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GENERALIZED NEKRASSOV–MEHMKE PROCEDURES FOR SOLVING LINEAR SYSTEM OF EQUATIONS

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Abstract

A generalized Nekrassov–Mehmke iterative method for finding solution of linear system of algebraic equations $Ax = b$ is given by the decomposition $A = T_m - E_m - F_m$, where T_m is a banded matrix of bandwidth $2m + 1$. We study the convergence of the new method based on the ideas given in [1–3]. Some successive overrelaxation modifications, symmetric and 2-stage schemes of the Nekrassov–Mehmke iterations are proposed. Interesting numerical examples are presented.

Key words: solving linear system of equations, Nekrassov–Mehmke 2-method (NM2), generalized Nekrassov–Mehmke method (GNM2), SOR–Nekrassov–Mehmke methods, 2-stage schemes, symmetric algorithms

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1. Introduction. Let us consider the linear system $Ax - b = 0$, $(det\, A \neq 0)$, or

$$(1)\qquad a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - b_i = 0 = f_i(x_1, x_2, \ldots, x_n),\ \ i = 1, 2, \ldots, n.$$

Suppose that the matrix A is strictly diagonally dominant (SDD), i.e. $|a_{ii}| > \sum_{j\neq i}^{n} |a_{ij}|,\quad i = 1, 2, \ldots, n$. In this paper we propose new iterative algorithms based on the classical methods of Nekrassov–Mehmke. Using Nekrassov–Mehmke iteration scheme, (or Gauss–Seidel scheme), see Nekrassov [4], Mehmke [5] and Mehmke and Nekrassov [6], the sequence of consecutive approximations x_i^k, is computed in this way:

$$(2)\quad x_i^{k+1} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^{k+1} - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{k} + \frac{b_i}{a_{ii}},\ \ i = 1, 2, \ldots, n;\ \ k = 0, 1, 2, \ldots\ .$$

Henceforth, we shall call the above scheme the Nekrassov–Mehmke 1-method (NM1). Let $A = (a_{ij})$ be an $n \times n$ matrix and $T_m = (t_{ij})$ be a banded matrix of bandwidth $2m + 1$ defined as

$$t_{ij} = \begin{cases} a_{ij}, & |i-j| \le m, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$T_m = \begin{pmatrix} a_{11} & \cdots & a_{1,m+1} & & \\ \vdots & \ddots & & \ddots & \\ a_{m+1,1} & & \ddots & & a_{n-m,n} \\ & \ddots & & \ddots & \vdots \\ & & a_{n,n-m} & \cdots & a_{n,n} \end{pmatrix},$$

$$E_m = \begin{pmatrix} & & & \\ & & & \\ -a_{m+2,1} & & & \\ \vdots & \ddots & & \\ -a_{n,1} & \cdots & -a_{n,n-m-1} & \end{pmatrix},$$

$$F_m = \begin{pmatrix} & -a_{1,m+2} & \cdots & -a_{1,n} \\ & & \ddots & \vdots \\ & & & -a_{n-m-1,n} \\ & & & \\ & & & \end{pmatrix}.$$

Applying the Nekrassov–Mehmke method (NM1) to system $Ax = b$ with the decomposition of $A = T_m - E_m - F_m$, i.e.

$$x^{k+1} = (T_m - E_m)^{-1} F_m x^k + (T_m - E_m)^{-1} b, \tag{3}$$

Salkuyeh [1] proved that the generalized Nekrassov–Mehmke method (GNM1) is convergent for any initial point x^0.

2. Main results. I. New generalized Nekrasov–Mehmke method. In a number of cases the success of the procedures of type (2) depends on the proper ordering of the equations (and x_i, $i = 1, \ldots, n$) in system (1). Despite this fact the following modification of the Nekrassov–Mehmke method is known (see Faddeev and Faddeeva [7]):

$$x_i^{k+1} = -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^k - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{k+1} + \frac{b_i}{a_{ii}}, \; i = n, n-1, \ldots, 1; \; k = 0, 1, 2, \ldots. \tag{4}$$

From now on, we shall call the above scheme the Nekrassov–Mehmke 2-method (NM2). In [7] Faddeev and Faddeeva especially pointed out that of certain interest are such iteration processes in which the cycles studied in the two Nekrassov–Mehmke methods (NM1) and (NM2) are alternated. The method (NM2) does not possess better convergence in comparison with the method (NM1). Under the circumstances if matrix A is positive definite, then the eigenvalues of matrix G in the matrix equations $x = Gx + t$ are real and this allows us to apply different methods for improving the rate of convergence, i.e., as an example, Abramov's technique [8].

