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Some Inequalities for Functions of Exponential Type Real on the Real Axis

M. A. Qazi and Q. I. Rahman

1. Introduction and Statement of Results

1.1. Functions of Exponential Type and the Indicator Function

Let f be an entire function and let $M(r) := \max_{|z|=r} |f(z)|$. Unless f is a constant of modulus less than or equal to 1, its *order* is defined [3, Chapter 2] to be

$$\rho := \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}\,.$$

A constant c such that $|c| \le 1$ has order 0, by convention. If f is of finite positive order ρ, then $T := \limsup_{r\to\infty} r^{-\rho} \log M(r)$ is called its *type*.

A function f, analytic in any unbounded subset S of the complex plane, like the sector $S := \{z = r\,\mathrm{e}^{\mathrm{i}\theta} : \alpha < \theta < \beta\}$, is said to be of *exponential type* τ *in* S if for each $\varepsilon > 0$ there exists a constant K, depending on ε but not on z, such that $|f(z)| < K\,\mathrm{e}^{(\tau+\varepsilon)|z|}$ for all $z \in S$.

In view of the preceding definitions, an entire function of order less than 1 is of exponential type τ for any $\tau \ge 0$; functions of order 1 type $T \le \tau$ are also of exponential type τ. It is easily seen that a trigonometric polynomial of degree at most n is the restriction of an entire function of exponential type n to $\mathbb{R}$. Trigonometric polynomials are bounded on the real axis and they are 2π-periodic. It is known (see [3, Theorem 6.10.1]) that if $f(z)$ is an entire function of exponential type τ which is periodic on the real axis with period Δ then it must be of the form $f(z) = \sum_{\nu=-n}^{n} a_\nu\,\mathrm{e}^{2\pi\mathrm{i}\nu z/\Delta}$ with $n \le \lfloor (\Delta/2\pi)\,\tau \rfloor$.

Let f be of exponential type in the sector $\{z = r\,\mathrm{e}^{\mathrm{i}\theta} : r > 0,\, \alpha < \theta < \beta\}$. The *indicator function* of f is defined to be

$$h_f(\theta) := \limsup_{r\to\infty} \frac{\log|f(r\mathrm{e}^{\mathrm{i}\theta})|}{r} \qquad (\alpha < \theta < \beta)\,.$$

Unless $h_f(\theta) \equiv -\infty$, it is continuous. For this and other properties of the indicator function see [3, Chapter 5]. For an entire function f of exponential type, the indicator function $h_f(\theta)$ is defined for all θ. It is clear that if f is an entire function of exponential type τ, then $h_f(\theta) \leq \tau$ for $0 \leq \theta < 2\pi$. It may be noted that if $P(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree at most n, then $f(z) := P(\mathrm{e}^{\mathrm{i}z})$ is an entire function of exponential type n such that $h_f(\pi/2) \leq 0$. Furthermore, $\max_{x\in\mathbb{R}} |f(x)| = \max_{|z|=1} |P(z)|$.

1.2. A Fundamental Property of Functions of Exponential Type

The following result [3, Theorem 6.2.4], a consequence of *the Phragmén-Lindelöf principle*, plays an important role in the study of functions of exponential type. As an alternative reference for this result we mention [11, Theorem 12.6.1], which should be read in conjunction with [11, Theorem 1.6.14].

Theorem A. *Let f be analytic and of exponential type in the open upper half-plane H_+ such that $h_f(\pi/2) \leq c$. Furthermore, let f be continuous in the closed upper half-plane and suppose that $|f(x)| \leq M$ on the real axis. Then*

$$|f(x + \mathrm{i}y)| < M\,\mathrm{e}^{cy} \qquad (-\infty < x < \infty,\ y > 0) \tag{1}$$

unless $f(z) \equiv M\,\mathrm{e}^{\mathrm{i}\gamma}\,\mathrm{e}^{-\mathrm{i}cz}$ for some real γ.

If f is analytic and of exponential type in the lower half-plane such that $h_f(-\pi/2) \leq c$ and $|f(x)| \leq M$ on the real axis, then Theorem A, applied to the function $\overline{f(\overline{z})}$, shows that $|f(x+\mathrm{i}y)| \leq M\,\mathrm{e}^{c|y|}$ for $y < 0$. Hence, by Liouville's theorem, an entire function f of exponential type 0 can be bounded on the real line (in fact, on any line) only if it is a constant.

1.3. The Operator $f \mapsto f'$ and Bernstein's Inequality

If f is an entire function of exponential type τ then so is f'. By a result of Bernstein ([2, p. 102], see septième corollaire), if f is an entire function of exponential type τ such that $|f(x)| \leq 1$ for all real x then $|f'(x)| \leq \tau$ for all real x. It may be added that $|f'(x_0)| = \tau$ for some $x_0 \in \mathbb{R}$ only if $f(z)$ is of the form $a\,\mathrm{e}^{\mathrm{i}\tau z} + b\,\mathrm{e}^{-\mathrm{i}\tau z}$, where $a \in \mathbb{C}$, $b \in \mathbb{C}$ and $|a| + |b| = 1$. In other words, the following result holds.

Theorem B. *Let f be an entire function of exponential type τ and suppose that $|f(x)| \leq 1$ on the real axis. Then, either f is a constant or else f is of order 1 type at most τ and*

$$|f'(x)| \leq \tau \qquad (-\infty < x < \infty)\,. \tag{2}$$

Besides, $|f'(x_0)| = \tau$ for some $x_0 \in \mathbb{R}$ only if $f(z) \equiv a\,\mathrm{e}^{\mathrm{i}\tau z} + b\,\mathrm{e}^{-\mathrm{i}\tau z}$, where $a \in \mathbb{C}$, $b \in \mathbb{C}$ and $|a| + |b| = 1$.

Amongst the extremals there are functions, like $\cos\tau(z+\alpha)$, $\alpha \in \mathbb{R}$, which are real at every point of the real axis and then there are those, like $\mathrm{e}^{\mathrm{i}\tau z}$, which are non-real except at the points $0, \pm\pi/\tau, \pm 2\pi/\tau, \ldots$. In other words, the sharp upper bound for $|f'(x_0)|$, $x_0 \in \mathbb{R}$, given by (2), is the same, whether $f(x)$ is real for all $x \in \mathbb{R}$ or it is non-real almost everywhere on $\mathbb{R}$.

1.4. The Operator $f \mapsto f' + e^{i\gamma}\tau f$, $\gamma \in \mathbb{R}$

It was shown by Duffin and Schaeffer [4, Theorem II] that if f is an entire function of exponential type τ such that $f(x)$ *is real and* $|f(x)| \le 1$ *on the real axis*, then, at any point $z \in \mathbb{C}$ with $\Im z = y$, we have

$$|f'(z)|^2 + \tau^2 |f(z)|^2 \le \tau^2 \cosh 2\tau\, y\,. \tag{3}$$

Unless $f(z)$ is of the form $\cos\tau\,(z+\alpha)$, the equality sign can occur only at points on the real axis where $f(x) = \pm 1$.

Inequality (3) says in particular that if a function f satisfies the conditions of Theorem B and is real on the real axis, then

$$|f'(z) + i\tau f(z)| \;\le\; \tau \qquad (z \in \mathbb{R})\,. \tag{4}$$

How large can $|f'(z)+i\tau f(z)|$ be at any given point z of the complex plane? We would like to know the answer to this question, first when f is simply an entire function of exponential type τ such that $|f(x)| \le 1$ on the real axis and then when $f(x)$ is, in addition, real for real x.

If f is any entire function of exponential type τ such that $|f(x)| \le 1$ on the real axis then, by Theorem B, $|f'(x)| \le \tau$ for all real x. Hence,

$$\Lambda(z) := f'(z) + i\,\tau\, f(z) \tag{5}$$

is an entire function of exponential type τ and $|\Lambda(x)| \le 2\tau$ on the real axis. The same can be said about the function $\overline{\Lambda}(z) := \overline{\Lambda(\overline{z})}$. Applying Theorem A with $c = \tau$ and $M = 2\tau$, to $\overline{\Lambda}$, we obtain

$$\left|\overline{f'(x-iy)} - i\tau\,\overline{f(x-iy)}\right| \le 2\tau\, e^{\tau\, y} \qquad (-\infty < x < \infty,\; y > 0)\,.$$

Hence

$$|f'(x+iy) + i\tau f(x+iy)| \le 2\tau\, e^{-\tau\, y} \qquad (-\infty < x < \infty,\; y < 0)\,. \tag{6}$$

In order to obtain an estimate for $|\Lambda(z)|$ at a point z of the open upper half-plane H_+, we note that $f'(z) + i\tau f(z)$ cannot be of the form $2\tau\, e^{i\gamma}\, e^{-i\tau z}$, since otherwise $f(z)$ would be of the form $(2\tau e^{i\gamma} z + d)\, e^{-i\tau z}$ for some constant d and $|f(x)|$ would not be bounded on the real axis. Hence, by applying (1) with $c = \tau$ and $M = 2\tau$, to Λ, we obtain

$$|f'(x+iy) + i\tau f(x+iy)| = |\Lambda(x+iy)| < 2\tau\, e^{\tau y} \quad (-\infty < x < \infty,\; y > 0)\,. \tag{7}$$

Whereas inequality (6) gives the sharp upper bound for $|f'(z) + i\tau f(z)|$ at any point z of the lower half-plane for a function f satisfying the conditions of Theorem B, inequality (7) leaves much to be desired. The following result (Theorem 1) says considerably more for $y > 1/4\tau$. We shall show by means of an example that the improved bound for $|f'(z) + \mathrm{i}\tau f(z)|$, given in (8), though not attained, cannot be replaced by anything smaller than $(1-3\,\mathrm{e}^{-2})\,(2y)^{-1}\,\mathrm{e}^{\tau y}$, at least for $y > 1/\tau$.

