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Zero-Distribution of a Class of Finite Fourrier Transforms

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Sufficient conditions the entire functions

$$E(F;z) = \int_{-1}^{1} F(t)\exp(izt)\,dt \tag{$*$}$$

to have finite many zeros outside a region of the kind

$$S(\sigma) = \{z = x + iy : -\infty < x < \infty, |y| < (1 + |x|)^{-\sigma}, 0 < \sigma < \infty\}$$

are proposed provided the complex function $F \in L(-1,1)$ is such that either $F(-t) = \overline{F(t)}$, or $F(-t) = -\overline{F(t)}, -1 \le t \le 1$. Asymptotics of the zeros of entire functions of the kind $(*)$, having only real zeros, is studied.

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0. Introduction

In this paper the Fourier transform of a complex function of a real variable having compact support is called finite. Let F be such a function and denote by $[a,b], a < b$ the minimal compact interval of $\mathbb{R}$ containing its support. If F is L-integrable, then its Fourier transform

$$\int_{a}^{b} F(t)\exp(izt)\,dt, \tag{0.1}$$

as a function of the complex variable z, is an entire function of exponential type. As it is well-known, such functions play an important role in the analysis. They find applications, e.g., in the theory of approximations, harmonic analysis, and in convolutional calculus.

A classical result due to R. Paley and N. Winer, says that an entire function of (normal) exponential type σ is in $L^2(-\infty,\infty)$ iff it has the form

$$\int_{-\sigma}^{\sigma} F(t)\exp(izt)\,dt, \tag{0.2}$$

where $F\in L^2(-\sigma,\sigma)$. It is quite evident that the study of zero-distribution of entire functions of the kind (0.1) can be reduced to that of the entire functions of the kind (0.2) with $F\in L(-\sigma,\sigma)$. Indeed, the linear substitution $t\longmapsto t+(a+b)/2$ transforms the entire function (0.1) into

$$\exp(i(a+b)z/2)\int_{-\sigma}^{\sigma} F(t+(a+b)/2)\exp(izt)\,dt,$$

where $\sigma=(b-a)/2$. Then, replacing t by t/σ and afterwards z by σz, one comes, in fact, to entire functions of the kind $(*)$.

It is evident that the zero-distribution of the entire functions of the kind (0.1) is "equivalent" to that of the function defined by the Laplace-type transform of the kind

$$\int_{a}^{b} F(t)\exp(zt)\,dt, \tag{0.3}$$

where $F\in L(a,b)$. More precissely, the zeros of the entire functions (0.1) and (0.3) are related by means of the correspondance $z\longleftrightarrow -iz$.

It seems that the first study of the zero-distribution of entire functions of the kind (0.3) with $a=0$ and $b=1$ is due to G. Pólya [4]. But the corresponding results play an auxiliary role in the paper just mentioned, since the main attention in it is paid to the zero-distribution of entire functions defined as finite cos- or sin-Fourier transforms. More precisely, sufficient conditions are given in order that such functions to have only real zeros as well as their zeros to be separated either by that of the entire function $\sin z$, or $\cos z$.

A more systematical study of the zero-distribution of the entire functions (0.3) is performed by E. Titchmarsh [7]. By the author, the main result in his paper is that if $n(r)$ is the number of the zeros of the function (0.3) in the disk $|z|\le r$, then $n(r)\sim\pi^{-1}(b-a)r, r\to\infty$.

Further, as R.E. Langer claimed [2], in M.L. Cartwright's paper [1] it is proved that if $a=-1, b=1$ and the function F is continuous at the points $-1,1$ and has bounded variation in the interval $[-1,1]$, then there exists a strip of the kind $|\Re z|<K$ containing all the zeros of the function (0.3). Moreover,

if the function F is continuous on the whole interval $[-1,1]$ and $\omega(F;\delta)$ is its modulus of continuity, then the zeros of the function (0.3) are located in a region determined by the inequality $|\Re z| < K|z|\omega(F;|z|^{-1})$.

The results in this paper concerning the asymptotics of the zeros of the entire functions $(*)$ may be regarded as such of Cartwright's type, but the "width" of the domain $S(\sigma)$, containing almost all the zeros of the functions $E(F;z)$, is determined by the growth of Fourier's coefficients of the function $F(t/\pi)$. The results for the asymptotics of the zeros of functions of the kind $(*)$ having only real zeros may be considered as refinements of Pólya's ones. They also generalize some results published in the papers [5] and [6].

1. Location of the zeros of a class of meromorphic functions

Let $a_n, n \in \mathbb{Z}$ be dinstinct real numbers such that $\lim_{|n|\to\infty} |a_n| = \infty$. Suppose that γ and $A_n, n \in \mathbb{Z}$ are real and the series in

$$A(z) = -\gamma + \sum_{n=-\infty}^{\infty} \frac{A_n}{z - a_n} \tag{1.1}$$

is uniformly convergent on each bounded subset of the region $\mathbb{C} \setminus \{a_n, n \in \mathbb{Z}\}$.