In this paper the following generalization of the (NM2) method– generalized Nekrassov–Mehmke method (GNM2) is proposed

$$x^{k+1} = (T_m - F_m)^{-1} E_m x^k + (T_m - F_m)^{-1} b, \quad k = 0, 1, 2, \dots \ . \tag{5}$$

We give a convergence theorem for the method (GNM2).

Theorem 1. *Let A be an (SDD) matrix. Then for any natural number $m < n$ the (GNM2) method is convergent for any initial guess x^0.*

Proof. The proof follows the ideas given in [1]. The following result obtained by Jin, Wei and Tam [3] is more often applicable: Let $M = (m_{ij})$ and $N = (n_{ij})$ be $n \times n$ matrices with M being strictly diagonally dominant. Then for the spectral radius we have

$$\rho(M^{-1}N) \le \rho = \max_i \rho_i, \tag{6}$$

where

$$\rho_i = \frac{\displaystyle\sum_{j=1}^{n} |n_{ij}|}{|m_{ii}| - \displaystyle\sum_{j \ne i}^{n} |m_{ij}|}, \quad i = 1, 2, \dots n.$$

Let $M = T_m - F_m$ and $N = E_m$ in the (GNM2) method. Obviously, the matrix $T_m - F_m$ is an (SDD) matrix. Hence M and N satisfy relation (6). Having in mind that matrix A is an (SDD) matrix, it can be easily verified that $\rho_i < 1$, $i = 1, \dots, n$. Therefore $\rho\left((T_m - F_m)^{-1} E_m\right) \le \rho < 1$ and this completes the proof. □

Remarks. 1. The definition of matrixes M and N in Theorem 1 depends on the parameter m. We denote ρ by $\rho^{(m)}$. By a little computation one can see that (see, also [1])

$$\rho^{(1)} \ge \rho^{(2)} \ge \dots \ge \rho^{(n)} = 0. \tag{7}$$

2. Let $R^{(m)} = (T_m - F_m)^{-1} E_m$ be the iteration matrix of the (GNM2) method. From relation (7) we cannot deduce that $\rho\left(R^{(m+1)}\right) \le \rho\left(R^{(m)}\right)$. But equation (7) shows that we can choose a natural number $m \le n$ such that $\rho\left(R^{(m)}\right)$

is sufficiently small. To illustrate this, let us consider the system (a classical example by Faddeev and Faddeeva [7]):

$$\left| \begin{array}{l} 0.78x_1 - 0.02x_2 - 0.12x_3 - 0.14x_4 = 0.76 \\ -0.02x_1 + 0.86x_2 - 0.04x_3 + 0.06x_4 = 0.08 \\ -0.12x_1 - 0.04x_2 + 0.72x_3 - 0.08x_4 = 1.12 \\ -0.14x_1 + 0.06x_2 - 0.08x_3 + 0.74x_4 = 0.68 \end{array} \right.$$

The exact solution of the system is $x = (1.534965, 0.122010, 1.975156, 1.412955)$. For Nekrassov–Mehmke method (NM2) applied to system $Ax = b$ with the decomposition $A = D - E - F$, i.e. $x^{k+1} = (D - F)^{-1}Ex^k + (D - F)^{-1}b$ (D is the diagonal of A, $-E$ its strict lower part and $-F$ its strict upper part) for the matrix $R^* = (D - F)^{-1} E$ we have

$$R^* = \begin{pmatrix} 0.0633138 & -0.00719144 & 0.021073 & 0 \\ 0.0187862 & 0.00782178 & -0.00698373 & 0 \\ 0.187688 & 0.0465466 & 0.012012 & 0 \\ 0.189189 & -0.081081 & 0.10818 & 0 \end{pmatrix},$$

and for the spectral radii ρ of R^*: $\rho(R^*) = 0.10569$. Let $m = 2$. For generalized Nekrassov–Mehmke method (GNM2) for the matrix $R^{(m)} = (T_m - F_m)^{-1} E_m$ we have

$$R^{(2)} = \begin{pmatrix} 0.0385524 & 0 & 0 & 0 \\ -0.0113048 & 0 & 0 & 0 \\ 0.0272475 & 0 & 0 & 0 \\ 0.193052 & 0 & 0 & 0 \end{pmatrix},$$

and for the spectral radii ρ of $R^{(2)}$: $\rho(R^{(2)}) = 0.0385524$, i.e., $\rho(R^*) = 0.10569 > \rho(R^{(2)}) = 0.0385524$.