Theorem 1. *Let f be an entire function of exponential type τ and suppose that $|f(x)| \le 1$ on the real axis. Then, the sharp estimate for $|f'(z) + \mathrm{i}\tau f(z)|$ at any point of the lower half-plane is given by* (6)*, whereas at points of the open upper half-plane H_+, we have*

$$|f'(z) + \mathrm{i}\tau f(z)| < \begin{cases} 2\tau\,\mathrm{e}^{\tau y}\,, & \text{if } \ 0 < y \le 1/(4\tau), \\ (2y)^{-1}\,\mathrm{e}^{\tau y}\,, & \text{if } \ 1/(4\tau) \le y < \infty\,. \end{cases} \tag{8}$$

These estimates for $|f'(z)+\mathrm{i}\tau f(z)|$ can be improved if f is, in addition, real on the real axis. In fact, the following result holds.

Theorem 2. *Let f satisfy the conditions of Theorem 1 and suppose that $f(x)$ is real for all real x. Then*

$$|f'(z) + \mathrm{i}\tau f(z)| \le \tau\,\mathrm{e}^{-\tau y} \qquad (y := \Im z < 0)\,, \tag{9}$$

whereas for $z \in H_+$, we have

$$|f'(z) + \mathrm{i}\tau f(z)| < \begin{cases} \tau\,\mathrm{e}^{\tau y}\,, & \text{if } \ 0 < y \le 1/(2\tau), \\ (2y)^{-1}\,\mathrm{e}^{\tau y}\,, & \text{if } \ 1/(2\tau) \le y < \infty\,. \end{cases} \tag{10}$$

The example $f(z) := \sin \tau z$ shows that $|f'(z) + \mathrm{i}\tau f(z)|$ can be equal to $\tau\,\mathrm{e}^{-\tau y}$ at any point of the lower half-plane and so (9) is sharp. The bound given in (10) is not attained but we shall give an example which shows that it cannot be replaced by anything smaller than $\big((1-3\,\mathrm{e}^{-2})/2\big)(2y)^{-1}\,\mathrm{e}^{\tau y}$, at least for $y > 1/(2\tau)$.

1.5. The Paley-Wiener Space

An entire function f is said to belong to the Paley-Wiener space $\mathcal{P}_\tau$ if it is of exponential type τ and is square integrable on the real axis. It is known that if $f \in \mathcal{P}_\tau$, then $f(x) \to 0$ as $x \to \pm\infty$ and that f' belongs to $\mathcal{P}_\tau$ also. For these results see [3, Chapters 6 and 11].

By a fundamental theorem of Plancherel, if $\varphi \in L^2(-\infty,\infty)$ then there exists a function $f \in L^2(-\infty,\infty)$ such that

$$\lim_{A\to\infty} \int_{-\infty}^{\infty} \left| f(x) - \int_{-A}^{A} \mathrm{e}^{\mathrm{i}xt}\,\varphi(t)\,\mathrm{d}t \right|^2 \mathrm{d}x = 0\,.$$

Furthermore,

$$\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x = 2\pi \int_{-\infty}^{\infty} |\varphi(t)|^2\,\mathrm{d}t$$

and

$$\lim_{A\to\infty}\int_{-\infty}^{\infty}\left|\varphi(t)-\frac{1}{2\pi}\int_{-A}^{A}\mathrm{e}^{-\mathrm{i}xt}\,f(x)\,\mathrm{d}x\right|^2\mathrm{d}t=0\,.$$

The function f is called *Fourier transform of* φ.

It was proved by Paley and Wiener (see [7, p. 13]; also see [3, p. 103]) that any function f belonging to $\mathcal{P}_\tau$ is the Fourier transform of a function φ whose support lies in $[-\tau,\tau]$. Their result may be stated as follows.

Theorem C. *The entire function f is of exponential type τ and belongs to L^2 on the real axis if and only if*

$$f(z)=\int_{-\tau}^{\tau}\mathrm{e}^{\mathrm{i}zt}\,\varphi(t)\,\mathrm{d}t\,, \tag{11}$$

where $\varphi\in L^2(-\tau,\tau)$ and

$$\int_{-\infty}^{\infty}|f(x)|^2\,\mathrm{d}x=2\pi\int_{-\tau}^{\tau}|\varphi(t)|^2\,\mathrm{d}t\,. \tag{12}$$

We shall refer to (11) as the "*Paley-Wiener representation*" and to (12) as the "*Parseval's formula*".

The following result is the L^2 analogue of Theorem 1.

Theorem 3. *Let f be an entire function belonging to the Paley-Wiener space $\mathcal{P}_\tau$. Then*

$$\frac{\int_{-\infty}^{\infty}|f'(x+\mathrm{i}y)+\mathrm{i}\tau f(x+\mathrm{i}y)|^2\,\mathrm{d}x}{\int_{-\infty}^{\infty}|f(x)|^2\,\mathrm{d}x}\le\begin{cases}4\tau^2\,\mathrm{e}^{-2\tau y}\,, & \text{if } -\infty<y\le\frac{1}{2\tau},\\ \left(\mathrm{e}^2y^2\right)^{-1}\mathrm{e}^{2\tau y}\,, & \text{if } \frac{1}{2\tau}\le y<\infty\,.\end{cases} \tag{13}$$

The estimates given in (13) cannot be improved. It is notable that the inequality $\int_{-\infty}^{\infty}|f'(x+\mathrm{i}y)+\mathrm{i}\tau f(x+\mathrm{i}y)|^2\,\mathrm{d}x\le 4\tau^2\,\mathrm{e}^{-2\tau y}\int_{-\infty}^{\infty}|f(x)|^2\,\mathrm{d}x$, which is trivial for any $y\le 0$, remains true for $0<y\le 1/(2\tau)$.

We also consider the L^2 analogue of Theorem 2.

Theorem 4. *Let f be an entire function belonging to the Paley-Wiener space $\mathcal{P}_\tau$ such that $f(x)$ is real for all real x. Furthermore, let*

$$U(y,\xi):=\xi^2(\mathrm{e}^{4y\tau})^{1-\xi}+(1-\xi)^2\,(\mathrm{e}^{4y\tau})^{\xi}\qquad(-\infty<y<\infty,\ 0\le\xi\le 1)\,. \tag{14}$$

Then, for any real y, we have

$$\frac{\int_{-\infty}^{\infty}|f'(x+\mathrm{i}y)+\mathrm{i}\tau f(x+\mathrm{i}y)|^2\,\mathrm{d}x}{\int_{-\infty}^{\infty}|f(x)|^2\,\mathrm{d}x}\le 2\,\tau^2\mathrm{e}^{-2\tau y}\max_{0\le\xi\le 1}U(y,\xi)\,. \tag{15}$$

In § 3 we give an example (see Example 5) which shows that inequality (15) is sharp.

The following result is to be compared with the first half of Theorem 2, namely (9); the thing to note is the restriction on y.

Corollary 1. *Let f be an entire function belonging to the Paley-Wiener space $\mathcal{P}_\tau$ such that $f(x)$ is real for all real x. Then*

$$\frac{\int_{-\infty}^{\infty} |f'(x+\mathrm{i}y)+\mathrm{i}\tau f(x+\mathrm{i}y)|^2\,\mathrm{d}x}{\int_{-\infty}^{\infty}|f(x)|^2\,\mathrm{d}x} \le 2\,\tau^2 \mathrm{e}^{-2\tau y} \qquad \left(y \le \frac{\ln 2}{2\tau}\right). \tag{15'}$$

The estimate given in (15′) cannot be improved and the inequality does not remain true for any $y > (\ln 2)/(2\tau)$.

It is quite tricky to determine the value of $\max_{0\le\xi\le 1} U(y,\xi)$, the quantity that appears on the right-hand side of (15). Our proof of Corollary 1 consists mainly in showing that $\max_{0\le\xi\le1} U(y,\xi) = 1$ for $y \le (\ln 2)/(2\tau)$, which we found quite hard to accomplish. However, it is fairly easy to determine $\max_{0\le\xi\le1} U(1/\tau,\xi)$, which we do and use it to find the maximum of $U(y,\xi)$ over $[0,1]$ for any $y \in \big((\ln 2)/(2\tau), 1/\tau\big)$.

Corollary 2. *Let f be an entire function belonging to the Paley-Wiener space $\mathcal{P}_\tau$ such that $f(x)$ is real for all real x. Then*

$$\frac{\int_{-\infty}^{\infty} |f'(x+\mathrm{i}y)+\mathrm{i}\tau f(x+\mathrm{i}y)|^2\,\mathrm{d}x}{\int_{-\infty}^{\infty}|f(x)|^2\,\mathrm{d}x} \le \tau^2 \qquad \left(\frac{\ln 2}{2\tau} \le y \le \frac{1}{\tau}\right). \tag{15''}$$

The estimate given in (15″) cannot be improved.