[1.1] *Suppose that the real number* $\gamma \neq 0$ *and let* $\mu_n, n \in \mathbb{Z}$ *be real and positive numbers such that* $\sum_{n=-\infty}^{\infty} \mu_n = 1$. *Define* $a_n^* = a_n + A_n(\gamma\mu_n)^{-1}, n \in \mathbb{Z}$ *and let* C_n *be the circle with diameter* $a_n ... a_n^*$ *if* $A_n \neq 0$ *and let* $C_n = \emptyset$, *otherwise. Then, the meromorphic function* (1.1) *has no zeros outside the union of the circles* $C_n, n \in \mathbb{Z}$.

The proof is rather elementary. Suppose that $\gamma > 0$. If $A_n \neq 0$, then the homografic transformation $w = A_n(z - a_n)^{-1}$ maps the outside of C_n onto the half-plane $\Re w < \gamma\mu_n$. If z_0 is outside $\cup_{n\in\mathbb{Z}} C_n$, then

$$\Re\left(\sum_{n=-\infty}^{\infty} \frac{A_n}{z - a_n}\right) = \sum_{n=-\infty}^{\infty} \Re\left(\frac{A_n}{z - a_n}\right) < \sum_{n=-\infty}^{\infty} \gamma\mu_n = \gamma.$$

i.e. z_0 cannot be a zero of the function (1.1). If $\gamma < 0$, then the above reasonings can be applied to the function $-A(z)$.

R e m a r k. In Part VI of N. Obrechkoff's paper [3] is studied the zero-distribution in the complex plane of some classes of rational functions. On p. 144 it is mentioned that the derived theorems could be applied also to meromorphic functions of the kind (1.1).

2. Mittag-Leffler's decomposition of a class of meromorphic functions

[2.1] *If the complex function $F \in L(-1,1)$, then*

$$\frac{E(F;z)}{\sin z} = \sum_{n=-\infty}^{\infty} (-1)^n \frac{E(F;\pi n)}{z-\pi n}, \quad z \in \mathbb{C}\setminus \pi\mathbb{Z}. \tag{2.1}$$

Moreover, the series in the above equality is uniformly convergent on each bounded subset of the region $\mathbb{C}\setminus\pi\mathbb{Z}$.

This assertion can be proved by means of a classical method due to CAUCHY. To that end denote by $R_N, N\in\mathbb{N}$ the rectangle with vortices $\pi(N+1/2)+iN, -\pi(N+1/2)+iN, -\pi(N+1/2)-iN, \pi(N+1/2)-iN$. If $B\subset\mathbb{C}\setminus\pi\mathbb{Z}$ is bounded, then there exists $N_0\in\mathbb{N}$ such that B is in the interior of R_N for each $N>N_0$. For such $N's$ and $z\in B$ define

$$I_N(F;z) = \frac{1}{2\pi i}\int_{R_N} \frac{E(F;\zeta)}{(\zeta-z)\sin\zeta}\,d\zeta.$$

Then,

$$I_N(F;z) = \frac{E(F;z)}{\sin z} - \sum_{n=-N}^{N} (-1)^n \frac{E(F;\pi n)}{z-\pi n}$$

and, hence, in order to verify the validity of (2.1), it is sufficient to prove that $\lim_{N\to\infty} I_N(F;z)=0$ uniformly with respect to $z\in B$.

There exists $q=q(B,N_0)\in(0,1)$ such that $|z\zeta^{-1}|<q$ for each $z\in B$ provided $\zeta\in R_N$ and $N>N_0$. Then,

$$|I_N(f;z)| \le \frac{p_N M_N(F)}{2\pi(1-q)N}, \quad z\in B, N>N_0,$$

where $p_N = 4(\pi(N+1/2)+N)$ is the perimeter of R_N and

$$M_N(F) = \max\{\zeta\in R_N : |E(F;\zeta)(\sin\zeta)^{-1}|\}.$$

It is clear it remains to show that $\lim_{N\to\infty} M_N(F)=0$. Since

$$E(F;-z) = \int_{-1}^{1} F(-t)\exp(izt)\,dt$$

and

$$\overline{E(F;z)} = \int_{-1}^{1} \overline{F(-t)}\exp(izt)\,dt$$

we have to prove that ($\zeta = \xi + i\eta$)

$$\lim_{N\to\infty} \max_{0\le\xi\le\pi(N+1/2),\eta=N} |E(F;\zeta)(\sin\zeta)^{-1}| = 0$$

and

$$\lim_{N\to\infty} \max_{\xi=\pi(N+1/2),0\le\eta\le N} |E(F;\zeta)(\sin\zeta)^{-1}| = 0$$

under the only assumption that the function $F \in L(-1,1)$.

