We have seen how the knot group, that is, the fundamental group of the knot exterior, gives us a knot invariant, whereas its abelianization, which by the Hurewicz theorem is also the first homology group of the knot exterior, does not help us in distinguishing knots because it is always $\mathbb{Z}$. Although the knot group is an obvious knot invariant, it is not the most useful invariant when studying knots because it is not that easy to work with nonabelian groups. However, we can obtain a polynomial (strictly speaking, Laurent polynomial) knot invariant, the Alexander polynomial, from the first homology group of the infinite cyclic cover of the knot exterior. This knot invariant turns out to have a close analogy with the Iwasawa polynomial in number theory.

1. HOMOLOGY GROUPS

By the Hurewicz theorem, which states that the first homology group of a space is isomorphic to the abelianization of its fundamental group, one could conceivably give a definition of the Alexander polynomial in terms of fundamental groups and abelianizations. However, in order to provide more intuition regarding the Alexander polynomial, we shall first introduce homology groups.

As we shall see, homology is directly related to the decomposition of a space into cells, and may be regarded as an algebraization of the most obvious geometry in a cell structure: how cells of dimension n attach to cells of dimension $n-1$. It provides information about the number of "holes of each dimension" in a space.

Example 1.1. Let us do an example to get an idea of what homology measures. Consider the graph X_1 shown in Figure 1, which consists of two vertices joined by four edges; for the purpose of computing homology, we shall make the edges directed. When studying the fundamental group of X_1, we consider loops formed by sequences of edges, starting and ending at a fixed basepoint. For example, we could first travel along the edge a, then backward along the edge b, to obtain a loop ab^{-1}. A more complicated loop would be $ab^{-1}ac^{-1}ba^{-1}cb^{-1}$. Now, a key feature of the fundamental group is that it is nonabelian in the sense that ab^{-1} is regarded as a different element from $b^{-1}a$ and $ab^{-1}ac^{-1}ba^{-1}cb^{-1}$, which enriches but simultaneously complicates the theory.

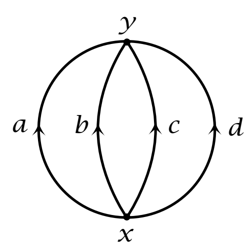


FIGURE 1. Directed graph X_1.

The idea of homology is to try to simplify matters by abelianizing. For example, the loops ab^{-1}, $b^{-1}a$ and $ab^{-1}ac^{-1}ba^{-1}cb^{-1}$ are regarded as equal if we make a, b, c and d all commute with each other. Thus, one consequence of abelianizing is that loops are no longer required to have a fixed basepoint; rather, they become *cycles*, without a chosen basepoint.

Having abelianized, let us switch to additive notation instead. Then cycles become linear combinations of edges with integer coefficients, for example $a-b+c-b=a-2b+c$. We call such linear combinations *chains* of edges. For example, $a-b+c$ is a chain that cannot arise from a cycle, whereas the chain $a-2b+c$ can arise from the loop $ab^{-1}cb^{-1}$. We will call a *cycle* (in the algebraic sense) any chain that can be decomposed into one or more cycles in the previous geometric sense. (Thus, for example, we can also think of the chain $a-2b+c$ as arising from a disjoint union of the (geometric) cycles $a-b$ and $c-b$.)

A natural question then arises: when is a chain a cycle (in the algebraic sense)? A disjoint union of geometric cycles is characterised by the property that the number of ingoing edges is equal to the number of outgoing edges at every vertex. For an arbitrary chain $ka+lb+mc+nd$, the number of ingoing edges at y is $k+l+m+n$, while the number of outgoing edges is $-k-l-m-n$ (and vice versa at x). Hence the condition for $ka+lb+mc+nd$ to be a cycle is that $k+l+m+n=0$.