1.6. Functions g Such That $g(z) \equiv \mathrm{e}^{\mathrm{i}\gamma}\,\mathrm{e}^{\mathrm{i}\sigma z}\,\overline{g(\overline{z})}$, $\sigma > 0$, $\gamma \in \mathbb{R}$

Let g be an entire function belonging to the Paley-Wiener space $\mathcal{P}_\sigma$ such that $g(z) \equiv \mathrm{e}^{\mathrm{i}\gamma}\,\mathrm{e}^{\mathrm{i}\sigma z}\,\overline{g(\overline{z})}$ for some $\gamma \in \mathbb{R}$. Then

$$h_g\left(\frac{\pi}{2}\right) = -\sigma + h_g\left(-\frac{\pi}{2}\right) \le 0$$

and so $f(z) := \mathrm{e}^{-\mathrm{i}\gamma/2}\,\mathrm{e}^{-\mathrm{i}\sigma z/2}\,g(z)$ is an entire function of exponential type $\sigma/2$. Besides, for any real x, we have

$$\begin{aligned} f(x) &= \mathrm{e}^{-\mathrm{i}\gamma/2}\,\mathrm{e}^{-\mathrm{i}\sigma x/2}\,g(x) \\ &= \mathrm{e}^{-\mathrm{i}\gamma/2}\,\mathrm{e}^{-\mathrm{i}\sigma x/2}\,\overline{\mathrm{e}^{-\mathrm{i}\gamma}\,\mathrm{e}^{-\mathrm{i}\sigma x}\,g(x)} = \overline{\mathrm{e}^{-\mathrm{i}\gamma/2}\,\mathrm{e}^{-\mathrm{i}\sigma x/2}\,g(x)} = \overline{f(x)}\,, \end{aligned}$$

which means that $f(x)$ is real on the real axis. Applying Corollaries 1 and 2 (with $\sigma/2$ in place of τ), to f, we obtain the following result.

Proposition 1. *Let g be an entire function belonging to the Paley-Wiener space $\mathcal{P}_\sigma$ such that $g(z) \equiv \mathrm{e}^{\mathrm{i}\gamma}\,\mathrm{e}^{\mathrm{i}\sigma z}\,\overline{g(\overline{z})}$ for some $\gamma \in \mathbb{R}$. Then*

$$\frac{\int_{-\infty}^{\infty}|g'(x+\mathrm{i}y)|^2\,\mathrm{d}x}{\int_{-\infty}^{\infty}|g(x)|^2\,\mathrm{d}x} \le \begin{cases} (\sigma^2/2)\,\mathrm{e}^{-2\sigma y}\,, & \text{if } y \le (\ln 2)/\sigma, \\ (\sigma^2/4)\,\mathrm{e}^{-\sigma y}\,, & \text{if } (\ln 2)/\sigma \le y \le 2/\sigma\,. \end{cases} \tag{16}$$

The estimates given in (16) *are the best possible.*

If p is a polynomial of degree n having all its zeros on the unit circle then $p(z) \equiv \mathrm{e}^{\mathrm{i}\gamma} z^n \overline{p(1/\overline{z})}$ for some $\gamma \in \mathbb{R}$. However, the class of all polynomials satisfying $p(z) \equiv \mathrm{e}^{\mathrm{i}\gamma} z^n \overline{p(1/\overline{z})}$, which are called "self-inversive" [6], is much wider than the class of all polynomials of degree n having all their zeros on the unit circle. If p is a self-inversive polynomial of degree at most n then $g(z) := p(\mathrm{e}^{\mathrm{i}z})$ is an entire function of exponential type such that $g(z) \equiv \mathrm{e}^{\mathrm{i}\gamma}\, \mathrm{e}^{\mathrm{i}nz}\, \overline{g(\overline{z})}$ for some $\gamma \in \mathbb{R}$. Proposition 1 is related to a result in [5] about entire functions of exponential type satisfying $f(z) \equiv \mathrm{e}^{\mathrm{i}\gamma}\, \mathrm{e}^{\mathrm{i}\tau z}\, \overline{f(\overline{z})}$ for some $\gamma \in \mathbb{R}$. It extends inequality (5) of that paper, in the case where the parameter p appearing therein is 2.

1.7. Functions Belonging to L^p, $p > 0$ on the Real Line

Although the Paley-Wiener space has some special significance it is natural and meaningful to wonder how large $\int_{-\infty}^{\infty} |f'(x+\mathrm{i}y) + \mathrm{i}f(x+\mathrm{i}y)|^p\, \mathrm{d}x$ can be for any given $y \in (-\infty, \infty)$ if $f \in L^p(\mathbb{R})$ for some positive p other than 2.

By a result of Plancherel and Pólya (see [8] or [3, Theorem 6.7.1]), if f is an entire function of exponential type τ belonging to $L^p(\mathbb{R})$ for some $p > 0$, then

$$\int_{-\infty}^{\infty} |f(x+\mathrm{i}y)|^p\, \mathrm{d}x \le \mathrm{e}^{p\tau|y|} \int_{-\infty}^{\infty} |f(x)|^p\, \mathrm{d}x \qquad (y \in \mathbb{R})\,. \tag{17}$$

It is also known [10, Theorem 1] that, for any two constants $A \in \mathbb{C}$, $B \in \mathbb{C}$, not both zero and $\Im(A/B) \ge 0$ in case $B \ne 0$, we have

$$\int_{-\infty}^{\infty} \Big|\frac{Af(x) + Bf'(x)}{A + \mathrm{i}\tau B}\Big|^p\, \mathrm{d}x \le \int_{-\infty}^{\infty} |f(x)|^p\, \mathrm{d}x \qquad (p > 0)\,;$$

in particular,

$$\int_{-\infty}^{\infty} |f'(x) + \mathrm{i}\tau f(x)|^p\, \mathrm{d}x \le (2\tau)^p \int_{-\infty}^{\infty} |f(x)|^p\, \mathrm{d}x \qquad (p > 0)\,. \tag{18}$$

Let f be an entire function of exponential type τ belonging to $L^p(\mathbb{R})$ for some $p > 0$. Then, also $f' + \mathrm{i}\tau f$ is an entire function of exponential type τ which, by (18), belongs to $L^p(\mathbb{R})$ for the same p. Applying (17) to $f' + \mathrm{i}\tau f$ and then making use of (18), we see that

$$\begin{aligned} \int_{-\infty}^{\infty} |f'(x+\mathrm{i}y) + \mathrm{i}\tau f(x+\mathrm{i}y)|^p\, \mathrm{d}x &\le \mathrm{e}^{p\tau|y|} \int_{-\infty}^{\infty} |f'(x) + \mathrm{i}f(x)|^p\, \mathrm{d}x \\ &\le (2\tau)^p\, \mathrm{e}^{p\tau|y|} \int_{-\infty}^{\infty} |f(x)|^p\, \mathrm{d}x \quad (y \in \mathbb{R})\,. \end{aligned} \tag{19}$$

Inequality (19) says in particular that for any $y \le 0$, we have

$$\int_{-\infty}^{\infty} |f'(x+\mathrm{i}y) + \mathrm{i}\tau f(x+\mathrm{i}y)|^p\, \mathrm{d}x \le (2\tau)^p\, \mathrm{e}^{-p\tau y} \int_{-\infty}^{\infty} |f(x)|^p\, \mathrm{d}x\,. \tag{19'}$$

Thus, if y_p, $p > 0$ denotes the largest number such that, for any entire function of exponential type τ belonging to $L^p(\mathbb{R})$, inequality (19′) holds for all $y \le y_p$, then $y_p \ge 0$. However, by (13), the precise value of y_2 is $1/(2\tau)$. This raises the following question.

Question 1. *What is the exact value of y_p for any given $p > 0$?*

If f is an entire function of exponential type belonging to $L^p(\mathbb{R})$ and taking only real values on the real axis then (see [10, Corollary 4])

$$\frac{\int_{-\infty}^{\infty} |f'(x) + \mathrm{i}\tau f(x)|^p \,\mathrm{d}x}{\int_{-\infty}^{\infty} |f(x)|^p \,\mathrm{d}x} \le (2\tau)^p \, \frac{2^{-p}\sqrt{\pi}\,\Gamma(\frac{1}{2}\,p+1)}{\Gamma(\frac{1}{2}\,p+\frac{1}{2})}\,. \tag{20}$$

From (20) and (17) it follows that if f is an entire function of exponential type τ which belongs to $L^p(\mathbb{R})$ for some $p > 0$ and is real on the real axis, then for $y \le 0$, we have

$$\int_{-\infty}^{\infty} |f'(x+\mathrm{i}y) + \mathrm{i}\tau f(x+\mathrm{i}y)|^p \,\mathrm{d}x \le (2\tau)^p \, \frac{2^{-p}\sqrt{\pi}\,\Gamma(\frac{1}{2}\,p+1)}{\Gamma(\frac{1}{2}\,p+\frac{1}{2})} \, \mathrm{e}^{-p\tau y} \int_{-\infty}^{\infty} |f(x)|^p \,\mathrm{d}x \qquad (p > 0)\,. \tag{20'}$$

Comparing this inequality with (15′) we see that for $p = 2$ the restriction on y can be relaxed; in fact, it can be replaced by "$y \le (\ln 2)/(2\tau)$". For any given $p > 0$, let η_p denote the largest number such that for any entire function of exponential type τ belonging to $L^p(\mathbb{R})$ and taking only real values on the real axis, inequality (20′) holds for all $y \le \eta_p$. We know that $\eta_2 = (\ln 2)/(2\tau)$ but for other values of p we only know that $\eta_p \ge 0$.