For each $\zeta = \xi + iN, \xi \in \mathbb{R}, |\sin\zeta| > (1/4)\exp N$ provided N is so large that $1 - \exp(-2N) > 1/2$. Then,

$$|E(F;\xi+iN)(\sin(\xi+iN))^{-1}| \le (1/4)\int_{-1}^{1} |F(t)|\exp(-N(1+t))\,dt$$

and, hence, $\lim_{N\to\infty} E(F;\xi+iN)(\sin(\xi+iN))^{-1} = 0$ uniformly with respect to $\xi \in \mathbb{R}$. Further,

$$\begin{aligned} &E(F;\pi(N+1/2)+i\eta)(\sin(\pi(N+1/2)+i\eta))^{-1} \\ &= 2(-1)^N(\exp\eta+\exp(-\eta))^{-1}\int_{-1}^{1} F(t)\exp(-\eta t)\exp(i\pi(N+1/2)t)\,dt \\ &= 2(-1)^N(1+\exp(-2\eta))^{-1}\int_{-1}^{1} F(t)\exp(-\eta(1+t))\exp(i\pi(N+1/2)t)\,dt \\ &= -2i(1+\exp(-2\eta))^{-1}\int_{0}^{2} F(t-1)\exp(-\eta t)\exp(i\pi(N+1/2)t)\,dt. \end{aligned}$$

The following assertion is a version of the classical lemma of Riemann-Lebesgue:

[2.2] *If the complex function* $\varphi \in L(0,a), 0 < a \le \infty$, *then*

$$\lim_{\lambda\in\mathbb{R},\lambda\to\infty} \int_0^a \varphi(t)\exp(-\eta t)\exp(i\lambda t)\,dt = 0 \tag{2.2}$$

uniformly on $\eta \in [0,\infty)$.

Suppose that $\varepsilon > 0$, then there exist $\tau = \tau(\varepsilon) \in (0,a)$ and $T = T(\varepsilon) > 0$ such that

$$\left|\int_0^\tau \varphi(t)\exp(-\eta t)\exp(i\lambda t)\,dt\right| < \varepsilon$$

for each $\lambda \in \mathbb{R}$ and $\eta \in [0,\infty)$ as well as

$$\left|\int_\tau^a \varphi(t)\exp(-\eta t)\exp(i\lambda t)\,dt\right| < \varepsilon$$

for each $\lambda \in \mathbb{R}$ and $\eta > T$. Hence, it has to be proved that whatever $0 < \tau < T < \infty, \tau < a$ may be, then (2.2) holds uniformly on $\eta \in [\tau, T]$. The last assertion may be established by approximating the function $\exp(-\eta t)$ in the L-norm by step-functions and afterwards applying the classical Riemann-Lebesgues lemma.

An immediately corollary of (2.2) is that

$$\lim_{N\to\infty} E(F;\pi(N+1/2)+i\eta)(\sin(\pi(N+1/2)+i\eta))^{-1} = 0$$

uniformly on $\eta \in [0,\infty])$. Thus the validity of representation (2.1) is established. In the same way it can be proved that:

[2.3] *If the complex function $F \in L(-1,1)$, then*

$$\frac{E(F;z)}{\cos z} = -\sum_{n=-\infty}^{\infty} (-1)^n \frac{E(F;\pi(n+1/2))}{z-\pi(n+1/2)}, \quad z \in \mathbb{C} \setminus \pi(\mathbb{Z}+1/2) \tag{2.3}$$

and, moreover, the series in (2.3) *is uniformly convergent on each bounded subset of the region* $\mathbb{C} \setminus \pi(\mathbb{Z}+1/2)$.

Remark. The expansions (2.1) and (2.2) can be regarded as generalizations of the representation of the meromorphic function

$$\frac{\int_{-1}^{1} f(t)\cos(zt+\alpha)\,dt}{\cos(z-\beta)}, \quad \alpha,\beta \in \mathbb{R},$$

as a sum of elementary fractions given in TSCHAKALOFF'S paper [8]. In order to get it he has proved that

$$\lim_{n\to\infty} \int_0^1 \psi(t)(\exp yt \pm \exp(-yt))(\exp y + \exp(-y))^{-1} \exp(i\pi nt)\,dt = 0,$$

uniformly with respect to $y \in (-\infty,\infty)$ provided the real function $\psi(t)$ is R-integrable on the interval $[0,1]$.

3. The results

[3.1] *Suppose the complex function $F \in L(-1,1)$ is such that either $F(-t) = \overline{F(t)}$ or $F(-t) = -\overline{F(t)}, -1 \le t \le 1$ and, moreover*

$$(a) \qquad E(F;\pi n) = O(|n|^{-2-\lambda}), \lambda > 0, \quad |n| \to \infty;$$

$$(b) \qquad \gamma(E,F) = \sum_{n=-\infty}^{\infty} (-1)^n E(F;\pi n) \neq 0\,.$$

Then, whatever $\varepsilon \in (0, \lambda)$ may be, the entire function $E(F; z)$ has finite many zeros outside the region $S(\lambda - \varepsilon)$.

