Chapter 1

Introduction

1.1 The volume of sets in $\mathbb{R}^n$

One of the oldest problems in mathematics is that of assigning a **measure** (מידה)
to a geometric shape, quantifying its **length** (in 1D), its **area** (in 2D) or its **volume**
(in 3D); in other applications, one may want to assign subsets of $\mathbb{R}^3$ a **mass**, a
charge and other physical attributes. More generally, we would like to assign a
measure to arbitrary subsets of $\mathbb{R}^n$; that is, to define a function

$$\mu : \mathscr{P}(\mathbb{R}^n) \to \bar{\mathbb{R}},$$

where for a set X, $\mathscr{P}(X)$ denotes its **power set** (קבוצת החזקה), i.e., the collection
of all of its subsets, and $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ (length, area and volume might be infinite).
Such a function should satisfy a collection of natural requirements, such as:

1. **Finite additivity**: If $E_1, \ldots, E_k$ are disjoint, then

$$\mu\left(\coprod_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} \mu(E_i),$$

   where throughout these notes $\sqcup$ denotes a disjoint union.

2. **Invariance under rigid motion**: For every $U \subset \mathbb{R}^n$ and every rigid motion,
   $f(x) = Qx + b$, where $Q \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ is orthogonal and $b \in \mathbb{R}^n$,

$$\mu(f(U)) = \mu(U).$$

Throughout these notes, for a function $f : A \to B$ and a set $U \subset A$, we denote

$$f(U) = \{f(x) \: : \: x \in U\}.$$

3. **Normalization**:

$$\mu([0, 1]^n) = 1.$$

The finite additivity is too restrictive. Think for example of the area of a disc or the area under the graph of a function: you would like to calculate these areas by covering the domains with a *countable* number of disjoint boxes. Therefore, the first condition may have to be replaced by:

1'. **Countable additivity**: If $E_1, E_2 \ldots$ is a countable collection of disjoint sets, then

$$\mu\left(\coprod_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

These plausible conditions on μ turn out, however, to be inconsistent. To see why, we will examine a classical example for $n = 1$.

Example: Endow $\mathbb{R}$ with an equivalence relation,

$$x \sim y \qquad \text{if and only if} \qquad x - y \in \mathbb{Q}.$$

Then, construct a set $N \subset [0, 1)$, which consists of exactly one representative of each equivalence class (this construction relies on the **axiom of choice**). That is,

$$\forall x \in [0, 1) \quad \exists! r \in \mathbb{Q}, \quad \text{such that} \quad x + r \in N.$$

For every $r \in [0, 1) \cap \mathbb{Q}$ define

$$N_r = N + r \mod 1.$$

We start by noting that if $r, q \in [0, 1) \cap \mathbb{Q}$ and $r \neq q$, then $N_r \cap N_q = \varnothing^1$. Secondly, every N_r is a translated copy of N, of which a part is then translated again. That is,

$$N_r = (N \cap [0, 1 - r) + r) \amalg (N \cap [1 - r, 1) + r - 1).$$

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[1] Indeed, suppose that $x \in N_r \cap N_q$ for $r \neq q$. Then, by definition

$$x - r \in N \qquad \text{and} \qquad x - q \in N,$$

however, $(x - q) \sim (x - r)$, contradicting the minimality condition in the definition of N.

By the desired properties of the measure,

$$\mu(N_r) = \mu(N \cap [0, 1-r) + r) + \mu(N \cap [1-r, 1) + r - 1)$$
$$= \mu(N \cap [0, 1-r)) + \mu(N \cap [1-r, 1))$$
$$= \mu(N).$$

Finally, by the maximality of N,

$$[0, 1) = \bigcup_{r \in \mathbb{Q}} N_r.$$

If the length measure μ satisfied the desired properties, we would have

$$1 = \mu([0, 1)) = \sum_{r \in \mathbb{Q}} \mu(N_r) = \sum_{r \in \mathbb{Q}} \mu(N).$$

Now, either $\mu(N) = 0$, in which case we get that $1 = 0$, or $\mu(N) > 0$, in which case we get that $1 = \infty$. Thus, the defining assumptions on μ are inconsistent. ▲ ▲ ▲

One could argue that countable additivity was too much of a requirement, yet without it, much of the limiting processes of calculus could not be carried out. It turns out that even finite-additivity would run us into problems. In 1924, Banach and Tarski proved what has become known as the **Banach-Tarski paradox**. They showed that the unit ball $B(0, 1)$ in $\mathbb{R}^3$ can be partitioned into five disjoint sets $E_1, \ldots, E_5$, and there exist rigid maps $f_k : E_k \to \mathbb{R}^3$, such that

$$\coprod_{i=1}^{5} f_k(E_k) = B(x, 1) \sqcup B(y, 1),$$

for a pair of points $x, y \in \mathbb{R}^3$. This would imply that the volume of the unit ball equals twice itself. (It should be pointed out that these sets are very strange and that their construction relies on the axiom of choice.)

Thus, the difficulty in defining a measure for subsets of $\mathbb{R}^n$ is not related to the countable additivity requirement, but rather to the possibility of constructing very peculiar sets. The remedy to this problem is restricting the collection of sets to which a measure can be assigned—sets that we shall call **measurable** (מדידים).

1.2 Integration theory

In the first calculus courses, you learned about the concept of integration, following the construction of Riemann (or the equivalent construction of Darboux).

Integration, as defined by Riemann has several shortcomings. One of them, is that it is not continuous under pointwise limits. For example, let (q_n) be an enumeration of $(0, 1) \cap \mathbb{Q}$, and for every $n \in \mathbb{N}$ define the function

$$f_n : (0, 1) \to \mathbb{R} \qquad f_n(x) = \begin{cases} 1 & x \in \{q_1, \ldots, q_n\} \\ 0 & \text{otherwise.} \end{cases}$$

In terms of Riemann integration, for every n,

$$\int_0^1 f_n(x) \, dx = 0,$$

whereas (f_n) converges pointwise to

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise,} \end{cases}$$

which isn't even Riemann-integrable.

We have very good reasons to expect the integral of the Dirichlet function to be zero. For example, we may cover $(0, 1) \cap \mathbb{Q}$ with open sets,

$$\bigcup_{n=1}^{\infty} (q_n - \varepsilon/2^n, q_n + \varepsilon/2^n).$$

These (possibly overlapping) segments have a total length of ε, and since this holds for every $\varepsilon > 0$, we would expect the integral of the Dirichlet function to vanish.

Moreover, we tend to think of integration and differentiation as opposite operations. Namely, if f is integrable and

$$F(x) = \int_a^x f(t) \, dt,$$

then F is differentiable and $F' = f$. This is not quite so if f is not continuous. As an example, consider the function

$$F(x) = \begin{cases} x^2 \sin(1/x^2) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

This function is differentiable, with

$$f(x) = F'(x) = \begin{cases} -(1/x)\cos(1/x^2) & x \neq 0 \\ 0 & x = 0, \end{cases}$$

however f, despite being a derivative is not even integrable.

Ideally, we would like a notion of integration that is truly inverse to differentiation and behaves well under limits. A sound concept of integration turns out to be intimately related to the above notion of measure.

1.3 Historical notes