Question 2. *What is the exact value of η_p for any given $p > 0$?*

2. Auxiliary Results

Inequalities (8) and (10) are mainly based on Lemma 1, which we deduce from a generalized version of Schwarz's lemma, due to Pick (see [1, p. 3]), known as *the invariant form of Schwarz's lemma.*

Let $\mathfrak{U}$ denote the open unit disk $|z| < 1$. Schwarz's lemma in its simplest form says that *if ω is holomorphic in $\mathfrak{U}$, $|\omega(z)| \le 1$ for all $z \in \mathfrak{U}$ and $\omega(0) = 0$, then $|\omega(z)| \le |z|$ for $|z| < 1$; in particular, $|\omega'(0)| \le 1$.* Here is its extension due to Pick. For sake of completeness, we also include an outline of its proof.

The Schwarz-Pick Theorem. *Let ψ be holomorphic in the open unit disk $\mathfrak{U}$ and $|\psi(\zeta)| \le 1$ for all $\zeta \in \mathfrak{U}$. Then*

$$(1 - |\zeta_0|^2)|\psi'(\zeta_0)| + |\psi(\zeta_0)|^2 \le 1 \qquad (\zeta_0 \in \mathfrak{U})\,. \tag{21}$$

At a point $\zeta_0 \in \mathfrak{U}$ *where* ψ *vanishes, equality holds in* (21) *only if*

$$\psi(\zeta) = \mathrm{e}^{\mathrm{i}\alpha}\,\frac{\zeta - \zeta_0}{\overline{\zeta}_0\zeta - 1}\,, \qquad \alpha \in \mathbb{R}\,. \tag{22}$$

Proof. We distinguish two cases.

(i) First let $|\psi(\zeta_0)| = 1$. Then, by the maximum modulus principle, ψ is a constant, so that $\psi'(\zeta_0) = 0$ and (21) holds.

(ii) Next let $\psi(\zeta_0) = b$, $|b| < 1$. Then the function

$$\omega(\zeta) := \frac{\psi\Big(\frac{\zeta+\zeta_0}{\overline{\zeta}_0\zeta+1}\Big) - b}{\overline{b}\,\psi\Big(\frac{\zeta+\zeta_0}{\overline{\zeta}_0\zeta+1}\Big) - 1}$$

satisfies the conditions of Schwarz's lemma enunciated above, namely, "it is holomorphic in $\mathfrak{U}$, $|\omega(\zeta)| \le 1$ for all $\zeta \in \mathfrak{U}$ and $\omega(0) = 0$". Hence $|\omega'(0)| \le 1$. Since

$$\omega'(0) = -\frac{(1 - |\zeta_0|^2)\psi'(\zeta_0)}{1 - |\psi(\zeta_0)|^2}\,,$$

as is easily seen, inequality (21) holds.

(a) Now, let us find all functions ψ, if there are any, which vanish at the origin and for which $|\psi'(0)| = 1$. We may write $\psi(\zeta) = \sum_{n=1}^{\infty} a_n\,\zeta^n$, where the expansion is valid for all $\zeta \in \mathfrak{U}$. Since $|\psi(\zeta)| \le 1$ for all $\zeta \in \mathfrak{U}$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\psi(r\,\mathrm{e}^{\mathrm{i}\theta})|^2\,\mathrm{d}\theta = \sum_{n=1}^{\infty} |a_n|^2\,r^{2n} \le 1 \qquad (0 \le r < 1)\,.$$

Hence, for $|a_1|$ (which is the same as $|\psi'(0)|$) to be equal to 1, the other coefficients a_2, a_3, … must all vanish, which means that $\psi(\zeta)$ must be of the form $\mathrm{e}^{\mathrm{i}\alpha}\,\zeta$ for some real α.

(b) Next, we shall use the observation made in (a) to determine the extremals when $\psi(\zeta_0) = 0$ but $\zeta_0 \ne 0$. For equality in (21), we must have

$$(1 - |\zeta_0|^2)\,|\psi'(\zeta_0)| = 1\,. \tag{23}$$

We recognize $(1 - |\zeta_0|^2)\,\psi'(\zeta_0)$ to be $\varphi'(0)$, where

$$\varphi(\zeta) := \psi\Big(\frac{\zeta + \zeta_0}{\overline{\zeta}_0\zeta + 1}\Big)\,.$$

The function φ satisfies the conditions of the Schwarz-Pick theorem and $\varphi(0)=0$. Hence, for (23) to hold, i.e. for $|\varphi'(0)|$ to be equal to 1, we must, in view of (a), have $\varphi(\zeta) \equiv \mathrm{e}^{\mathrm{i}\alpha}\,\zeta$ for some real α. This means that if $\psi(\zeta_0) = 0$, where $0 < |\zeta_0| < 1$, then equality holds in (21) if and only if $\psi(\zeta)$ is as in (22). □

Lemma 1. *Let F be holomorphic and $|F(z)| \le 1$ in the open upper half-plane H_+. Then*

$$2y\,|F'(z)| + |F(z)|^2 \le 1 \qquad (y := \Im z > 0)\,. \tag{24}$$

At a point $z = z_0 \in H_+$ where F vanishes, equality holds in (24) *only if*

$$F(z) = \mathrm{e}^{\mathrm{i}\beta}\,\frac{z - z_0}{z - \overline{z}_0}\,, \qquad \beta \in \mathbb{R}\,. \tag{25}$$

Proof. The function

$$\psi(\zeta) := F\Big(\frac{1+\zeta}{1-\zeta}\,\mathrm{i}\Big)$$

is holomorphic and $|\psi(\zeta)| \le 1$ in the open unit disk. Applying (21) to ψ and noting that

$$\psi'(\zeta) = \frac{2\,\mathrm{i}}{(1-\zeta)^2}\,F'\Big(\frac{1+\zeta}{1-\zeta}\,\mathrm{i}\Big)\,,$$

we obtain

$$(1 - |\zeta|^2)\,\frac{2}{|1-\zeta|^2}\,\Big|F'\Big(\frac{1+\zeta}{1-\zeta}\,\mathrm{i}\Big)\Big| + \Big|F\Big(\frac{1+\zeta}{1-\zeta}\,\mathrm{i}\Big)\Big|^2 \le 1 \qquad (|\zeta| < 1)\,.$$

Setting

$$z := \frac{1+\zeta}{1-\zeta}\,\mathrm{i}, \quad \text{so that} \quad \zeta = \frac{z - \mathrm{i}}{z + \mathrm{i}}\,,$$

we see that

$$\Big(1 - \Big|\frac{z-\mathrm{i}}{z+\mathrm{i}}\Big|^2\Big)\frac{2}{\big|\frac{2\,\mathrm{i}}{z+\mathrm{i}}\big|^2}\,|F'(z)| + |F(z)|^2 \le 1 \qquad (\Im z > 0)\,,$$

that is,

$$\frac{1}{2}(|z+\mathrm{i}|^2 - |z-\mathrm{i}|^2)\,|F'(z)| + |F(z)|^2 \le 1 \qquad (\Im z > 0)\,.$$

Now note that

$$|z+\mathrm{i}|^2 - |z-\mathrm{i}|^2 = (z+\mathrm{i})(\overline{z}-\mathrm{i}) - (z-\mathrm{i})(\overline{z}+\mathrm{i}) = 2\,\mathrm{i}\,(\overline{z} - z) = 4\,\Im z\,.$$

Hence (24) holds.

Note that $F(z) = 0$ for $z = z_0$ if and only if

$$\psi(\zeta) = F\Big(\frac{1+\zeta}{1-\zeta}\,\mathrm{i}\Big) = 0 \;\text{ for }\; \zeta = \zeta_0 = \frac{z_0 - \mathrm{i}}{z_0 + \mathrm{i}}$$

and then $(1 - |\zeta_0|^2)|\psi'(\zeta_0)| = 1$ if and only if

$$F\Big(\frac{1+\zeta}{1-\zeta}\,\mathrm{i}\Big) = \psi(\zeta) = \mathrm{e}^{\mathrm{i}\alpha}\,\frac{\zeta - \zeta_0}{\overline{\zeta}_0\zeta - 1}\,.$$

Hence, if $F(z_0) = 0$ and $y_0 := \Im z_0$, then $2\, y_0\, |F'(z_0)| = 1$ if and only if

$$F(z) = \mathrm{e}^{\mathrm{i}\alpha}\, \frac{\frac{z-\mathrm{i}}{z+\mathrm{i}} - \frac{z_0-\mathrm{i}}{z_0+\mathrm{i}}}{\frac{\overline{z}_0+\mathrm{i}}{\overline{z}_0-\mathrm{i}} \cdot \frac{z-\mathrm{i}}{z+\mathrm{i}} - 1} = \mathrm{e}^{\mathrm{i}\alpha}\, \frac{\overline{z}_0 - \mathrm{i}}{z_0 + \mathrm{i}} \cdot \frac{(z-\mathrm{i})(z_0+\mathrm{i}) - (z+\mathrm{i})(z_0-\mathrm{i})}{(z-\mathrm{i})(\overline{z}_0+\mathrm{i}) - (z+\mathrm{i})(\overline{z}_0-\mathrm{i})}$$
$$= \mathrm{e}^{\mathrm{i}\alpha}\, \frac{\overline{z}_0 - \mathrm{i}}{z_0 + \mathrm{i}} \cdot \frac{z - z_0}{z - \overline{z}_0} = \mathrm{e}^{\mathrm{i}\beta}\, \frac{z - z_0}{z - \overline{z}_0}\,, \quad \text{where } \beta := \arg\left(\mathrm{e}^{\mathrm{i}\alpha}\, \frac{\overline{z}_0 - \mathrm{i}}{z_0 + \mathrm{i}}\right).$$

□

The next lemma plays a crucial role in the proof of Theorem 4.

