Th 6/22

Stokes' theorem

Recall: Green's theorem

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Stokes' theorem is a generalization of Green's theorem

Figure 1 shows an oriented surface with unit normal vector **n**. The orientation of S induces the **positive orientation of the boundary curve** C shown in the figure. This means that if you walk in the positive direction around C with your head pointing in the direction of **n**, then the surface will always be on your left.

> **Stokes' Theorem** Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let **F** be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then
>
> $$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

positive orientation: when your head point in the direction of $\vec{n}$
surface is on your left

Stokes' theorem recovers Green's theorem

In fact, in the special case where the surface S is flat and lies in the xy-plane with upward orientation, the unit normal is **k**, the surface integral becomes a double integral, and Stokes' Theorem becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dA$$

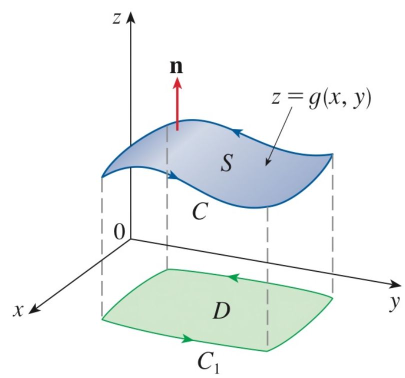
This is precisely the vector form of Green's Theorem given in Equation 16.5.12. Thus we see that Green's Theorem is really a special case of Stokes' Theorem.

$$\left.\begin{aligned} &\vec{F} = (P, Q, 0), \quad d\vec{r} = \vec{i}\,dx + \vec{j}\,dy + \vec{k}\,dz \\ &\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k} \end{aligned}\right\} \Rightarrow \int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Goal: Give a proof of Stokes' theorem in some simple cases

Assumption: S is the graph of a function $z = g(x,y)$, $(x,y) \in D$

S has boundry C $\leftrightsquigarrow$ D has boundary C_1



Let $\vec{F} = (P, Q, R)$, then $\text{curl}\,\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

then
$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_D \left[-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] dA$$

For the line integral

where the partial derivatives of P, Q, and R are evaluated at $(x, y, g(x, y))$. If

$$x = x(t) \qquad y = y(t) \qquad a \leqslant t \leqslant b$$

is a parametric representation of C_1, then a parametric representation of C is

$$x = x(t) \qquad y = y(t) \qquad z = g(x(t), y(t)) \qquad a \leqslant t \leqslant b$$

This allows us, with the aid of the Chain Rule, to evaluate the line integral as follows:

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right) dt \\
&= \int_a^b \left[P\frac{dx}{dt} + Q\frac{dy}{dt} + R\left(\frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}\right)\right] dt \\
&= \int_a^b \left[\left(P + R\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(Q + R\frac{\partial z}{\partial y}\right)\frac{dy}{dt}\right] dt \\
&= \int_{C_1} \left(P + R\frac{\partial z}{\partial x}\right) dx + \left(Q + R\frac{\partial z}{\partial y}\right) dy \\
&= \iint_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right)\right] dA
\end{aligned}$$

where we have used Green's Theorem in the last step. Then, using the Chain Rule again and remembering that P, Q, and R are functions of x, y, and z and that z is itself a function of x and y, we get

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left[\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial x}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + R\frac{\partial^2 z}{\partial x\, \partial y} \right) - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + \frac{\partial R}{\partial y}\frac{\partial z}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial y\, \partial x} \right) \right] dA$$

Four of the terms in this double integral cancel and the remaining six terms can be arranged to coincide with the right side of Equation 2. Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

Example:

EXAMPLE 1 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = -y^2\,\mathbf{i} + x\,\mathbf{j} + z^2\,\mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)

SOLUTION The curve C (an ellipse) is shown in Figure 3. Although $\int_C \mathbf{F} \cdot d\mathbf{r}$ could be evaluated directly, it's easier to use Stokes' Theorem. We first compute