From **[2.1]** it follows that

$$\frac{zE(F;z)}{\sin z} = \gamma(E,F) + \sum_{n=-\infty}^{\infty} (-1)^n \frac{\pi n E(F;\pi n)}{z - \pi n} \tag{3.1}$$

for each $z \in \mathbb{C} \setminus \pi\mathbb{Z}^*$ $(\mathbb{Z}^* = \mathbb{Z} \setminus \{0\})$ and, moreover, the series in (3.1) is uniformly convergent on each bounded subset of the region $\mathbb{C} \setminus \pi\mathbb{Z}^*$. Suppose that $F(-t) = \overline{F(t)}, -1 \le t \le 1$, then

$$\overline{E(F;x)} = \int_{-1}^{1} \overline{F(t)} \exp(-ixt)\, dt = \int_{-1}^{1} \overline{F(-t)} \exp(ixt)\, dt$$

$$= \int_{-1}^{1} f(t) \exp(ixt)\, dt = E(F;x)$$

for each $x \in \mathbb{R}$, i.e. all $E(F; n\pi), n \in \mathbb{Z}^*$ are real.

Let $\varepsilon \in (0, \lambda), \tau \in (0, \varepsilon)$ and define $D(\tau) = \sum_{k \in \mathbb{Z}^*} (1 + |k|)^{-1-\tau}, \mu_n = (D(\tau))^{-1}(1 + |n|)^{-1-\tau}, n \in \mathbb{Z}^*$ and $a_n^* = \pi n + (\gamma(E,F)\mu_n)^{-1} \pi n E(F; \pi n), n \in \mathbb{Z}^*$. Let C_n be the circle with diameter $\pi n ... a_n^*$ if $E(F; \pi n) \neq 0$ and $C_n = \emptyset$, otherwise. Then, by assertion **[1.1]** the entire function $E(F; z)$ has no zeros outside the union of the circles $C_n, n \in \mathbb{Z}^*$.

Further, all but finite many of the circles $C_n, n \in \mathbb{Z}^*$ are in the region $S(\lambda - \varepsilon)$. If this is not the case, then there exists a sequence of points $z_{n_\nu} = x_{n_\nu} + iy_{n_\nu}, \nu \in \mathbb{N}$ such that $z_{n_\nu} \in C_{n_\nu} \cap \partial S(\lambda - \varepsilon)$, i.e. $x_\nu \in (\pi n_\nu, a_{n_\nu}^*)$ and $|y_{n_\nu}| = (1 + |x_{n_\nu}|)^{-\lambda+\varepsilon}, \nu \in \mathbb{N}$. But this leads to a contradiction. Indeed, if $r_{n_\nu} = (1/2)|\pi n_\nu - a_{n_\nu}^*|$ is the radius of the circle C_{n_ν}, then on one hand $|y_{n_\nu}| \le r_{n_\nu} = O(|n_\nu|^{-\lambda+\tau}), \nu \to \infty$, but on the other, $|y_{n_\nu}| = |x_{n_\nu}|^{-\lambda+\varepsilon}(1 + |x_{n_\nu}|^{-1})^{\lambda+\varepsilon} \sim |\pi n_\nu|^{-\lambda+\varepsilon}, \nu \to \infty$.

A simple example illustrating assertion **[3.1]** provides the function

$$F(t) = \sum_{k=-\infty}^{\infty} \frac{\exp(-ik\pi t)}{(1 + |k|)^{2+\lambda}}, \quad \lambda > 0, t \in \mathbb{R}.$$

Indeed,

$$E(F; \pi n) = \sum_{k=-\infty}^{\infty} (1 + |k|)^{-2-\lambda} \int_{-1}^{1} \exp(i(n-k)\pi t)\, dt = 2(1 + |n|)^{-2-\lambda}, n \in \mathbb{Z}.$$

Moreover,

$$\gamma(E,F) = 2 - 2\sum_{n\in\mathbb{Z}^*}(-1)^{n-1}(1+|n|)^{-2-\lambda}$$

$$= 2 - 4\sum_{n=1}^{\infty}(-1)^{n-1}(1+n)^{-2-\lambda} > 2 - 2^{-\lambda} > 0.$$

Remark. The validity of the assertion **[3.1]** when $F(-t) = -\overline{F(t)}, -1 \leq t \leq 1$ is a corollary of the fact that $iF(t) = -i\overline{F(t)} = \overline{(iF(t))}, -1 \leq t \leq 1$.