Lemma 2. *Let* $u(x) := x^2\, 4^{1-x} + (1-x)^2\, 4^x$. *Then* $u(x) \le 1$ *for* $0 \le x \le 1$.

Proof. Note that $u(x) = u(1-x)$, and hence, it is enough to prove that $u(x) \le 1$ for $0 \le x \le 1/4$ and for $1/2 \le x \le 3/4$.

In order to prove the desired inequality for $0 \le x \le 1/4$ we observe that $u(0) = 1$ and $u(1/4) < 1$. We then show that, for $0 < x < 1/4$, the graph of $u(x)$ lies below the line segment joining the points $(0, 1)$ and $(1/4, u(1/4))$ in $\mathbb{R}^2$. In fact, we prove that $u''(x) \ge 0$ for $0 \le x \le 1/4$, which means that the function $u(x)$ is convex on the interval $[0, 1/4]$. For the sake of brevity, we set

$$w = \ln 4 = 1.38629\cdots$$

Straightforward calculation gives

$$u''(x) = (8 - 16w\, x + 4w^2\, x^2)\, 4^{-x} + [2 - 4\, w + w^2 + (4w - 2w^2)\, x + w^2\, x^2]\, 4^x\,.$$

Since

$$\sqrt{2}\, u''\left(\frac{1}{4}\right) = 8 - 4w + \frac{1}{4}\, w^2 + 2\left[2 - 4w + w^2 + \frac{1}{2}\,(2w - w^2) + \frac{1}{16}\, w^2\right]$$
$$= 12 - 10w + \frac{11}{8}\, w^2 \approx 12 - 13.86294361 + 2.642491577 > 0\,,$$

it suffices to show that $u'''(x) \le 0$ for $0 \le x \le 1/4$. On calculating $u'''(x)$ we see that

$$4^x\, u'''(x) = -24w + 24w^2 x - 4w^3\, x^2 + [6w - 6w^2 + w^3 + (6w^2 - 2w^3)x + w^3 x^2]4^{2x}\,.$$

Now, note that $6w - 6w^2 + w^3 < 0$ and that 4^{2x} increases with x. Consequently,

$$(6w - 6w^2 + w^3)4^{2x} \le 6w - 6w^2 + w^3 \qquad (x \ge 0)$$

and so

$$\frac{4^x u'''(x)}{w} \le -18 - 6w + w^2 + (24w\, x - 4w^2 x^2) + [(6w - 2w^2)x + w^2 x^2]4^{2x} =: v(x)\,.$$

Since $24w\,x - 4w^2\,x^2$ is an increasing function of x for $x < 3/w$, the same can be said about $v(x)$. Hence, in order to prove that $u'''(x) < 0$ for $0 \le x \le 1/4$, it is sufficient to check that $v(1/4) < 0$. This is indeed so. In fact

$$\begin{aligned} v\Big(\frac{1}{4}\Big) &= -18 - 6w + w^2 + 6w - \frac{1}{4}\,w^2 + 2\left[\frac{1}{2}\,(3w - w^2) + \frac{1}{16}\,w^2\right] \\ &= -18 + 3w - \frac{1}{8}\,w^2 \approx -18 + 4.158883083 - 0.240226507 < 0\,. \end{aligned}$$

This completes the proof of the fact that $u(x) \le 1$ for $0 \le x \le 1/4$.

Now, we turn to the proof of the inequality "$u(x) \le 1$" for $1/2 \le x \le 3/4$. Since $u(1/2) = 1$, it is sufficient to prove that $u(x)$ is a decreasing function of x for $1/2 \le x \le 3/4$. We shall show that

$$\phi(t) := 2\,u\Big(\frac{1}{2} + t\Big) = (1 + 2t)^2\,4^{-t} + (1 - 2t)^2\,4^t$$

is a decreasing function of t for $0 \le t \le 1/4$. It is easily checked that

$$4^t\,\phi'(t) = 4 - w + (8 - 4w)t - 4w\,t^2 - [4 - w - (8 - 4w)t - 4w\,t^2]\,4^{2t}\,.$$

Since $4 - w - (8 - 4w)\,t - 4w\,t^2$ is positive for $0 < t \le 1/4$ and

$$4^{2t} = \mathrm{e}^{2\,t\,w} = 1 + \sum_{k=1}^{\infty} \frac{1}{k!}\,(2\,t\,w)^k > 1 + 2w\,t + 2w^2\,t^2 \qquad (t > 0)\,,$$

we find that if $0 < t \le 1/4$, then

$$\begin{aligned} 4^t\,\phi'(t) &< 2(8 - 4w)t + [4 - w - (8 - 4w)t - 4w\,t^2]\,[1 - (1 + 2w\,t + 2w^2 t^2)] \\ &= 2(8 - 4w)t + 2w\,t(1 + w\,t)[-4 + w + (8 - 4w)t + 4w\,t^2] \\ &= 2t\,q(t)\,, \end{aligned}$$

where

$$q(t) = w^2 - 8w + 8 + w(w^2 - 8w + 8)t + 4w^2(3 - w)t^2 + 4w^3 t^3\,.$$

Since $w^2 - 8w + 8 < 0$, the cubic polynomial $q(t)$ has sign pattern $\{-1, -1, 1, 1\}$ of its coefficients, and by Descartes' rule of signs, $q(t)$ has a unique positive root, say θ. Since

$$q\Big(\frac{1}{4}\Big) = 8 - 6w - \frac{1}{4}\,w^2 + \frac{1}{16} w^3 = 2(4 - 3w) + \frac{1}{16} w^2 (w - 4) < 0,$$

it follows that $\theta > 1/4$, and consequently $q(t) < 0$ for $t \in (0, 1/4]$. Hence, $\phi'(t)$ is negative for $0 < t \le 1/4$, and so $\phi(t)$ is a decreasing function of t on the interval $(0, 1/4]$. $\square$

3. Proofs of Theorems 1–4 and of Corollaries 1–2

Proof of Theorem 1. As already explained, inequality (6) may be obtained by applying Theorem A, with $c = \tau$ and $M = 2\tau$, to the function $\overline{\Lambda(\overline{z})}$, where $\Lambda(z) := f'(z) + \mathrm{i}\tau f(z)$. The example $f(z) := \mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\tau z}$, $\alpha \in \mathbb{R}$, shows that the upper bound for $|f'(z) + \mathrm{i}\tau f(z)|$, given in (6), is attained at every point of the lower half-plane.

For (8), it is enough to prove that $|f'(z)+\mathrm{i}\tau f(z)| < \min\{2\tau\, \mathrm{e}^{\tau y}, (2y)^{-1}\, \mathrm{e}^{\tau y}\}$ at any point of the open upper half-plane. From (7) we already know that $|f'(z) + \mathrm{i}\tau f(z)| < 2\tau\, \mathrm{e}^{\tau y}$ for $y > 0$. So, we only need to show that

$$|f'(z_0) + \mathrm{i}\tau f(z_0)| < \frac{1}{2\, y_0}\, \mathrm{e}^{\tau y_0} \qquad (y_0 := \Im z_0 > 0)\,. \tag{26}$$

For this, we note that $F(z) := \mathrm{e}^{\mathrm{i}\tau z}\, f(z)$ is an entire function of exponential type such that $|F(x)| \le 1$ for all real x and $h_F(\pi/2) \le 0$. Hence, by Theorem A, $|F(z)| \le 1$ in the upper half-plane and Lemma 2 applies. It is clear from (24) that $2\, y_0\, |F'(z_0)| < 1$ if $F(z_0) \ne 0$ and *also if* $F(z_0) = 0$ *unless* $F(z)$ *is of the form* $\mathrm{e}^{\mathrm{i}\beta}\, (z - z_0)/(z - \overline{z}_0)$ *for some* $\beta \in \mathbb{R}$, as stipulated in (25). However, $F(z)$ cannot be of this form, since it is an entire function and so cannot have any poles. Hence (26) holds. □

As regards the bound given in (8), the following example shows that, except for a constant factor, its dependence on y is the right one, for large values of y.

Example 1. For any $\eta > 1/\tau$, let

$$f_1(z) := \mathrm{e}^{-\mathrm{i}(\tau z - z/\eta)}\, \frac{\sin(z/\eta)}{z}\,. \tag{27}$$

Then f_1 is an entire function of exponential type τ such that

$$\max_{x\in\mathbb{R}} |f_1(x)| = |f_1(0)| = \frac{1}{\eta}$$

and

$$f_1'(z) + \mathrm{i}\tau f_1(z) = \Big(\frac{\mathrm{i}}{\eta z}\, \sin\frac{z}{\eta} + \frac{1}{\eta z}\, \cos\frac{z}{\eta} - \frac{1}{z^2}\, \sin\frac{z}{\eta}\Big)\, \mathrm{e}^{-\mathrm{i}(\tau z - z/\eta)}\,.$$

In particular, we have

$$f_1'(\mathrm{i}\eta) + \mathrm{i}\tau\, f_1(\mathrm{i}\eta) = \frac{1}{\mathrm{e}\eta^2}(2\,\sin \mathrm{i} - \mathrm{i}\,\cos \mathrm{i})\, \mathrm{e}^{\tau\eta} = \frac{\mathrm{i}}{2\eta^2}\left(1 - \frac{3}{\mathrm{e}^2}\right)\mathrm{e}^{\tau\eta} \tag{28}$$

and so

$$|f_1'(\mathrm{i}\eta) + \mathrm{i}\tau\, f_1(\mathrm{i}\eta)| = \Big(1 - \frac{3}{\mathrm{e}^2}\Big)\frac{1}{2\eta}\, \mathrm{e}^{\tau\eta}\, \max_{x\in\mathbb{R}} |f_1(x)| \qquad \Big(\eta > \frac{1}{\tau}\Big)\,.$$

Hence, in the class of functions f satisfying the conditions of Theorem 1, the upper bound $(2y)^{-1}\,\mathrm{e}^{\tau y}$ for $|f'(z)+\mathrm{i}f(z)|$ given in (8) cannot be replaced by anything smaller than $\left(1-3\,\mathrm{e}^{-2}\right)(2y)^{-1}\,\mathrm{e}^{\tau y}$, at least for $y>1/\tau$.