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1 + 2y)\,\mathbf{k}$$

Stokes' Theorem allows us to choose any (oriented, piecewise-smooth) surface with boundary curve C. Among the many possible such surfaces, the most convenient choice is the elliptical region S in the plane $y + z = 2$ that is bounded by C. If we orient S upward, then C has the induced positive orientation. The projection D of S onto the xy-plane is the disk $x^2 + y^2 \leqslant 1$ and so using Equation 16.7.10 with $z = g(x, y) = 2 - y$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D (1 + 2y)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} + 2\frac{r^3}{3}\sin\theta \right]_0^1 d\theta = \int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{2}{3}\sin\theta \right) d\theta$$

$$= \tfrac{1}{2}(2\pi) + 0 = \pi$$

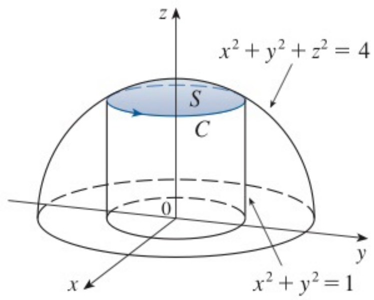


FIGURE 4

EXAMPLE 2 Use Stokes' Theorem to compute the integral $\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + xy\,\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane. (See Figure 4.)

SOLUTION 1 To find the boundary curve C we solve the equations $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$. Subtracting, we get $z^2 = 3$ and so $z = \sqrt{3}$ (since $z > 0$). Thus C is the circle given by the equations $x^2 + y^2 = 1$, $z = \sqrt{3}$. A vector equation of C is

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sqrt{3}\,\mathbf{k} \qquad 0 \leqslant t \leqslant 2\pi$$

so
$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

Solution 1: Find the boundary curve of S:

$$x^2+y^2+z^2=4 \quad \& \quad x^2+y^2=1 \quad \Rightarrow \quad z=\sqrt{3}$$

then the boundary curve can be parametrized by:

$$x=\cos\theta, \quad y=\sin\theta, \quad z=\sqrt{3}, \qquad 0\le\theta\le 2\pi$$

then
$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}\cdot\vec{r}\,'(\theta)\,d\theta$$

$$\vec{F}(\vec{r}(\theta))\cdot\vec{r}\,'(\theta) = (\sqrt{3}\cos\theta,\ \sqrt{3}\sin\theta,\ \sin\theta\cos\theta)\cdot(-\sin\theta,\ \cos\theta,\ 0)$$

$$= -\sqrt{3}\cos\theta\sin\theta + \sqrt{3}\sin\theta\cos\theta = 0$$

hence
$$\iint_S \text{Curl}\,\vec{F}\cdot d\vec{S} = 0$$

SOLUTION 2 Let S_1 be the disk in the plane $z = \sqrt{3}$ inside the cylinder $x^2 + y^2 = 1$, as shown in Figure 5. Since S_1 and S have the same boundary curve C, it follows by Stokes' Theorem that

$$\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \text{curl}\,\mathbf{F}\cdot d\mathbf{S}$$

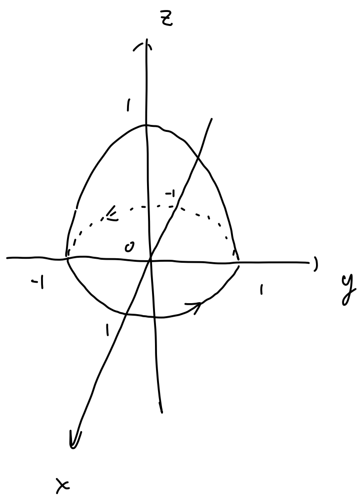
Because S_1 is part of a horizontal plane, its upward normal is $\mathbf{k}$. We calculate that $\text{curl}\,\mathbf{F} = (x - y)\,\mathbf{i} + (x - y)\,\mathbf{j}$, so

$$\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS$$

$$= \iint_{S_1} [(x - y)\,\mathbf{i} + (x - y)\,\mathbf{j}]\cdot\mathbf{k}\,dS = \iint_{S_1} 0\,dS = 0 \qquad ■$$

