\})}{\partial z_k}\right) \cdot \det_N\left(\frac{\partial\tau(z_k|\{\lambda\})}{\partial\lambda_j}\right)}{\prod_{a=1}^{N} \mathcal{Y}_\kappa(z_a|\{z\}) \cdot \det_N\left(\frac{\partial\mathcal{Y}(\lambda_k|\{\lambda\})}{\partial\lambda_j}\right)}.$$

Notations:

$\tau_\kappa(\mu|\{\lambda\})$ = eigenvalue of the κ-twisted transfer matrix $t_\kappa(\mu)$ on the eigenstate $|\psi_\kappa\rangle = \prod_k B(\lambda_k)|0\rangle$, for $\{\lambda\}$ solution of the (twisted) Bethe equations : $\mathcal{Y}_\kappa(\lambda_j|\{\lambda\}) = 0, \quad j = 1, \dots, N.$
($\kappa = 1 \to$ no subscript)

The integration contour is such that the only singularities of the integrand within the contour $\Gamma\{\xi\} \cup \Gamma\{\lambda\}$ which contribute to the integral are the points $\{\xi\}$ and $\{\lambda\}$.

2 ways to evaluate the integrals:

- compute the residues in the poles **inside** Γ
  - $\longrightarrow$ representation of $\langle \sigma_1^z \sigma_{m+1}^z \rangle$ as sum of m multiple integrals (previous resummation obtained with approach *(i)*)
- compute the residues in the poles **outside** Γ (within strips of width $i\pi$)
  - $\longrightarrow$ sum over (admissible) solutions of (twisted) Bethe equations
  - $\longrightarrow$ form factor expansion of $\langle \sigma_1^z \sigma_{m+1}^z \rangle$ (approach *(ii)*)

$\hookrightarrow$ **link between the two approaches**

Time-dependent master equation

$$\langle Q^{\kappa}_{1,m}(t)\rangle = \frac{1}{N!} \oint\limits_{\Gamma\{\pm\frac{\eta}{2}\}\cup\Gamma\{\lambda\}} \prod_{j=1}^{N} \frac{dz_j}{2\pi i} \cdot \prod_{b=1}^{N} e^{it\left(E(z_b)-E(\lambda_b)\right)+im\left(p(z_b)-p(\lambda_b)\right)}$$

$$\times \prod_{a,b=1}^{N} \sinh^2(\lambda_a - z_b) \cdot \frac{\det_N \left(\frac{\partial \tau_\kappa(\lambda_j|\{z\})}{\partial z_k}\right) \cdot \det_N \left(\frac{\partial \tau(z_k|\{\lambda\})}{\partial \lambda_j}\right)}{\prod_{a=1}^{N} \mathcal{Y}_\kappa(z_a|\{z\}) \cdot \det_N \left(\frac{\partial \mathcal{Y}(\lambda_k|\{\lambda\})}{\partial \lambda_j}\right)}$$

with

$$E(z) = \frac{2\sinh^2\eta}{\sinh(z-\frac{\eta}{2})\sinh(z+\frac{\eta}{2})}$$

$$p(\lambda) = i\log\left(\frac{\sinh(\lambda-\frac{\eta}{2})}{\sinh(\lambda+\frac{\eta}{2})}\right)$$

Explicit results at $\Delta = \frac{1}{2}$

Generating function at $\Delta = \frac{1}{2}$

Partial resummation in the inhomogeneous case
$\longrightarrow$ multiple integrals can be separated and computed:

$$\langle Q_\kappa(m)\rangle = \frac{3^m}{2^{m^2}} \prod_{a>b}^{m} \frac{\sinh 3(\xi_a-\xi_b)}{\sinh^3(\xi_a-\xi_b)} \sum_{n=0}^{m} \kappa^{m-n} \sum_{\substack{\{\xi\}=\{\xi_{\gamma_+}\}\cup\{\xi_{\gamma_-}\}\\ |\gamma_+|=n}} \det_m \hat{\Phi}^{(n)}$$

$$\times \prod_{a\in\gamma_+} \prod_{b\in\gamma_-} \frac{\sinh(\xi_b-\xi_a-\frac{i\pi}{3})\sinh(\xi_a-\xi_b)}{\sinh^2(\xi_b-\xi_a+\frac{i\pi}{3})},$$

with

$$\hat{\Phi}^{(n)}(\{\xi_{\gamma_+}\},\{\xi_{\gamma_-}\}) = \left(\begin{array}{c|c} \Phi(\xi_j-\xi_k) & \Phi(\xi_j-\xi_k-\frac{i\pi}{3}) \\ \hline \Phi(\xi_j-\xi_k+\frac{i\pi}{3}) & \Phi(\xi_j-\xi_k) \end{array}\right), \quad \Phi(x) = \frac{\sinh\frac{x}{2}}{\sinh\frac{3x}{2}}.$$

→ If the lattice distance m is not too large, the representations can be successfully used to compute $\langle Q_\kappa(m)\rangle$ explicitely.