Proof of Theorem 2. Let f satisfies the conditions of Theorem 2 and, as in (5), let $\Lambda(z) := f'(z)+\mathrm{i}\tau f(z)$. Then Λ is an entire function of exponential type τ and by (4), $|\Lambda(x)| \le \tau$ for all real x. Hence, Theorem A with $c=\tau$, $M=\tau$ applies to $\overline{\Lambda}(z) := \overline{\Lambda(\overline{z})}$ and gives (9).

Inequality (10) says that if f satisfies the conditions of Theorem 2, then at any point z of the open upper half-plane, we have $|\Lambda(z)| < \min\left\{\tau\,\mathrm{e}^{\tau y}, (2y)^{-1}\,\mathrm{e}^{\tau y}\right\}$. So, in view of (26), we only need to show that $|\Lambda(z)| < \tau\,\mathrm{e}^{\tau y}$ for $y>0$. As observed above, Λ is an entire function of exponential type τ such that $|\Lambda(x)| \le \tau$ for all real x. Furthermore, $\Lambda(z)$ cannot be of the form $\tau\,\mathrm{e}^{\mathrm{i}\gamma}\,\mathrm{e}^{-\mathrm{i}\tau z}$ since otherwise $f(z)$ would be of the form $(\tau\,\mathrm{e}^{\mathrm{i}\gamma}\,z+d)\,\mathrm{e}^{-\mathrm{i}\tau z}$ for some constant d and $|f(x)|$ would not be bounded on the real axis. Now, apply Theorem A to Λ in order to obtain the desired estimate. □

Example 2. With f_1 as in (27), let $f_2(z) := \overline{f_1(\overline{z})}$. Then

$$f(z) := \frac{1}{2}\left\{f_1(z)+f_2(z)\right\}$$

is an entire function of exponential type τ which is real on the real axis and $\max_{x\in\mathbb{R}}|f(x)| = |f(0)| = 1/\eta$. Since

$$f_2'(z)+\mathrm{i}\tau f_2(z) = \mathrm{e}^{\mathrm{i}(\tau z - z/\eta)}\left\{\left(2\tau-\frac{1}{\eta}\right)\frac{\mathrm{i}}{z}\,\sin\frac{z}{\eta}+\frac{1}{\eta z}\,\cos\frac{z}{\eta}-\frac{1}{z^2}\,\sin\frac{z}{\eta}\right\},$$

we have

$$\begin{aligned} f_2'(\mathrm{i}\eta)+\mathrm{i}\tau\, f_2(\mathrm{i}\eta) &= \mathrm{e}\,\mathrm{e}^{-\tau\eta}\left\{\left(2\tau-\frac{1}{\eta}\right)\frac{1}{\eta}\,\sin\mathrm{i}+\left(\frac{1}{\mathrm{i}\eta^2}\,\cos\mathrm{i}+\frac{1}{\eta^2}\,\sin\mathrm{i}\right)\right\}\\ &= \mathrm{i}\,\frac{\tau}{\eta}\,\mathrm{e}^{-\tau\eta}\,(\mathrm{e}^2-1)\left(1-\frac{1}{2\tau\eta}\cdot\frac{\mathrm{e}^2-1}{\mathrm{e}^2+1}\right). \end{aligned}$$

Taking this and (28) into account, we see that

$$\begin{aligned} \frac{f'(\mathrm{i}\eta)+\mathrm{i}\tau f(\mathrm{i}\eta)}{\mathrm{i}} &= \frac{f_1'(\mathrm{i}\eta)+\mathrm{i}\tau f_1(\mathrm{i}\eta)+f_2'(\mathrm{i}\eta)+\mathrm{i}\tau f_2(\mathrm{i}\eta)}{2\mathrm{i}}\\ &= \frac{1}{2\eta^2}\left(\frac{1}{2}-\frac{3}{2\mathrm{e}^2}\right)\mathrm{e}^{\tau\eta}+\frac{\tau}{2\eta}\,\mathrm{e}^{-\tau\eta}\,(\mathrm{e}^2-1)\left(1-\frac{1}{2\tau\eta}\cdot\frac{\mathrm{e}^2-1}{\mathrm{e}^2+1}\right)\\ &> \frac{1}{2\eta^2}\left(\frac{1}{2}-\frac{3}{2\mathrm{e}^2}\right)\mathrm{e}^{\tau\eta} \qquad \text{for } \eta>\frac{1}{2\tau} \end{aligned}$$

and so

$$|f'(\mathrm{i}\eta)+\mathrm{i}\tau\, f(\mathrm{i}\eta)| > \left(\frac{1}{2}-\frac{3}{2\mathrm{e}^2}\right)\frac{1}{2\eta}\,\mathrm{e}^{\tau\eta}\,\max_{x\in\mathbb{R}}|f(x)| \qquad \text{for } \eta>\frac{1}{2\tau}\,.$$

Hence, in the class of functions f satisfying the conditions of Theorem 2, the upper bound $(2y)^{-1}\,\mathrm{e}^{\tau y}$ for $|f'(z)+\mathrm{i}f(z)|$ given in (9) cannot be replaced by anything smaller than $((1-3\,\mathrm{e}^{-2})/2)(2y)^{-1}\,\mathrm{e}^{\tau y}$, at least for $y>1/(2\tau)$.

Proof of Theorem 3. By the Paley-Wiener representation (11), there exists a function $\varphi\in L^2(-\tau,\tau)$ such that $f(z)=\int_{-\tau}^{\tau}\mathrm{e}^{\mathrm{i}zt}\,\varphi(t)\,\mathrm{d}t$. Then

$$f'(z)+\mathrm{i}\tau f(z)=\int_{-\tau}^{\tau}\mathrm{i}\,(t+\tau)\,\mathrm{e}^{\mathrm{i}zt}\,\varphi(t)\,\mathrm{d}t \tag{29}$$

and by Parseval's formula (12), we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}|f'(x+\mathrm{i}y)+\mathrm{i}\tau f(x+\mathrm{i}y)|^2\,\mathrm{d}x &= 2\pi\int_{-\tau}^{\tau}(t+\tau)^2\,\mathrm{e}^{-2yt}\,|\varphi(t)|^2\,\mathrm{d}t\\ &= 2\pi\,\mathrm{e}^{2y\tau}\int_{-\tau}^{\tau}(t+\tau)^2\,\mathrm{e}^{-2y(t+\tau)}\,|\varphi(t)|^2\,\mathrm{d}t\\ &= 2\pi\,\mathrm{e}^{2y\tau}\int_{0}^{2\tau}s^2\,\mathrm{e}^{-2ys}\,|\varphi(s-\tau)|^2\,\mathrm{d}s\,.\end{aligned}$$

We claim that if $-\infty<y\le 1/(2\tau)$, then $s\,\mathrm{e}^{-ys}$ is a non-decreasing function of s on $(0,2\tau]$. This is obvious if $y\le 0$. So, let $0<y\le 1/(2\tau)$. Then $2\tau\le 1/y$ and it suffices to note that

$$\frac{\mathrm{d}}{\mathrm{d}s}(s\,\mathrm{e}^{-ys})=\mathrm{e}^{-ys}(1-ys)\ge 0 \qquad \left(s\le\frac{1}{y}\right).$$

Hence, if $-\infty<y\le 1/(2\tau)$ then $s^2\,\mathrm{e}^{-2ys}\le 4\tau^2\,\mathrm{e}^{-4y\tau}$ for $0\le s\le 2\tau$ and so

$$\begin{aligned}\int_{-\infty}^{\infty}|f'(x+\mathrm{i}y)+\mathrm{i}\tau f(x+\mathrm{i}y)|^2\,\mathrm{d}x &\le 4\tau^2\mathrm{e}^{-2y\tau}\,2\pi\int_{0}^{2\tau}|\varphi(s-\tau)|^2\,\mathrm{d}s\\ &= 4\tau^2\mathrm{e}^{-2y\tau}\,2\pi\int_{-\tau}^{\tau}|\varphi(t)|^2\,\mathrm{d}t\\ &= 4\tau^2\,\mathrm{e}^{-2y\tau}\int_{-\infty}^{\infty}|f(x)|^2\,\mathrm{d}x\,,\end{aligned}$$

which proves (13) for $y\in(-\infty,1/(2\tau))$.