[3.2] *Suppose the complex function* $F \in L(-1,1)$ *is such that either* $F(-t) = \overline{F(t)}$ *or* $F(-t) = -\overline{F(t)}, -1 \leq t \leq 1$ and, moreover, that:

$$(a) \qquad E(F;\pi(n+1/2)) = O(|n|^{-2-\lambda}), \lambda > 0, \quad |n| \to \infty,$$

$$(b) \qquad \delta(E,F) = \sum_{n=-\infty}^{\infty}(-1)^n E(F;\pi(n+1/2)) \neq (0).$$

Then, whatever $\varepsilon \in (0,\lambda)$ *may be, the entire function* $E(f;z)$ *has finite many zeros outside the region* $S(\lambda-\varepsilon)$.

The proof is based on the representation

$$\frac{zE(F;z)}{\cos z} = -\delta(E,F) - \sum_{n=-\infty}^{\infty}(-1)^n \frac{\pi(n+1/2)E(F;\pi(n+1/2))}{z-\pi(n+1/2)}, \tag{3.2}$$

which is a corollary of the expansion (2.3). The requirements (a) and (b) of the above assertion are satisfied, e.g. by the function

$$F(t) = \sum_{k=-\infty}^{\infty}\frac{\exp(-i(k+1/2)\pi t)}{(1+|k|)^{2+\lambda}}, \lambda > 0, t \in \mathbb{R}.$$

Suppose that the series

$$\sum_{n=-\infty}^{\infty}(-1)^n E(F;\pi n) \tag{3.3}$$

is convergent and define ($|r| < 1$)

$$\gamma(E,F;r) = E(F;0) + \sum_{n=1}^{\infty}(-1)^n E(F;\pi n)r^n + \sum_{n=1}^{\infty}(-1)^n E(F;-\pi n)r^n, |r| \leq 1.$$

A simple calculation yields that

$$\gamma(E,F;r) = \int_{-1}^{1} \frac{1-r^2}{1+2r\cos\pi t + r^2} F(t)\,dt.$$

By Abel's theorem for the power series $\gamma(E,F) = \lim_{r\to 1-0}\gamma(E,F;r)$. Hence, the requirement that $\gamma(E,F)\neq 0$, under which the assertion **[3.1]** was proved, may be replaced by $\lim_{r\to 1-0}\gamma(E,F;r)\neq 0$, i.e. by

$$\lim_{r\to 1-0}\int_{-1}^{1} \frac{1-r^2}{1+2r\cos\pi t + r^2} F(t)\,dt \neq 0. \tag{3.4}$$

Similarly ($|r|<1$),

$$\begin{aligned}\delta(E,F;r) &= \sum_{n=-\infty}^{\infty} (-1)^n E(F;\pi(n+1/2))r^n \\ &= \int_{-1}^{1} \frac{1-r^2}{1+2r\cos\pi t + r^2} F(t)\exp(i\pi t/2)\,dt\end{aligned} \tag{3.5}$$

and, hence, $\delta(E,F)\neq 0$ iff

$$\lim_{r\to 1-0}\int_{-1}^{1} \frac{1-r^2}{1+2r\cos\pi t + r^2} F(t)\exp(i\pi t/2)\,dt \neq 0, \tag{3.6}$$

provided the series

$$\sum_{n=-\infty}^{\infty} (-1)^n E(F;\pi(n+1/2)) \tag{3.7}$$

is convergent.

It is not so "hopeless" to check the validity of (3.4) and (3.6). Indeed, suppose that F is real and, moreover, $m(F) = \inf_{-1\le t\le 1} F(t) > 0$. Then,

$$\begin{aligned}&\int_{-1}^{1} \frac{1-r^2}{1+2r\cos\pi t + r^2} F(t)\,dt \\ \geq\ & \frac{m(F)}{\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+2r\cos t + r^2}\,dt = \frac{m(F)}{\pi}2\pi = 2m(F) > 0.\end{aligned}$$

If, in addition, F is even, then

$$\int_{-1}^{1} \frac{1-r^2}{1+2r\cos\pi t + r^2} F(t)\exp(i\pi t/2)\,dt$$

$$\geq \frac{4m(F)}{\pi}\int_0^{\pi/2}\frac{1-r^2}{1+2r\cos 2t+r^2}\cos t\,dt$$

$$= \frac{4m(F)}{\pi}\int_0^1\frac{1-r^2}{(1-r)^2+4rt^2}\,dt = \frac{2m(F)}{\pi}\frac{1+r}{\sqrt{r}}\arctan\frac{2\sqrt{r}}{1-r}$$

and, hence,

$$\lim_{r\to 1-0}\int_{-1}^1\frac{1-r^2}{1+2r\cos\pi t+r^2}F(t)\exp(i\pi t/2)\,dt$$

$$\geq \frac{2m(F)}{\pi}\lim_{r\to 1-0}\frac{1+r}{\sqrt{r}}\arctan\frac{2\sqrt{r}}{1-r} = 2m(F) > 0.$$