First results for $P_m(\kappa) = 2^{m^2}\langle Q_\kappa(m)\rangle$ up to $m = 9$:

$$P_1(\kappa) = 1 + \kappa,$$
$$P_2(\kappa) = 2 + 12\kappa + 2\kappa^2,$$
$$P_3(\kappa) = 7 + 249\kappa + 249\kappa^2 + 7\kappa^3,$$
$$P_4(\kappa) = 42 + 10004\kappa + 45444\kappa^2 + 10004\kappa^3 + 42\kappa^4$$
$$P_5(\kappa) = 429 + 738174\kappa + 16038613\kappa^2 + 16038613\kappa^3 + 738174\kappa^4 + 429\kappa^5,$$
$$P_6(\kappa) = 7436 + 96289380\kappa + 11424474588\kappa^2 + 45677933928\kappa^3 + 11424474588\kappa^4 + 96289380\kappa^5 + 7436\kappa^6.$$

→ Two-point functions $\langle\sigma_1^z\sigma_{m+1}^z\rangle$ at $\Delta = \frac{1}{2}$

m	$\langle\sigma_1^z\sigma_{m+1}^z\rangle$ Exact		$\langle\sigma_1^z\sigma_{m+1}^z\rangle$ Asympt.
1	-2^{-1}	-0.5000000000	-0.5805187860
2	$7 \cdot 2^{-6}$	0.1093750000	0.1135152692
3	$-401 \cdot 2^{-12}$	-0.0979003906	-0.0993588501
4	$184453 \cdot 2^{-22}$	0.0439770222	0.0440682654
5	$-95214949 \cdot 2^{-31}$	-0.0443379157	-0.0444087865
6	$1758750082939 \cdot 2^{-46}$	0.0249933420	0.0249365346
7	$-30283610739677093 \cdot 2^{-60}$	-0.0262668452	-0.0262404925
8	$5020218849740515343761 \cdot 2^{-78}$	0.0166105110	0.0165641239

and comparison with the values given by the asymptotic prediction:

$$\langle\sigma_1^z\sigma_{m+1}^z\rangle = -\frac{1}{\pi(\pi-\zeta)}\frac{1}{m^2} + (-1)^m \frac{A_z}{m^{\frac{\pi}{\pi-\zeta}}} + \cdots$$

with value of A_z conjecture by S. Lukyanov

Some other models

- **Non periodic boundary conditions**
  Open XXZ chain (with diagonal boundary conditions):
  $$H = \sum_{m=1}^{M-1} \{\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta(\sigma_m^z \sigma_{m+1}^z - 1)\} + h_- \sigma_1^z + h_+ \sigma_M^z$$
  no translation invariance $\longrightarrow$ revisit solution of the inverse Problem

  $\hookrightarrow$ multiple integral formulas for elementary blocks, partial resummation for 2-point correlation functions
  Master equation ?

- **Continuum field theory**
  Master equation valid for all models with the same R-matrix
  (depend only on commutation relations of the Yang-Baxter algebra)

  $\hookrightarrow$ density-density correlation functions of the
  quantum non-linear Schrödinger model (or one-dimensional Bose gas):
  $$H = \int_0^L \Big(\partial_x \psi^\dagger(x)\, \partial_x \psi(x) + c\, \psi^\dagger(x) \psi^\dagger(x) \psi(x) \psi(x) - h\, \psi^\dagger(x) \psi(x) \Big)\, dx$$

Some open problems...

- Asymptotic behavior of correlation functions: challenging the conformal limit from the lattice models
- Continuum (Field theory) models (NLS, ShG,...) :
  ⋆ Approach from the lattice
  ⋆ Inverse problem for infinite dimensional representations
  ⋆ Link to Q operator and SOV methods
- Even more "sophisticated" models :
  ⋆ XYZ model
  ⋆ Hubbard : needs extended Yang-Baxter and ABA or FBA understanding