For any given $y>1/(2\tau)$, say $y=\eta$, the function $s\,\mathrm{e}^{-ys}$ attains its maximum for $s=1/\eta\in(0,2\tau)$, that is $s^2\,\mathrm{e}^{-2ys}\le(\mathrm{e}^2\eta^2)^{-1}$ for $0\le s\le 2\tau$ and so, in this case, we have

$$\begin{aligned}\int_{-\infty}^{\infty}|f'(x+\mathrm{i}y)+\mathrm{i}\tau f(x+\mathrm{i}y)|^2\,\mathrm{d}x &\le 2\pi\mathrm{e}^{2y\tau}\,\frac{1}{\mathrm{e}^2\eta^2}\int_{0}^{2\tau}|\varphi(s-\tau)|^2\,\mathrm{d}s\\ &= \frac{1}{\mathrm{e}^2\eta^2}\mathrm{e}^{2y\tau}\,2\pi\int_{-\tau}^{\tau}|\varphi(t)|^2\,\mathrm{d}t\\ &= \frac{1}{\mathrm{e}^2\eta^2}\,\mathrm{e}^{2y\tau}\int_{-\infty}^{\infty}|f(x)|^2\,\mathrm{d}x\,,\end{aligned}$$

which proves (13) for $y>1/(2\tau)$. □

The following example shows that $4\tau^2\,\mathrm{e}^{-2\tau y}$, which is given in (13) as an upper bound for

$$\frac{\int_{-\infty}^{\infty}|f'(x+\mathrm{i}y)+\mathrm{i}\tau f(x+\mathrm{i}y)|^2\,\mathrm{d}x}{\int_{-\infty}^{\infty}|f(x)|^2\,\mathrm{d}x}$$

in the case where $y \le 1/(2\tau)$, is sharp. For $y > 1/(2\tau)$, we need another example, namely Example 4, which is presented after Example 3.

Example 3. For any $\varepsilon \in (0, 2\tau)$, let

$$f(z) := \int_{\tau-\varepsilon}^{\tau} \mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t = \mathrm{e}^{\mathrm{i}(\tau-\frac{\varepsilon}{2})z}\,\frac{\sin(\varepsilon z/2)}{z/2}\,.$$

Then $\int_{-\infty}^{\infty}|f(x)|^2\,\mathrm{d}x = 2\pi\int_{\tau-\varepsilon}^{\tau}\mathrm{d}t = 2\pi\varepsilon$. Since $f'(z)+\mathrm{i}\tau f(z) = \int_{\tau-\varepsilon}^{\tau}\mathrm{i}\,(t+\tau)\,\mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t$, we have

$$\int_{-\infty}^{\infty}|f'(x+\mathrm{i}y)+\mathrm{i}\tau f(x+\mathrm{i}y)|^2\,\mathrm{d}x = 2\pi\int_{\tau-\varepsilon}^{\tau}(t+\tau)^2\,\mathrm{e}^{-2yt}\,\mathrm{d}t = (\xi+\tau)^2\,\mathrm{e}^{-2y\xi}2\pi\varepsilon$$

for some $\xi \in [\tau-\varepsilon, \tau]$, by the mean value theorem on integration. Thus

$$\frac{\int_{-\infty}^{\infty}|f'(x+\mathrm{i}y)+\mathrm{i}\tau f(x+\mathrm{i}y)|^2\,\mathrm{d}x}{\int_{-\infty}^{\infty}|f(x)|^2\,\mathrm{d}x} = (\xi+\tau)^2\,\mathrm{e}^{-2y\xi} \to 4\tau^2\,\mathrm{e}^{-2\tau y} \quad \text{as} \quad \varepsilon \to 0\,.$$

Example 4. Let $y=\eta>1/(2\tau)$ so that $0<1/\eta<2\tau$ and $-\tau<(1/\eta)-\tau<\tau$. Now, let

$$f(z) := \int_{(1/\eta)-\tau-\varepsilon}^{(1/\eta)-\tau+\varepsilon} \mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t\,, \quad \text{where} \quad \varepsilon < \min\{1/\eta, 2\tau-1/\eta\}\,.$$

Then

$$f'(z)+\mathrm{i}\tau f(z) = \int_{(1/\eta)-\tau-\varepsilon}^{(1/\eta)-\tau+\varepsilon} \mathrm{i}\,(t+\tau)\,\mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t\,.$$

By Parseval's formula,

$$\int_{-\infty}^{\infty}|f(x)|^2\,\mathrm{d}x = 2\pi\int_{(1/\eta)-\tau-\varepsilon}^{(1/\eta)-\tau+\varepsilon}\mathrm{d}t = 4\pi\varepsilon$$

and

$$\begin{aligned}\int_{-\infty}^{\infty}|f'(x+\mathrm{i}\eta)+\mathrm{i}\tau f(x+\mathrm{i}\eta)|^2\,\mathrm{d}x &= 2\pi\int_{(1/\eta)-\tau-\varepsilon}^{(1/\eta)-\tau+\varepsilon}(t+\tau)^2\,\mathrm{e}^{-2\eta t}\,\mathrm{d}t \\ &= 2\pi\,\mathrm{e}^{2\tau\eta}\int_{(1/\eta)-\varepsilon}^{(1/\eta)+\varepsilon} s^2\,\mathrm{e}^{-2\eta s}\,\mathrm{d}s \\ &= \mathrm{e}^{2\tau\eta}\,\xi^2\,\mathrm{e}^{-2\eta\xi}\,4\pi\varepsilon\end{aligned}$$

for some $\xi \in [(1/\eta) - \varepsilon, (1/\eta) + \varepsilon]$. Hence,

$$\frac{\int_{-\infty}^{\infty} |f'(x + \mathrm{i}\eta) + \mathrm{i}\tau f(x + \mathrm{i}\eta)|^2 \,\mathrm{d}x}{\int_{-\infty}^{\infty} |f(x)|^2 \,\mathrm{d}x} = \xi^2 \,\mathrm{e}^{-2\eta\xi} \,\mathrm{e}^{2\tau\eta} \to \frac{1}{\mathrm{e}^2\eta^2} \,\mathrm{e}^{2\tau\eta} \quad \text{as} \quad \varepsilon \to 0\,.$$

Proof of Theorem 4. We again start with the Paley-Wiener representation (11) of $f(z)$ and note that

$$\overline{f(\overline{z})} = \int_{-\tau}^{\tau} \mathrm{e}^{-\mathrm{i}zt} \,\overline{\varphi(t)} \,\mathrm{d}t = \int_{-\tau}^{\tau} \mathrm{e}^{\mathrm{i}zt} \,\overline{\varphi(-t)} \,\mathrm{d}t\,.$$

Since $f(x)$ is real for real x, we have $f(z) \equiv \overline{f(\overline{z})}$, that is

$$\int_{-\tau}^{\tau} \mathrm{e}^{\mathrm{i}zt} \big(\varphi(t) - \overline{\varphi(-t)}\big) \,\mathrm{d}t = 0\,.$$

Consequently, $\phi(t)$ is equal to $\overline{\phi(-t)}$, almost everywhere in $(-\tau, \tau)$. In particular, we have

$$\begin{aligned}
\int_{-\tau}^{\tau} (\tau + t)^2 \,\mathrm{e}^{-2yt} \,|\varphi(t)|^2 \,\mathrm{d}t &= \int_{-\tau}^{\tau} (\tau - t)^2 \,\mathrm{e}^{2yt} \,|\varphi(t)|^2 \,\mathrm{d}t \\
&= \int_{-\tau}^{\tau} \frac{(\tau + t)^2 \,\mathrm{e}^{-2yt} + (\tau - t)^2 \,\mathrm{e}^{2yt}}{2} \,|\varphi(t)|^2 \,\mathrm{d}t\,.
\end{aligned}$$

Applying Parseval's formula to (29) and using this identity, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} |f'(x + \mathrm{i}y) + \mathrm{i}\tau f(x + \mathrm{i}y)|^2 \,\mathrm{d}x \\
&\qquad = 2\pi \int_{-\tau}^{\tau} \frac{(\tau + t)^2 \,\mathrm{e}^{-2yt} + (\tau - t)^2 \,\mathrm{e}^{2yt}}{2} \,|\varphi(t)|^2 \,\mathrm{d}t \\
&\qquad = 2\pi \int_{0}^{2\tau} \frac{s^2 \,\mathrm{e}^{-2y(s-\tau)} + (2\tau - s)^2 \,\mathrm{e}^{2y(s-\tau)}}{2} \,|\varphi(s - \tau)|^2 \,\mathrm{d}s \\
&\qquad = 2\tau^2 \,\mathrm{e}^{-2\tau y} \cdot 2\pi \int_{0}^{2\tau} U\Big(y, \frac{s}{2\tau}\Big) \,|\varphi(s - \tau)|^2 \,\mathrm{d}s\,, \qquad (30)
\end{aligned}$$

where $U(y, \cdot)$ is the function defined in (14).

Consequently, for any real y, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} |f'(x + \mathrm{i}y) + \mathrm{i}\tau f(x + \mathrm{i}y)|^2 \,\mathrm{d}x \\
&\qquad \leq 2\,\tau^2 \,\mathrm{e}^{-2y\tau} \cdot \max_{0 \leq s \leq 2\tau} U\Big(y, \frac{s}{2\tau}\Big) \cdot 2\pi \int_{0}^{2\tau} |\varphi(s - \tau)|^2 \,\mathrm{d}s \\
&\qquad = 2\,\tau^2 \,\mathrm{e}^{-2y\tau} \cdot \max_{0 \leq \xi \leq 1} U(y, \xi) \cdot 2\pi \int_{-\tau}^{\tau} |\varphi(t)|^2 \,\mathrm{d}t
\end{aligned}$$

and so (15) holds. □

The following example shows that inequality (15) gives the sharp upper bound for

$$\frac{\int_{-\infty}^{\infty} |f'(x+\mathrm{i}y)+\mathrm{i}\tau f(x+\mathrm{i}y)|^2\,\mathrm{d}x}{\int_{-\infty}^{\infty}|f(x)|^2\,\mathrm{d}x}$$

for any real y if f belongs to the Paley-Wiener space $\mathcal{P}_\tau$ and is real on the real axis.