Let us try to describe this result in a way that generalizes to all directed graphs. Let C_1 be the free abelian group on the edges (in this case a, b, c and d) and let C_0 be the free abelian group on the vertices (in this case x and y). The elements of C_1 are chains of edges (1-dimensional chains), while the elements of C_0 are chains of vertices (0-dimensional chains). Define a homomorphism $\partial_1 : C_1 \to C_0$ by

$$\partial_1(\text{edge}) = (\text{vertex at head of edge}) - (\text{vertex at tail of edge})$$

(in this case, ∂_1 sends all four of a, b, c and d to $y - x$). Then the (algebraic) cycles are precisely the kernel of ∂_1. In our example, one can easily check that the kernel of ∂_1 is generated by the three cycles $a - b$, $b - c$ and $c - d$. This conveys the information that the graph X_1 has three visible "holes" bounded by these cycles.

Let us now enlarge the graph X_1 by attaching a 2-cell, that is, an open disc A along the cycle $a - b$ to produce a 2-dimensional *CW complex* or *cell complex*, as in Figure 2. (We shall not define a CW complex rigorously, but it should be clear from this construction what is allowed.) If we think of the 2-cell A as being oriented clockwise, then we can think of its boundary as the cycle $a-b$. This cycle is now homotopically trivial since we have filled in the "hole" bounded by the cycle $a - b$. This suggests that we should form a quotient group of cycles by modding out by the subgroup generated by $a - b$. In this quotient the cycles $a - c$ and $b - c$ are equivalent, which is consistent with the fact that they are homotopic in X_2.

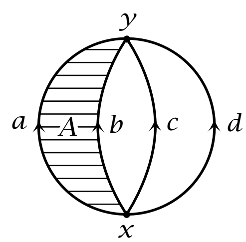


FIGURE 2. Directed graph X_2 obtained by attaching a 2-cell to X_1.

Once again, we can describe this result in a way that generalizes to all 2-dimensional cell complexes as follows. Let C_2 be the free abelian group generated by the 2-cells, and define ∂_2 to be the homomorphism taking a cell to its boundary (in this case, $\partial_2(A) = a - b$). We thus have a sequence of homomorphisms $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$, and the quotient group that we are interested in is $\ker \partial_1 / \operatorname{im} \partial_2$, the 1-dimensional cycles modulo boundaries. This quotient group is the first *homology group* $H_1(X_2)$.

We can continue this procedure by considering CW complexes obtained by adding cells in higher dimensions. It is clear what the general pattern should be: for a CW complex X, one has chain groups $C_n(X)$ which are the free abelian groups generated by the n-cells (i.e. open n-discs) of X, and homomorphisms $\partial_n : C_n(X) \to C_{n-1}(X)$ that are defined by linearity and the property that a cell is sent to its boundary.

Definition 1.2. The nth *homology group* $H_n(X)$ is the quotient group $\ker \partial_n / \operatorname{im} \partial_{n+1}$.

We call chains in the kernel of ∂_n cycles and chains in the image of ∂_{n+1} boundaries, so one should think of homology as the quotient group of cycles modulo boundaries; as in the 1-dimensional example, it should be the case that boundaries are always cycles. The major difficulty lies in defining ∂_n in general. We have seen how ∂_1 and ∂_2 should be defined, but it is less clear how to define ∂_n for $n \geq 3$. One solution to this problem is to consider CW complexes built from simplices (the interval, triangle and tetrahedron are the 1-, 2- and 3-dimensional instances of simplices respectively); these are called Δ-*complexes*. In this case, there is a easy formula for the boundary map ∂_n, and the homology groups thus defined are called *simplicial homology groups*. The drawback of this approach, however, is that this is a rather restrictive class of spaces. (For instance, the CW complex structure in Example 1.1 is not a Δ-complex structure. The type of homology we have computed in Example 1.1 is called *cellular homology*, which turns out to be equivalent to simplicial homology.) Moreover, an obvious question arises: given two different CW or Δ-complex structures on X, do they give rise to isomorphic homology groups? To address this problem, one introduces *singular homology groups*, which are defined for all spaces X, not just CW or Δ-complexes, and then shows that the cellular, simplicial and singular homology groups coincide whenever defined. We shall not define singular homology groups, since we are more interested in the *properties* of homology groups instead of their various equivalent definitions. (It turns out that singular homology can be characterized by a set of axioms consisting of the main properties it satisfies.) Here we simply state some of the main properties of homology groups that one can deduce using singular homology:

- The homology groups $H_n(X)$ only depend on the space X and not on the CW or Δ-complex structure.