II. New successive overrelaxation modification of the Nekrassov–Mehmke method. Let

$$L = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

Consider the iteration scheme $x^{k+1} = Bx^k + d$. Since we move from the current point x^k to the updated point x^{k+1}, we may think of it as the addition of a displacement to the old approximation $x^{k+1} = x^k + r^k$. Even though this method will converge if $\rho(B) < 1$, convergence will be slow if the spectral radius of B is close to 1. We could try to speed up convergence by modifying the iteration $x^{k+1} = x^k + \omega r^k = \omega x^{k+1} + (1-\omega)x^k$. Intuitively, if r^k is a good direction, we might think of accelerating the movement by setting $\omega > 1$. We may form a convex combination of the new and the old point as follows:

$$\begin{aligned} \hat{x}^{k+1} &= \omega x^{k+1} + (1-\omega)x^k \\ &= \omega\left(Bx^k + d\right) + (1-\omega)x^k = B_\omega x^k + \omega d. \end{aligned} \tag{8}$$

The iteration procedure (8) is stable if $\rho(B_\omega) < 1$. For instance, we may try the idea on Nekrassov–Mehmke 2-method (NM2) – (4). We may replace (4) by the following iteration:

$$\begin{aligned} z_i^{k+1} &= -\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} x_j^k - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j^{k+1} + \frac{b_i}{a_{ii}}, \\ x_i^{k+1} &= \omega z_i^{k+1} + (1-\omega)x_i^k, \;\; i = n, n-1, \ldots, 1; \;\; k = 0, 1, 2, \ldots \; . \end{aligned} \tag{9}$$

Hence, we shall call the above scheme the successive overrelaxation Nekrassov–Mehmke 2-method (SORNM2). We give the following convergence theorem for the relaxation method (9).

Theorem 2. *The iteration procedure* (9) *is stable if*

$$\rho\left(\left(I + \omega P^{-1}R\right)^{-1}\left((1-\omega)I - \omega P^{-1}L\right)\right) < 1. \tag{10}$$

Proof. In order to analyze the effect of this modification, let us rewrite method (9) in a compact form, based on the following decomposition of A: $A = L + P + R$. With this notation the (SORNM2) scheme may be rewritten in matrix form as

$$\begin{aligned} z^{k+1} &= P^{-1}\left(b - Rx^{k+1} - Lx^k\right), \\ x^{k+1} &= \omega z^{k+1} + (1-\omega)x^k. \end{aligned} \tag{11}$$

Eliminating z^{k+1} and rearranging yields

$$\left(I + \omega P^{-1}R\right)x^{k+1} = \left((1-\omega)I - \omega P^{-1}L\right)x^k + \omega P^{-1}b,$$

i.e.,

$$x^{k+1} = \left(I + \omega P^{-1}R\right)^{-1}\left((1-\omega)I - \omega P^{-1}L\right)x^k + \omega\left(I + \omega P^{-1}R\right)^{-1}P^{-1}b$$

we arrive at inequality (10), which completes the proof of Theorem 2. □

For other results, see NIETHAMMER and SCHADE [9], NIETHAMMER [10], EIERMANN, NIETHAMMER and VARGA [11], GAITANOS, HADJIDIMOS and YEYIOS [12], KINASHI, SAWAMI and NIKI [13], BREZINSKI and REDIVO-ZAGLIA [14].