Let the real function $f \in L(0,1)$ and define

$$U(f;z) = \int_0^1 f(t)\cos zt\,dt \tag{3.8}$$

and

$$V(f;z) = \int_0^1 f(t)\sin zt\,dt \tag{3.9}$$

Evidently, $U(f;z) = E(F;z)$ and $V(f;z) = E(-iG;z)$, where $F(t) = G(t) = (1/2)f(t)$ and $F(-t) = -G(-t) = (1/2)f(t)$ for $0 \leq t \leq 1$. Then, the following assertions are corollaries of **[3.1]** and **[3.2]**, respectively:

[3.3] *Let the real function $f \in L(0,1)$ is such that:*

(a) $U(f;\pi n) = O(|n|^{-2-\lambda}), \lambda > 0, \quad |n| \to \infty;$

(b) $\gamma(U,f) = \sum_{n=-\infty}^{\infty}(-1)^U(f;\pi n) \neq 0.$

Then, whatever $\varepsilon \in (0,\lambda)$ may be, the entire function $U(f;z)$ has finite many zeros outside the region $S(\lambda-\varepsilon)$.

[3.4]. *Let the real function $f \in L(0,1)$ is such that:*

(a) $V(f;\pi(n+1/2)) = O(|n|^{-2-2\lambda}), \lambda > 0, \quad |n| \to \infty;$

(b) $\delta(V,f) = \sum_{n=-\infty}^{\infty}(-1)^n V(f;\pi(n+1/2)) \neq 0.$

Then whatever $\varepsilon \in (0,\lambda)$ may be, the entire function $V(f;z)$ has finite many zeros outside the region $S(\lambda-\varepsilon)$.

More precisely, the proofs of assertions **[3.3]** and **[3.4]** are based on the expansions

$$\frac{zU(f;z)}{\sin z} = \gamma(U;f) + \sum_{n=-\infty}^{\infty} (-1)^n \frac{\pi n U(f;\pi n)}{z - \pi n}, z \in \mathbb{C} \setminus \pi\mathbb{Z}, \tag{3.10}$$

$$\frac{zV(f;z)}{\cos z} = -\delta(V,f) - \sum_{n=-\infty}^{\infty} (-1)^n \frac{\pi(n+1/2)V(f;\pi(n+1/2))}{z - \pi(n+1/2)}, \tag{3.11}$$

$$z \in \mathbb{C} \setminus \pi(\mathbb{Z} + 1/2),$$

which are corollaries of the expansions (3.1) and (3.2) respectively.

Examples:

1. Suppose the real function $f(t), 0 \le t \le 1$ has second derivative with bounded variation and, moreover, $f'(0) = f'(1) = 0$. Then,

$$U(f;n\pi) = \frac{1}{n^2\pi^2} \int_0^1 f''(t) \cos \pi n t \, dt = -\frac{1}{\pi^3 n^3} \int_0^1 \sin \pi n t df''(t)$$

and, hence, $U(f;\pi n) = O(n^{-3}), n \to \infty$. Since

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\pi^3 n^3} \sin \pi n t = (1/12)(t - t^3), \quad |t| < 1,$$

$$\gamma(U;f) = U(f;0) - 2\sum_{n=1}^{\infty} (-1)^{n-1} U(f;\pi n)$$

$$= \int_0^1 f(t)\, dt - (1/6) \int_0^1 (t - t^3) df''(t) = f(1),$$

i.e. under the additional condition that $f(1) \neq 0$, the entire function $U(f;z)$ has finite many zeros outside the region $S(1-\varepsilon)$ whatever $\varepsilon \in (0,1)$ may be.

2. Suppose the real function $f(t), 0 \le t \le 1$ has second derivative with bounded variation and, moreover, $f(0) = f'(1) = 0$. Then,

$$V(f;\pi(n+1/2)) = \frac{1}{((n+1/2)\pi)^3} \left\{ f''(0) + \int_0^1 \cos \pi(n+1/2) t df''(t) \right\},$$

i.e. $V(f;\pi(n+1/2)) = O(n^{-3}), n \to \infty$. Since,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\pi^3(n+1/2)^3} \cos\pi(n+1/2)t = (1/4)(1-t^2), \quad |t| \le 1,$$

$$\delta(V;f) = (1/2)f'(0) - f(1) - \int_0^1 f(t)\,dt.$$

Hence, if $\delta(V;f) \neq 0$, then, the entire function $V(f;z)$ has finite many zeros outside the region $S(1-\varepsilon)$ whatever $\varepsilon \in (0,1)$ may be.

[3.5] *Suppose that* $\{a_n, n \in \mathbb{N}\}$ *is an increasing sequence of real and positive numbers such that* $\lim_{n\to\infty} a_n = \infty, A_n > 0, n \in \mathbb{N}, \gamma \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ *and that at least one of the series in*

$$A(z) = -\gamma - \sum_{n=1}^{\infty} \frac{A_n}{z+a_n} + \sum_{n=1}^{\infty} \frac{A_n}{z-a_n} \tag{3.12}$$

is absolutely uniformly convergent on each bounded subset of the region $\mathbb{C} \setminus \{-a_n, a_n, n \in \mathbb{N}\}$.