More examples:

2. $\mathbf{F}(x, y, z) = x^2 \sin z\,\mathbf{i} + y^2\,\mathbf{j} + xy\,\mathbf{k}$,
S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, oriented upward



Solution 1: boundary curve:

$$\vec{r}(\theta) = (\cos\theta,\ \sin\theta,\ 0), \quad 0 \le \theta \le 2\pi$$

then

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_C \vec{F}\ d\vec{r}$$

$$= \int_0^{2\pi} \vec{F}(\vec{r}(\theta)) \cdot \vec{r}'(\theta)\, d\theta$$

$$\vec{F}(\vec{r}(\theta)) = (0,\ \sin^2\theta,\ \sin\theta\cos\theta)$$

$$\vec{r}'(\theta) = (-\sin\theta,\ \cos\theta,\ 0)$$

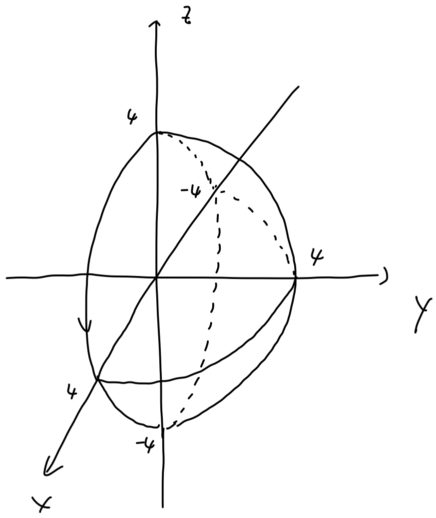
then $$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \sin^2\theta \cos\theta\, d\theta$$

Solution 2: Consider S_1: $x^2 + y^2 \le 1$, $z = 0$

then $$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_{S_1} \operatorname{curl} \vec{F} \cdot d\vec{S_1}$$

$$\left.\begin{array}{l} \operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 \sin z & y^2 & xy \end{vmatrix} = (x,\ x^2\cos z - y,\ 0) \\ \vec{k} = (0, 0, 1) \end{array}\right\} \Rightarrow 0$$

3. $\mathbf{F}(x, y, z) = ze^y\,\mathbf{i} + x\cos y\,\mathbf{j} + xz\sin y\,\mathbf{k}$,
S is the hemisphere $x^2 + y^2 + z^2 = 16$, $y \geqslant 0$, oriented in the direction of the positive y-axis



Solution 1 : boundary curve :

$$\vec{r}(\theta) = (4\cos\theta, 0, -4\sin\theta), \quad \theta \in [0, 2\pi]$$

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{s} = \int_0^{2\pi} \vec{F}(\vec{r}(\theta))\cdot\vec{r}\,'(\theta)\, d\theta$$

$$\vec{F}(\vec{r}(\theta)) = (-4\sin\theta,\ 4\cos\theta,\ 0)$$

$$\vec{r}\,'(\theta) = (-4\sin\theta,\ 0,\ -4\cos\theta)$$

$$\Rightarrow \iint_S \operatorname{curl}\vec{F}\cdot d\vec{s} = \int_0^{2\pi} 16\sin^2\theta\, d\theta$$

Solution 2 : $S_1 = x^2 + z^2 \leq 16,\ y = 0$

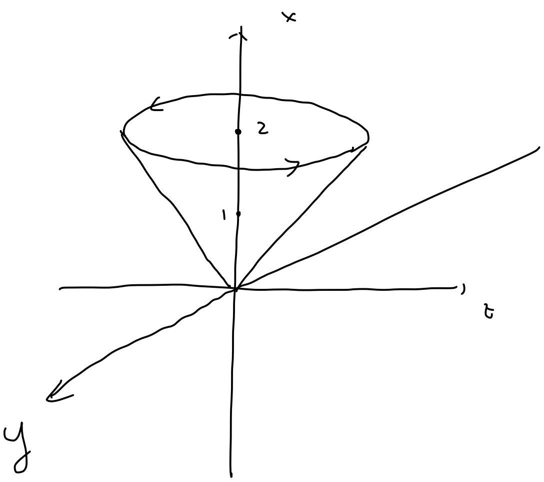
$$\vec{k} = (0, 1, 0)$$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^y & x\cos y & xz\sin y \end{vmatrix} = (xz\cos y,\ e^y - z\sin y,\ \cos y - ze^y)$$