III. New 2-stage modification of the Nekrassov–Mehmke method (4). Niethammer and Schade [9] proposed the relaxed SOR method applied to the linear system $Ax = b$, with coefficient matrix being the identity plus the skew–symmetric matrix. Kinashi, Sawami and Niki [13] consider the iterative method for numerical solution of the linear system $Ax = b$, with the regular splitting $A = D - L + U$, where D is a diagonal matrix, L and U are strictly lower and strictly upper triangular matrices, respectively, and $L \geq 0$, $U \geq 0$ on the base of Nekrassov–Mehmke 1-method (NM1) – (2):

$$x^{k+1} = -(D-L)^{-1}Ux^k + (D-L)^{-1}b.$$

Kinashi, Sawami and Niki [13] construct the following 2-stage Nekrassov–Mehmke method Nekrassov–Mehmke 2-method (NM1) – (2-NM1) method.

$$x^{k+1} = \frac{1}{2}(D-L)^{-1}(D-L-U)x^k + \frac{1}{2}(D-L)^{-1}b.$$

Following the ideas given by Kinashi, Sawami and Niki [13] we construct 2-stage Nekrassov method based on the Nekrassov–Mehmke 2-method (NM2) – (4). Let $A = D + L - U$. Substituting $L = A - D + U$ in the (NM2)-method we have

$$\begin{aligned} x^{k+1} &= -(D-U)^{-1}(A-(D-U))x^k + (D-U)^{-1}b \\ &= x^k - (D-U)^{-1}Ax^k + (D-U)^{-1}b \\ &= x^k - (D-U)^{-1}(Ax^k - b). \end{aligned} \tag{12}$$

Applying the Nekrassov–Mehmke 2-method (NM2) to system $Ax^k - b = 0$ again, we obtain

$$\begin{aligned} x^{k+1} &= x^k - (D-U)^{-1}((D-U)x^{k+1} + Lx^k - b) \\ &= x^k - x^{k+1} - (D-U)^{-1}Lx^k + (D-U)^{-1}b; \\ x^{k+1} &= \frac{1}{2}x^k - \frac{1}{2}(D-U)^{-1}Lx^k + \frac{1}{2}(D-U)^{-1}b, \end{aligned}$$

i.e.,

$$x^{k+1} = \frac{1}{2}(D-U)^{-1}(D-U-L)x^k + \frac{1}{2}(D-U)^{-1}b, \;\; k = 0, 1, 2, \dots \; . \tag{13}$$

Hereafter, we shall call the above scheme the 2-stage Nekrassov–Mehmke method based on the Nekrassov–Mehmke 2-method (NM2) – (2-NM2) method. Evidently, the (2-NM2) method yields considerable improvement in the rate of convergence for Nekrassov–Mehmke iterative method (NM2). For the (NM2) method we have the following expression:

$$x^{k+1} = -(D-U)^{-1}Lx^k + (D-U)^{-1}b, \ \ k = 0, 1, 2, \dots. \tag{14}$$

Here, we introduce the following convergence theorem for the (2-NM2) method.

Theorem 3. *If the* (*NM2*)*-method is convergent, then the* (*2-NM2*) *procedure* (13) *is also convergent.*

Proof. The proof follows the ideas given in [13]. We denote $H_{NM2} = -(D-U)^{-1}L$ as the (NM2)-iteration matrix and $H_{2-NM2} = \frac{1}{2}(I-(D-U)^{-1}L)$ for the iteration matrix of (2-NM2)-method. Thus,

$$H_{2-NM2} = \frac{1}{2}\left(I + H_{NM2}\right). \tag{15}$$

Denoting $\rho(H_{NM2})$ and $\rho(H_{2-NM2})$ as the spectral radii of H_{NM2} and H_{2-NM2}, from (15) we obtain $\rho(H_{2-NM2}) = \frac{1}{2}\rho\left(I + H_{NM2}\right) \leq \frac{1}{2}\left(1 + \rho(H_{NM2})\right)$. Since $0 \leq \rho(H_{NM2}) < 1$, the following inequality holds:

$$\rho(H_{2-NM2}) \leq \frac{1+\rho(H_{NM2})}{2} < 1$$

which completes the proof of Theorem 3. □

Remarks. 1. Let $\Delta(H_{NM2}) = D(0, \rho(H_{NM2})) \cap D(-1, 2\rho(H_{NM2}))$, where $D(O, r)$ is the disk of centre O and radius r. Then the following theorem is valid:

Theorem 4. *If the eigenvalue* $\lambda(H_{NM2})$ *of the NM2-iteration matrix satisfies the following condition* $\lambda(H_{NM2}) \in \Delta(H_{NM2})$, *then the following inequality holds:* $\rho(H_{2-NM2}) \leq \rho(H_{NM2})$.

The proof follows the ideas given in [13], and will be omitted. For the convergence region, see Fig. 1.