Example 5. For any given y, say $y=\eta$, there exists $\xi_\eta \in [0,1]$ (not necessarily unique) such that

$$m(\eta) := \max_{0\le\xi\le 1} U(\eta,\xi) = U(\eta,\xi_\eta) = U(\eta, 1-\xi_\eta)\,.$$

First let $0 \le \xi_\eta < 1/2$, so that $s_\eta := 2\,\tau\xi_\eta \in [0,\tau)$. Now, for any positive $\varepsilon < \tau - s_\eta$, let

$$f_\varepsilon(z) := \int_{s_\eta-\tau}^{s_\eta-\tau+\varepsilon} \mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t + \int_{\tau-s_\eta-\varepsilon}^{\tau-s_\eta} \mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t\,.$$

Then $\int_{-\infty}^{\infty}|f_\varepsilon(x)|^2\,\mathrm{d}x = 4\pi\varepsilon$ and, from (30), we have

$$\int_{-\infty}^{\infty} |f_\varepsilon'(x+\mathrm{i}\eta)+\mathrm{i}\tau f_\varepsilon(x+\mathrm{i}\eta)|^2\,\mathrm{d}x = 2\tau^2\,\mathrm{e}^{-2\eta\tau}\,2\pi\Big\{\int_{s_\eta}^{s_\eta+\varepsilon} + \int_{2\tau-s_\eta-\varepsilon}^{2\tau-s_\eta}\Big\} U\Big(\eta, \frac{s}{2\tau}\Big)\,\mathrm{d}s\,.$$

Since $U(\eta, s/(2\tau)) \to m(\eta)$ as $s\to s_\eta$ or as $s \to 2\tau - s_\eta$, we see that

$$\frac{\int_{-\infty}^{\infty} |f_\varepsilon'(x+\mathrm{i}\eta)+\mathrm{i}\tau f_\varepsilon(x+\mathrm{i}\eta)|^2\,\mathrm{d}x}{\int_{-\infty}^{\infty}|f_\varepsilon(x)|^2\,\mathrm{d}x} \to 2\tau^2\mathrm{e}^{-2\eta\tau} m(\eta) \quad \text{as} \quad \varepsilon\to 0\,.$$

In the case where $\xi_\eta = 1/2$ so that $s_\eta = \tau$, we may consider the function

$$f(z) := \int_{-\varepsilon}^{\varepsilon} \mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t = 2\,\frac{\sin\varepsilon z}{z}\,, \qquad \varepsilon\in(0,\tau)$$

to draw the same conclusion.

Proof of Corollary 1. It is clear from (14) that $U(y,0)=U(y,1)=1$ for all real y and that $U(y,\xi)$ is an increasing function of y for any $\xi\in(0,1)$. In particular,

$$U(y,\xi) \le U\Big(\frac{\ln 2}{2\tau},\xi\Big) \qquad \Big(-\infty < y \le \frac{\ln 2}{2\tau}\Big)\,.$$

Since

$$U\Big(\frac{\ln 2}{2\tau},\xi\Big) = \xi^2\,4^{(1-\xi)} + (1-\xi)^2\,4^{\xi}\,,$$

it follows from Lemma 2 that $\max_{0\le\xi\le1} U(y,\xi) = 1$ for all $y \le (\ln 2)/(2\tau)$. Hence, (15) can be replaced by (15′) for such values of y. □

The next example shows that inequality (15′) does not hold for any y larger than $(\ln 2)/(2\tau)$.

Example 6. Let $y > (\ln 2)/(2\tau)$ and for any $\varepsilon \in (0,\tau)$, let

$$f(z) := \int_{-\varepsilon}^{\varepsilon} \mathrm{e}^{\mathrm{i}zt}\,\mathrm{d}t = 2\,\frac{\sin \varepsilon z}{z}\,.$$

Then $\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x = 4\pi\varepsilon$ and

$$\int_{-\infty}^{\infty} |f'(x+\mathrm{i}y) + \mathrm{i}\tau f(x+\mathrm{i}y)|^2\,\mathrm{d}x = 2\pi \int_{-\varepsilon}^{\varepsilon} (\tau+t)^2\,\mathrm{e}^{-yt}\,\mathrm{d}t$$

so that

$$\frac{\int_{-\infty}^{\infty} |f'(x+\mathrm{i}y) + \mathrm{i}\tau f(x+\mathrm{i}y)|^2\,\mathrm{d}x}{\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x} \to \tau^2 \quad \text{as} \quad \varepsilon \to 0\,,$$

which proves our assertion, since $\tau^2 > 2\,\tau^2\,\mathrm{e}^{-2\tau y}$ for any $y > (\ln 2)/(2\tau)$.

Proof of Corollary 2. First, let us prove (16) for $y = 1/\tau$. In view of (15), this amounts to showing that $\max_{0\le\xi\le1} U(1/\tau,\xi) = \mathrm{e}^2/2$. Clearly,

$$U\Big(\frac{1}{\tau},\xi\Big) = \xi^2\,\mathrm{e}^{4(1-\xi)} + (1-\xi)^2\,\mathrm{e}^{4\xi}\,.$$

A simple calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\Big\{\xi^2\,\mathrm{e}^{4(1-\xi)} + (1-\xi)^2\,\mathrm{e}^{4\xi}\Big\} = 2(1-2\xi)\big\{\xi\mathrm{e}^{4(1-\xi)} + (1-\xi)\mathrm{e}^{4\xi}\big\}\,.$$

Since $\xi\mathrm{e}^{4(1-\xi)} + (1-\xi)\mathrm{e}^{4\xi} > 0$ for $0 \le \xi \le 1$, the function $\xi^2\,\mathrm{e}^{4(1-\xi)} + (1-\xi)^2\,\mathrm{e}^{4\xi}$ increases from 1 to $\mathrm{e}^2/2$ as ξ increases from 0 to 1/2 and then decreases to 1 as ξ increases to 1. Thus, we have

$$\max_{0\le\xi\le1} U\Big(\frac{1}{\tau},\xi\Big) = U\Big(\frac{1}{\tau},\frac{1}{2}\Big) = \frac{1}{2}\,\mathrm{e}^2\,. \tag{31}$$

Let $\mathcal{P}_{\tau,\mathbb{R}} := \{f \in \mathcal{P}_\tau : f(x) \in \mathbb{R} \text{ if } x \in \mathbb{R}\}$. For any function $f \in \mathcal{P}_{\tau,\mathbb{R}}$, let

$$I_f(y) := \int_{-\infty}^{\infty} |f'(x+\mathrm{i}y) + \mathrm{i}\tau f(x+\mathrm{i}y)|^2\,\mathrm{d}x \qquad (-\infty < y < \infty)\,.$$

Then, for any $f \in \mathcal{P}_{\tau,\mathbb{R}}$ such that $\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x = 1$, inequality (15) in conjunction with (31), implies that

$$I_f\Big(\frac{1}{\tau}\Big) \le \tau^2\,. \tag{32}$$

As a special case of (15′), we also have

$$I_f\Big(\frac{\ln 2}{2\tau}\Big) \le \tau^2\,, \tag{33}$$

under the same restrictions on f. Inequalities (32) and (33) say that inequality (16) holds for $y = 1/\tau$ and for $y = (\ln 2)/(2\tau)$. So, in order to prove (16) for the intermediate values of y it is sufficient to show that I_f is a convex function of y. However, this is trivial. Indeed, by the Paley-Wiener representation, there exists a function φ belonging to $L^2(-\tau,\tau)$ such that $f(z) = \int_{-\tau}^{\tau} \mathrm{e}^{\mathrm{i}zt}\,\varphi(t)\,\mathrm{d}t$, where φ cannot be zero almost everywhere on $[-\tau,\tau]$ since $\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x = 1$. Furthermore,

$$I_f(y) = 2\pi \int_{-\tau}^{\tau} (t+\tau)^2\,\mathrm{e}^{-2yt}\,|\varphi(t)|^2\,\mathrm{d}t\,,$$

so that

$$I_f''(y) = \frac{\mathrm{d}^2 I_f}{\mathrm{d}y^2} = 2\pi \int_{-\tau}^{\tau} 4\,t^2\,(t+\tau)^2\,\mathrm{e}^{-2yt}\,|\varphi(t)|^2\,\mathrm{d}t > 0\,.$$

□

Remark. Since (15) is sharp and

$$\max_{0\le\xi\le 1} U(y,\xi) \ge U\Big(y,\frac{1}{2}\Big) = \frac{1}{2}\,\mathrm{e}^{2\tau y}\,,$$

it follows that

$$\max_{0\le\xi\le 1} U(y,\xi) = \frac{1}{2}\,\mathrm{e}^{2\tau y} \qquad \Big(\frac{\ln 2}{2\tau} \le y \le \frac{1}{\tau}\Big)\,.$$

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M. A. Qazi
Department of Mathematics
Tuskegee University
Tuskegee, AL 36088
U. S. A.
E-mail: qazima@aol.com

Q. I. Rahman
Département de Mathématiques et de Statistique
Université de Montréal
Montréal, Québec H3C 3J7
CANADA
E-mail: rahmanqi@dms.umontreal.ca