hence
$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{s} = \iint_{S_1} \operatorname{curl}\vec{F}\cdot d\vec{s}_1$$

$$= \iint_{S_1} 1 \quad dS_1 = 16\pi$$

4. $\mathbf{F}(x, y, z) = \tan^{-1}(x^2yz^2)\,\mathbf{i} + x^2y\,\mathbf{j} + x^2z^2\,\mathbf{k}$,
S is the cone $x = \sqrt{y^2 + z^2}$, $0 \leqslant x \leqslant 2$, oriented in the direction of the positive x-axis



boundary curve:

$$x = 2, \quad y = 2\cos\theta, \quad z = -2\sin\theta$$

$$\theta \in [0, 2\pi]$$

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \vec{F}(\vec{r}(\theta)) \cdot \vec{r}\,'(\theta)\, d\theta$$

$$\vec{F}(\vec{r}(\theta)) = \left(\tan^{-1}(32 \cos\theta \sin^2\theta),\ 8\cos\theta,\ 16 \sin^2\theta \right)$$

$$\vec{r}\,'(\theta) = (0, -2\sin\theta, -2\cos\theta)$$

$$\Rightarrow \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \left(-16 \sin\theta \cos\theta - 32 \sin^2\theta \cos\theta \right) d\theta$$

20. Let C be a simple closed smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C z\,dx - 2x\,dy + 3y\,dz$$

depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

$$\int_C z\,dx - 2x\,dy + 3y\,dz = \int_C \overset{\vec{F}}{(z, -2x, 3y)} \cdot d\vec{r}$$

$$= \iint_S \text{curl}\,\vec{F} \cdot d\vec{S}$$

$$= \iint_S \left(\text{curl}\,\vec{F} \cdot \vec{n}\right) dS$$

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -2x & 3y \end{vmatrix} = (3, 1, -2)$$

$\Rightarrow$ $\text{Curl}\,\vec{F} \cdot \vec{n} = 2$ $\Rightarrow$

$$\int_C z\,dx - 2x\,dy + 3y\,dz = 2 \cdot \text{area}(S)$$

The Divergence Theorem

Another expression of Green's theorem

$D \subseteq \mathbb{R}^2$ be a region, we consider $\mathbb{R}^2$ as the xy-plane, let $\vec{F} = (P, Q)$ be a vector field

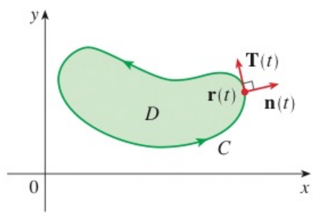


FIGURE 4

If C is given by the vector equation

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} \qquad a \leqslant t \leqslant b$$

then the unit tangent vector (see Section 13.2) is

$$\mathbf{T}(t) = \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}$$

You can verify that the outward unit normal vector to C is given by

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\,\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\,\mathbf{j}$$

(See Figure 4.) Then, from Equation 16.2.3, we have

$$\begin{aligned}
\oint_C \mathbf{F} \cdot \mathbf{n}\, ds &= \int_a^b (\mathbf{F} \cdot \mathbf{n})(t)\, |\mathbf{r}'(t)|\, dt \\
&= \int_a^b \left[\frac{P(x(t), y(t))\, y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t), y(t))\, x'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)|\, dt \\
&= \int_a^b P(x(t), y(t))\, y'(t)\, dt - Q(x(t), y(t))\, x'(t)\, dt \\
&= \int_C P\, dy - Q\, dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA
\end{aligned}$$

by Green's Theorem. But the integrand in this double integral is just the divergence of $\mathbf{F}$. So we have a second vector form of Green's Theorem.