To illustrate Theorem 4, consider the system

$$\left| \begin{array}{l} x_1 - 0.1x_2 = 0.8 \\ 14x_1 + 2x_2 = 18. \end{array} \right.$$

The exact solution of the system is $x(1, 2)$. For initial approximation we choose $x^0(0.9, 1.9)$. We give results of numerical experiments for each of the methods (14) and (13). In Table 1 the following notations are used:

– in the first column a serial number of iteration step has been used;

– in the second column results are given (array $x[\,]$) using the 2-stage Nekrassov–Mehmke method which is based on the Nekrassov–Mehmke 2-method (NM2). The convergence test is $\|x^{k+1} - x^k\|_2 < 10^{-5}$;

– in the third column results are given (array $y[\,]$) using the Nekrassov–Mehmke 2-method (NM2). The convergence test is $\|y^{k+1} - y^k\|_2 < 10^{-5}$.

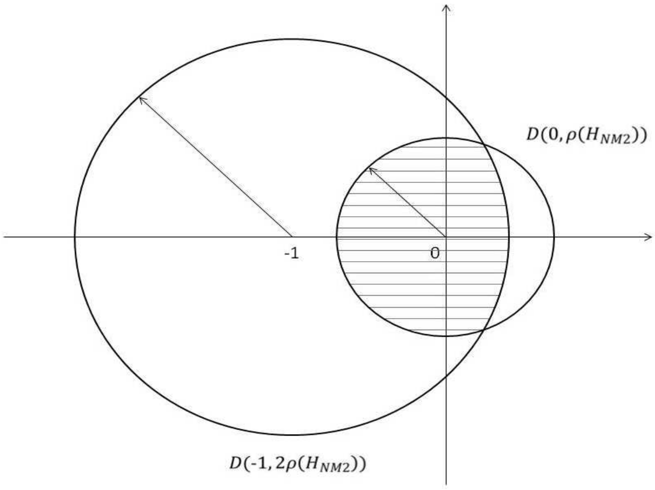

Fig. 1. Convergence region $\Delta(H_{NM2})$

Evidently, 2-NM2 method is better than NM2 method. The eigenvalues of H_{NM2} are $\lambda_1 = -0.7$; $\lambda_2 = 0$, $\lambda(H_{NM2}) \in \Delta(H_{NM2}) = D(0, \rho(H_{NM2})) \cap D(-1, 2\rho(H_{NM2}))$ and $\rho(H_{2-NM2}) \leq \rho(H_{NM2})$. We note that the eigenvalues of H_{2-NM2} are $\mu_1 = 0.15$; $\mu_2 = 0.5$.

2. Now define the splitting $\omega A = (T_m - \omega E_m) - (\omega F_m + (1-\omega)T_m)$, where T_m is a banded matrix of bandwidth $2m+1$.

We define the new

Successive Over Relaxation Generalized Nekrassov–Mehmke method (GNM1) – (SORGNM1):

$$x^{k+1} = (T_m - \omega E_m)^{-1}(\omega F_m + (1-\omega)T_m)x^k + (T_m - \omega E_m)^{-1}\omega b, \quad k = 0, 1, 2, \dots, \tag{16}$$

based on method (3) [1].

Successive Over Relaxation Generalized Nekrassov–Mehmke method (GNM2) – (SORGNM2):

$$x^{k+1} = (T_m - \omega F_m)^{-1}(\omega E_m + (1-\omega)T_m)x^k + (T_m - \omega F_m)^{-1}\omega b, k = 0, 1, 2, \dots \tag{17}$$

based on method (5) and

Symmetric Successive Over Relaxation Generalized Nekrassov–Mehmke method (SSORNM) consists from the cyclic procedures

$$\begin{aligned} x^{k+1/2} &= (T_m - \omega E_m)^{-1}(\omega F_m + (1-\omega)T_m)x^k + (T_m - \omega E_m)^{-1}\omega b, \\ x^{k+1} &= (T_m - \omega F_m)^{-1}(\omega E_m + (1-\omega)T_m)x^{k+1/2} + (T_m - \omega F_m)^{-1}\omega b. \end{aligned} \tag{18}$$