If $\gamma < 0$, *then the meromorphic function* A *has no nonreal zeros and each of the intervals* $-I_n = (-a_{n+1}, a_n), (-a_1, 0), (0, a_1), I_n = (a_n, a_{n+1}), n \in \mathbb{N}$ *contains only one of its zeros. If* $\gamma > 0$, *then* A *has a pair of conjugate pure imaginary zeros, a simple zero in each of the intervals* $-I_n, I_n, n \in \mathbb{N}$ *and has no other zeros.*

Since $\lim_{x\to a_n+0} A_N(x) = \lim_{x\to -a_n-0} A_n(x) = \infty$ and $\lim_{x\to a_{n+1}-0} A_N(x) = \lim_{x\to -a_{n+1}+0} A_N(x) = -\infty$ for each $n = 1, 2, 3, \ldots, N-1$, the rational function

$$A_N(z) = -\gamma - \sum_{n=1}^{N} \frac{A_n}{z+a_n} + \sum_{n=1}^{N} \frac{A_n}{z-a_n}$$

has an odd number of zeros in each of the intervals $-I_n, I_n,\ n = 1, 2, 3, \ldots, N-1$. Moreover, since it is even, the number of its zeros in the interval $-I_n$ is the same as in the interval I_n for each $n = 1, 2, 3, \ldots, N-1$.

In the representation

$$A_N(z) = \frac{P_N(z)}{Q_N(z)}, \quad Q_n(z) = \prod_{n=1}^{N} (z^2 - a_n^2)$$

the real polynomial P_N, which is of degree $2N$, has at least $2(N-1) = 2N-2$ real zeros. Hence, the function A_N has exactly one zero in each of the intervals $-I_n, I_n, n = 1, 2, 3, \ldots, N-1$.

Further, $\lim_{x\to -a_1+0} A_N(x) = \lim_{x\to a_1-0} A_N(x) = -\infty$ and $A_N(0) = -\gamma$. Hence, either A_N has no nonreal zeros and each of the intervals $-I_n, (-a_1, 0), (0, a_1), I_n, n = 1, 2, 3, \ldots, N-1$ contains only one of its zeros if $\gamma < 0$, or A_N has a pair of conjugate pure imaginary zeros, a simple zero in each of the intervals $-I_n, I_n, n = 1, 2, 3, \ldots, N-1$ and no other nonreal zeros if $\gamma > 0$. Since $\lim_{N\to\infty} A_N(z) = A(z)$ uniformly on each bounded subset of the region $\mathbb{C} \setminus \{-a_n, a_n, n \in \mathbb{N}\}$, the same conclusions hold for the meromorphic function A because of a classical theorem due to A. Hurwitz.

[3.6] *Suppose that the real function $f \in L(0,1)$ is such that:*

$$(a) \qquad \int_0^1 f(t)\cos \pi nt\, dt = O(n^{-2-\lambda}), \lambda > 0, \quad n \to \infty;$$

$$(b) \qquad (-1)^n \int_0^1 f(t)\cos \pi nt)\, dt > 0, n \in \mathbb{N} \cup \{0\}.$$

Then, the entire function $U(f; z)$ has no nonreal zeros and each of the intervals $I_n = (\pi n, \pi(n+1)), n \in \mathbb{N} \cup \{0\}$ contains exactly one of its zeros. Moreover, if $x_n, n \in \mathbb{N}\cup\{0\}$ is its zero in I_n, then $0 < x_n - \pi n = O(n^{-\lambda+\varepsilon}), n \to \infty$ whatever $\varepsilon \in (0, \lambda)$ may be.

Since the requirement (b) yields that

$$\gamma(U, f) = U(f; 0) + 2\sum_{n=1}^{\infty}(-1)^n U(f; \pi n) > 0,$$

the validity of the above assertion is a corollary of the expansion (3.10) and the assertions **[3.5]** and **[3.1]**.

[3.7] *Suppose that the real function $f \in L(0,1)$ is such that*

$$(a) \qquad \int_0^1 f(t)\sin \pi(n+1/2)t\, dt = O(n^{-2-\lambda}), \lambda > 0, \quad n \to \infty;$$

$$(b) \qquad (-1)^n \int_0^1 f(t)\sin \pi(n+1/2)t\, dt > 0, \quad n \in \mathbb{N} \cup \{0\}.$$

Then, the entire function $V(f; z)$ has no nonreal zeros and each of the intervals $I_n = (\pi(n+1/2), \pi(n+3/2)), n \in \mathbb{N} \cup \{0\}$ contains exactly one of its zeros. Moreover, if x_n is its zero in I_n, then $0 < x_n - \pi(n+1/2) = O(n^{-\lambda+\varepsilon}), n \to \infty$ whatever $\varepsilon \in (0, \lambda)$ may be.