13 $$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}(x, y)\, dA$$

This version says that the line integral of the normal component of $\mathbf{F}$ along C is equal to the double integral of the divergence of $\mathbf{F}$ over the region D enclosed by C.

The Divergence Theorem Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \underbrace{\mathbf{F} \cdot d\mathbf{S}}_{\vec{F} \cdot \vec{n}\, dS} = \iiint_E \operatorname{div} \mathbf{F}\, dV$$

Proof of the Divergence Theorem : E is a simple solid region, it's type 1, 2 & 3

PROOF Let $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. Then

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

so
$$\iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E \frac{\partial P}{\partial x}\, dV + \iiint_E \frac{\partial Q}{\partial y}\, dV + \iiint_E \frac{\partial R}{\partial z}\, dV$$

If $\mathbf{n}$ is the unit outward normal of S, then the surface integral on the left side of the Divergence Theorem is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \mathbf{n}\, dS$$

$$= \iint_S P\,\mathbf{i} \cdot \mathbf{n}\, dS + \iint_S Q\,\mathbf{j} \cdot \mathbf{n}\, dS + \iint_S R\,\mathbf{k} \cdot \mathbf{n}\, dS$$

Therefore, to prove the Divergence Theorem, it suffices to prove the following three equations:

2
$$\iint_S P\,\mathbf{i} \cdot \mathbf{n}\, dS = \iiint_E \frac{\partial P}{\partial x}\, dV$$

3
$$\iint_S Q\,\mathbf{j} \cdot \mathbf{n}\, dS = \iiint_E \frac{\partial Q}{\partial y}\, dV$$

4
$$\iint_S R\,\mathbf{k} \cdot \mathbf{n}\, dS = \iiint_E \frac{\partial R}{\partial z}\, dV$$

Claim: $\boxed{4}$ is true if E is of type 1

To prove Equation 4 we use the fact that E is a type 1 region:

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

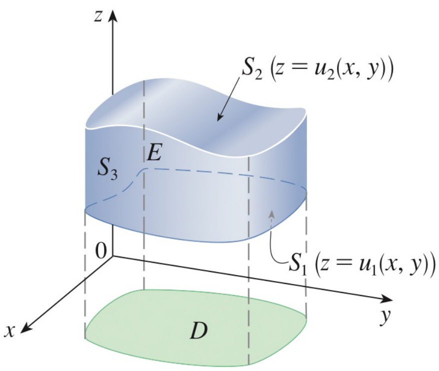
where D is the projection of E onto the xy-plane. By Equation 15.6.6, we have

$$\iiint_E \frac{\partial R}{\partial z}\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} \frac{\partial R}{\partial z}(x, y, z)\, dz \right] dA$$

and therefore, by the Fundamental Theorem of Calculus,

5
$$\iiint_E \frac{\partial R}{\partial z}\, dV = \iint_D \left[R\big(x, y, u_2(x, y)\big) - R\big(x, y, u_1(x, y)\big) \right] dA$$

Let's now try to compute $\iint_S R\,\vec{k}\cdot\vec{n}\,dS$



S consists of three parts:

S_1, S_2 & S_3

on S_3: $\vec{k}\cdot\vec{n} = 0$, then we only need to evaluate

$$\underbrace{\iint_{S_1}}_{\text{bottom surface}} R\,\vec{k}\cdot\vec{n}\,dS \quad \& \quad \underbrace{\iint_{S_2}}_{\text{top surface}} R\,\vec{k}\cdot\vec{n}\,dS$$

The equation of S_2 is $z = u_2(x, y)$, $(x, y) \in D$, and the outward normal $\mathbf{n}$ points upward, so from Equation 16.7.10 (with $\mathbf{F}$ replaced by $R\,\mathbf{k}$) we have

$$\iint_{S_2} R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_D R(x, y, u_2(x, y))\,dA$$