Table 1

1	$X[1] = 0.98500000000000000000$ $X[2] = 2.3000000000000000000$	$Y[1] = 1.0700000000000000000$ $Y[2] = 2.7000000000000000000$
2	$X[1] = 0.99775000000000000000$ $X[2] = 2.2025000000000000000$	$Y[1] = 0.95100000000000000000$ $Y[2] = 1.5100000000000000000$
3	$X[1] = 0.99966250000000000000$ $X[2] = 2.1091250000000000000$	$Y[1] = 1.0343000000000000000$ $Y[2] = 2.3430000000000000000$
4	$X[1] = 0.99994937500000000000$ $X[2] = 2.0557437500000000000$	$Y[1] = 0.97599000000000000000$ $Y[2] = 1.7599000000000000000$
5	$X[1] = 0.99999240625000000000$ $X[2] = 2.0280490625000000000$	$Y[1] = 1.0168070000000000000$ $Y[2] = 2.1680700000000000000$
6	$X[1] = 0.99999886093750000000$ $X[2] = 2.0140511093750000000$	$Y[1] = 0.98823510000000000000$ $Y[2] = 1.8823510000000000000$
7	$X[1] = 0.99999982914062500000$ $X[2] = 2.0070295414062500000$	$Y[1] = 1.0082354300000000000$ $Y[2] = 2.0823543000000000000$
8	$X[1] = 0.99999997437109375000$ $X[2] = 2.0035153687109375000$	$Y[1] = 0.99423519900000000000$ $Y[2] = 1.9423519900000000000$
9	$X[1] = 0.99999999615566406250$ $X[2] = 2.0017577740566406250$	$Y[1] = 1.0040353607000000000$ $Y[2] = 2.0403536070000000000$
10	$X[1] = 0.99999999942334960938$ $X[2] = 2.0008789004834960937$	$Y[1] = 0.99717524751000000000$ $Y[2] = 1.9717524751000000000$
11	$X[1] = 0.99999999991350244141$ $X[2] = 2.0004394522600244140$	$Y[1] = 1.0019773267430000000$ $Y[2] = 2.0197732674300000000$
12	$X[1] = 0.99999999998702536621$ $X[2] = 2.0002197264327536621$	$Y[1] = 0.99861587127990000000$ $Y[2] = 1.9861587127990000000$
13	$X[1] = 0.99999999999805380493$ $X[2] = 2.0001098632617880493$	$Y[1] = 1.0009688901040700000$ $Y[2] = 2.0096889010407000000$
14	$X[1] = 0.99999999999970807074$ $X[2] = 2.0000549316377057073$	$Y[1] = 0.99932177692715100000$ $Y[2] = 1.9932177692715100000$
15	$X[1] = 0.99999999999995621061$ $X[2] = 2.0000274658198746060$	$Y[1] = 1.0004747561509943000$ $Y[2] = 2.0047475615099430000$
16	$X[1] = 0.99999999999999343159$ $X[2] = 2.0000137329100905659$	$Y[1] = 0.99966767069430399000$ $Y[2] = 1.9966767069430399000$
17	$X[1] = 0.99999999999999901474$ $X[2] = 2.0000068664550682724$	$Y[1] = 1.0002326305139872070$ $Y[2] = 2.0023263051398720700$
18		$Y[1] = 0.99983715864020895510$ $Y[2] = 1.9983715864020895510$
19		$Y[1] = 1.0001139889518537314$ $Y[2] = 2.0011398895185373143$
20		$Y[1] = 0.99992020773370238802$ $Y[2] = 1.9992020773370238802$
39		$Y[1] = 1.0000000909543680130$ $Y[2] = 2.0000009095436801300$

This gives the recurrence

$$x^{k+1} = R_\omega x^k + r_\omega, \quad k = 0, 1, 2, \ldots, \tag{19}$$

where

(20)

$$R_\omega = (T_m - \omega F_m)^{-1}(\omega E_m + (1-\omega)T_m)(T_m - \omega E_m)^{-1}(\omega F_m + (1-\omega)T_m),$$

$$r_\omega = \omega(T_m - \omega F_m)^{-1}b + (T_m - \omega F_m)^{-1}(\omega E_m + (1-\omega)T_m)(T_m - \omega E_m)^{-1}\omega b$$

$$= \omega(T_m - \omega F_m)^{-1}\left(I + (\omega E_m + (1-\omega)T_m)\left(T_m - \omega E_m^{-1}\right) b.\right.$$

For other results, see [15].

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