This assertion is a corollary of the expansion (3.11) and the assertions **[3.5]** and **[3.1]**. Indeed, from the requirement (b) it follows that

$$\delta(V,f) = 2\sum_{n=0}^{\infty}(-1)^n V(f;\pi(n+1/2)) > 0.$$

Examples:

1. Denote $\varphi(t) = 2t^2 - t^4, 0 \le t \le 1$, then a simple calculation yields that

$$\int_0^1 \varphi(t)\cos zt\,dt = \frac{8}{z^3}\left(1-\frac{3}{z^2}\right)\sin z + \frac{24\cos z}{z^4}.$$

Hence, $U(\varphi;\pi n) = 24(-1)^n(\pi n)^{-4}$, i.e. the function φ satisfies the conditions of assertion **[3.6]**. Moreover, if ξ_n is the zero of $U(\varphi;z)$ in the interval $(\pi n, \pi(n+1)), n \in \mathbb{N}\cup\{0\}$, then $0 < \xi_n - \pi n = O(n^{-2+\varepsilon})$ whatever $\varepsilon \in (0,2)$ may be.

2. Let $\psi(t) = 2t^3/3 - t^5/5$, then

$$V(\psi;z) = \frac{1}{z}U(\varphi;z) = \frac{8}{z^4}\left(1-\frac{3}{z^3}\right)\sin z + \frac{24\cos s}{z^5}.$$

Hence, $V(\psi;\pi(n+1/2)) = 8(-1)^n(\pi(n+1/2))^{-4}\left(1-3((n+1/2)\pi)^{-2}\right)$, i.e. the function ψ satisfies the conditions of assertion **[3.7]**. Therefore, if η_n is the zero of $V(\psi;z)$ in the interval $(\pi(n+1/2).\pi(n+3/2)), n \in \mathbb{N}\cup\{0\}$, then $0 < \eta_n - \pi(n+1/2) = O(n^{-2+\varepsilon})$ whatever $\varepsilon \in (0,2)$ may be.

It is not surprising that the requirement (b) of assertions **[3.5]** and **[3.7]** is satisfied by the functions $U(\varphi;z)$ and $V(\psi;z)$, respectively. It can be proved that, in fact, it holds under the only assumption that the function f is nonnegative and increasing in the interval $[0,1]$. This may be established in the same way as in Pólya's paper [4]. That means the following assertion holds:

[3.8]. *Suppose that the function $f(t), 0 \le t \le 1$ is nonnegative and increasing. Then, both functions $U(f;z)$ and $V(f;z)$ have only real and simple zeros. Moreover:*

(A) If $U(f;n\pi) = O(n^{-2-\lambda}), \lambda > 0, n \to \infty$ and α_n is the only zero of $U(f;z)$ in the interval $(\pi n, \pi(n+1)), n \in \mathbb{N}\cup\{0\}$, then $0 < \alpha_n - \pi n = O(n^{-\lambda+\varepsilon})$ whatever $\varepsilon \in (0,\lambda)$ may be.

(B) If $V(f;(n+1/2)\pi) = O(n^{-\lambda+\varepsilon}), \lambda > 0, n \to \infty$ and β_n is the only zero of $V(f;z)$ in the interval $(\pi(n+1/2), \pi(n+3/2)), n \in \mathbb{N}\cup\{0\}$, then $0 < \beta_n - \pi(n+1/2) = O(n^{-\lambda+\varepsilon})$ whatever $\varepsilon \in (0,\lambda)$ may be.

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References

[1] M. L. Cartwright. The zeros of certain integral functions, *Quarterly J. Math.*, **1**, (1930), 38-59; doi: 10.1093/qmath/os-1.1.28.

[2] R. E. Langer. On the zeros of exponential sums and integrals, *Amer. Math. Soc.*, **37**, (1930), 213-239.

[3] N. Obrechkoff. On some classes of polynomials and rational functions, *Annuaire Univ. Sofia Phys.-Math. Fac.* **33**, (1936/37) Livre 1. Math. Phys., 39-148 (Bulgarian, French Summary).

[4] G. Pólya. Über die Nullstellen gewisser ganzer Funktionen, *Math. Z.*, **2**, (1918), 352-383.

[5] P. Rusev. On the asumptotic behaviour of the zeros of a class of entire functions, *Proc. Inst. Math. Acad. Bulgare Sci.*, **4**, No 2, (1960), 67-73 (Bulgarian, German Summary)

[6] P. Rusev. Asymptotic properties of the zeros of a class of meromorphic functions, *Annuaire Univ. Sofia Math. Fac.*, **58**, (1963/64), 241-271 (Bulgarian, English Summary).

[7] E. Titchmarsh. The zeros of certain integral functions, *Proc. London Math. Soc.*, **25**, (1926), 283-302.

[8] L. Tschakaloff. On a class of entire functions, *Journal Acad. Bulgare Sci.*, **36**, (1927), 51-92 (Bulgarian, German Summary).

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