On S_1 we have $z = u_1(x, y)$, but here the outward normal $\mathbf{n}$ points downward, so we multiply by -1:

$$\iint_{S_1} R\,\mathbf{k}\cdot\mathbf{n}\,dS = -\iint_D R(x, y, u_1(x, y))\,dA$$

Therefore Equation 6 gives

$$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iint_D \left[R(x, y, u_2(x, y)) - R(x, y, u_1(x, y))\right] dA$$

Comparison with Equation 5 shows that

$$\iint_S R\,\mathbf{k}\cdot\mathbf{n}\,dS = \iiint_E \frac{\partial R}{\partial z}\,dV$$

Equations 2 and 3 are proved in a similar manner using the expressions for E as a type 2 or type 3 region, respectively. ■

Examples :

EXAMPLE 1 Find the flux of the vector field $\mathbf{F}(x, y, z) = z\,\mathbf{i} + y\,\mathbf{j} + x\,\mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION First we compute the divergence of $\mathbf{F}$:

$$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$$

The unit sphere S is the boundary of the unit ball B given by $x^2 + y^2 + z^2 \leqslant 1$. Thus the Divergence Theorem gives the flux as

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_B \text{div}\,\mathbf{F}\,dV = \iiint_B 1\,dV = V(B) = \tfrac{4}{3}\pi(1)^3 = \frac{4\pi}{3}$$ ■

EXAMPLE 2 Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy\,\mathbf{i} + \left(y^2 + e^{xz^2}\right)\mathbf{j} + \sin(xy)\,\mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$. (See Figure 2.)

description of the solid

$$E = \left\{(x, y, z) \mid -1 \leqslant x \leqslant 1,\ 0 \leqslant z \leqslant 1 - x^2,\ 0 \leqslant y \leqslant 2 - z\right\}$$

divergence is:

$$\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}\left(y^2 + e^{xz^2}\right) + \frac{\partial}{\partial z}(\sin xy) = y + 2y = 3y$$

Then we have

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div}\,\mathbf{F}\,dV = \iiint_E 3y\,dV$$

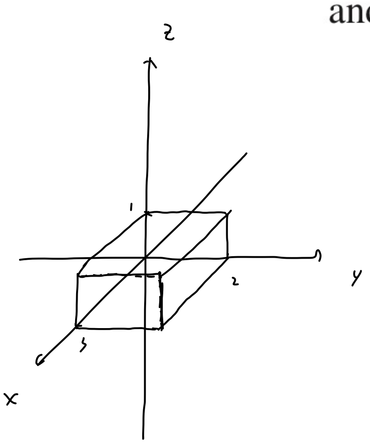
$$= 3\int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} y\,dy\,dz\,dx = 3\int_{-1}^{1}\int_0^{1-x^2} \frac{(2 - z)^2}{2}\,dz\,dx$$

$$= \frac{3}{2}\int_{-1}^{1}\left[-\frac{(2 - z)^3}{3}\right]_0^{1-x^2} dx = -\tfrac{1}{2}\int_{-1}^{1}\left[(x^2 + 1)^3 - 8\right]dx$$

$$= -\int_0^1 (x^6 + 3x^4 + 3x^2 - 7)\,dx = \frac{184}{35}$$ ■

More examples:

5. $\mathbf{F}(x, y, z) = xye^z\,\mathbf{i} + xy^2z^3\,\mathbf{j} - ye^z\,\mathbf{k}$,
S is the surface of the box bounded by the coordinate planes and the planes $x = 3$, $y = 2$, and $z = 1$