- A map of spaces $f : X \to Y$ induces a homomorphism of homology groups $f_* : H_n(X) \to H_n(Y)$ such that $(fg)_* = f_* g_*$ and $\mathrm{id}_* = \mathrm{id}$; in particular, homeomorphic spaces have isomorphic homology groups. In fact, homotopy equivalent spaces have isomorphic homology groups.
- If $f \simeq g : X \to Y$, then $f_* = g_* : H_n(X) \to H_n(Y)$.
- A pair of topological spaces (X,A), $A \subseteq X$ induces a long exact sequence in homology via the inclusions $i : A \to X$ and $j : (X,\emptyset) \to (X,A)$ (we can think of a space X as a pair $(X,\emptyset)$):

$$\cdots \to H_1(X) \xrightarrow{j_*} H_1(X,A) \to H_0(A) \xrightarrow{i_*} H_0(X) \xrightarrow{j_*} H_0(X,A) \to 0.$$

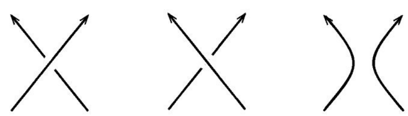
(Here $H_n(X,A)$ is a *relative homology* group. We shall not define relative homology groups, but if A and X are CW complexes with $A \neq \emptyset$ a subcomplex of X, then $H_n(X,A) = H_n(X/A)$ for $n \geq 1$ and $H_0(X,A) \oplus \mathbb{Z} \cong H_0(X/A)$.)

2. SKEIN RELATION DEFINITION OF THE ALEXANDER POLYNOMIAL

There are several equivalent ways of defining the Alexander polynomial. The most elementary way is in terms of a skein relation, that is, a relationship between three knot diagrams that are identical except at the same one crossing. Specifically, the Alexander polynomial $\Delta(t)$ is defined by the following rules:

- $\Delta_{\text{unknot}}(t) = 1$.
- Given three (oriented) links L_+, L_- and L_0 that are identical except as depicted in Figure 3 at one particular crossing, the Alexander polynomials of the links L_+, L_- and L_0 satisfy the relation

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) + (t - t^{-1})\Delta_{L_0}(t) = 0.$$



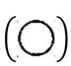
L_+ L_- L_0

FIGURE 3. Variations in crossing of L_+, L_- and L_0.

Example 2.1 (Alexander polynomial of trefoil). Treating the trefoil as L_+, with the circled crossing as the one that appears in Figure 3, we have

$\Delta\Big($  $\Big) - \Delta\Big($  $\Big) + (t - t^{-1})\Delta\Big($  $\Big) = 0,$

where

$\Delta\Big($  $\Big) = \Delta\Big($  $\Big) = 1$

and

$\Delta\Big($  $\Big) - \Delta\Big($  $\Big) + (t - t^{-1})\Delta\Big($  $\Big) = 0.$

Exercise 2.2. Show that $\Delta\Big($  $\Big) = 0.$

Hence

$\Delta\Big($  $\Big) = -t + t^{-1}$

and thus

$\Delta\Big($  $\Big) = t^2 - 1 + t^{-2}.$

This definition in terms of a skein relation allows one to express both the Alexander polynomial (discovered surprisingly early in 1923) and the Jones polynomial (discovered much later in 1984) as special cases of a more general polynomial knot invariant, the HOMFLY polynomial (discovered almost immediately after in 1985). In general, knot invariants that can be defined in terms of a skein relation are usually quantum invariants of interest to quantum topologists. However, to establish the analogy between the Alexander polynomial and the Iwasawa polynomial, we shall use a different definition of the Alexander polynomial, in terms of homology. Historically, the Alexander polynomial was first formulated by Alexander in terms of homology, although he also showed that the Alexander polynomial satisfies a similar skein relation; Conway later rediscovered the skein relation in a different form and proved that the skein relation together with a choice of value for the unknot suffice to determine the polynomial.