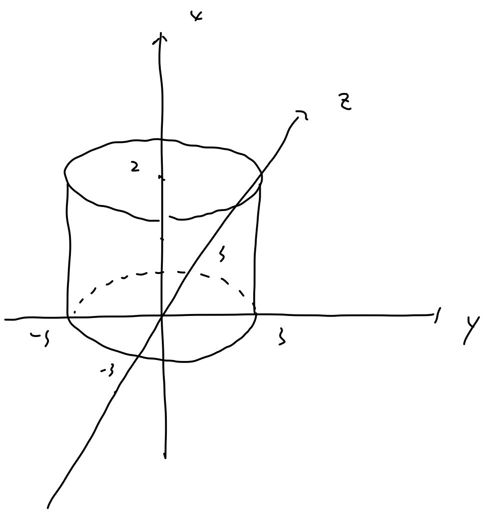


$$\operatorname{div} \vec{F} = ye^z + 2xyz^3 - ye^z$$

then $$\iint_S \vec{F}\, d\vec{S} = \iiint_E ye^z + 2xyz^3 - ye^z \, dV$$

$$= \int_0^3 \int_0^2 \int_0^1 ye^z + 2xyz^3 - ye^z \, dz\, dy\, dx$$

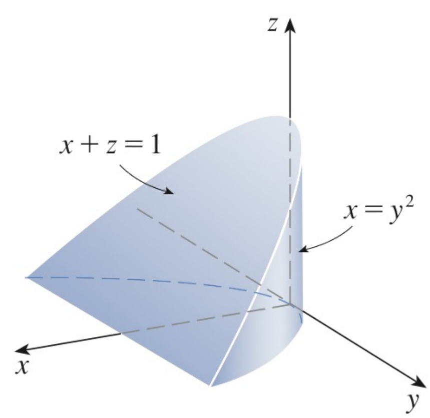
4. $\mathbf{F}(x, y, z) = \langle x^2, -y, z \rangle$,
E is the solid cylinder $y^2 + z^2 \leqslant 9,\ 0 \leqslant x \leqslant 2$



type 1, 2, 3 ✓

$\operatorname{div} \vec{F} = 2x - 1 + 1 = 2x$

14. $\mathbf{F}(x, y, z) = (xy - z^2)\,\mathbf{i} + x^3\sqrt{z}\,\mathbf{j} + (xy + z^2)\,\mathbf{k}$,
S is the surface of the solid bounded by the cylinder $x = y^2$ and the planes $x + z = 1$ and $z = 0$

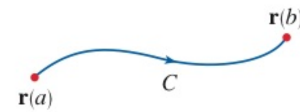
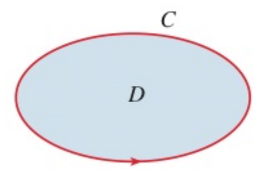
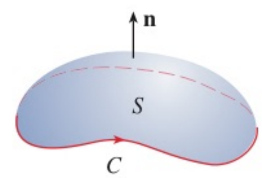
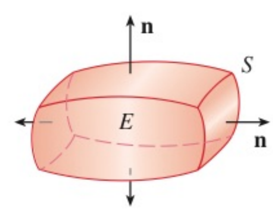


$\operatorname{div} \vec{F} = y + 0 + 2z = 2z + y$

$E:\ -1 \leq y \leq 1,\quad y^2 \leq x \leq 1,\quad 0 \leq z \leq 1 - x$

$$\Rightarrow \iint_S \vec{F}\, d\vec{S} = \iiint_E (2z + y)\, dV$$

Review of previous sections

Curves and their boundaries (endpoints)		
Fundamental Theorem of Calculus	$\int_a^b F'(x)\,dx = F(b) - F(a)$	
Fundamental Theorem for Line Integrals	$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$	
Surfaces and their boundaries		
Green's Theorem	$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C P\,dx + Q\,dy$	
Stokes' Theorem	$\iint_S \text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$	
Solids and their boundaries		
Divergence Theorem	$\iiint_E \text{div}\,\mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$	

line integral

- scalar functions $\int_C f\, ds$
- vector fields $\int_C \vec{F} \cdot d\vec{r}$

surface integral

- scalar functions $\int_S f\, ds$
- vector fields $\int_S \vec{F} \cdot d\vec{s}$