A COMBINATORIAL DETERMINANT DUAL TO THE GROUP DETERMINANT

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Abstract. We define the commuting algebra determinant of a finite group action on a finite set, a notion dual to the group determinant of Dedekind. We give the following combinatorial example of a commuting algebra determinant.

Let $B_q(n)$ denote the set of all subspaces of an n-dimensional vector space over $\mathbb{F}_q$. The *type* of an ordered pair (U, V) of subspaces, where $U, V \in B_q(n)$, is the ordered triple $(\dim U, \dim V, \dim U \cap V)$ of nonnegative integers. Assume that there are independent indeterminates corresponding to each type. Let $X_q(n)$ be the $B_q(n) \times B_q(n)$ matrix whose entry in row U, column V is the indeterminate corresponding to the type of (U, V). We factorize the determinant of $X_q(n)$ into irreducible polynomials.

Key words. Group determinant, Combinatorial determinant.

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Dedicated to Ravindra B. Bapat, on the occasion of his 60th birthday

1. Introduction. In this note we revisit certain classical and recent results in algebraic combinatorics from the viewpoint of determinants, connecting the topic to the group determinants of Dedekind **[2]**. Our motivation comes from the paper *Combinatorial matrices* by Knuth **[8]** where the following notion is studied.

For $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, let $B(n)$ denote the set of all subsets of $[n] = \{1, 2, \ldots, n\}$ and, for $0 \le i \le n$, let $B(n)_i$ denote the set of all i-element subsets of the set $[n]$.

Let $i, n \in \mathbb{N}$ with $i \le n/2$. Given $A, B \in B(n)_i$, the *type* of the pair (A, B) is the nonnegative integer $|A \cap B|$. Knuth **[8]** studies $B(n)_i \times B(n)_i$ real matrices with the property that the entry in row A, column B depends only on the type of (A, B). This suggests that we consider generic such matrices, defined as follows. Let $y_0, y_1, \ldots, y_i$ be independent indeterminates corresponding to the $i + 1$ distinct types and let $Y(n, i)$ denote the $B(n)_i \times B(n)_i$ matrix whose entry in row A, column B is given by the indeterminate corresponding to the type of (A, B), i.e., $y_{|A \cap B|}$. We say that $Y(n, i)$ is a *combinatorial matrix of type* (n, i). Note that $Y(n, i)$ is symmetric.

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For example, $Y(4,2)$ is the following matrix:

	12	13	14	23	24	34
12	y_2	y_1	y_1	y_1	y_1	y_0
13	y_1	y_2	y_1	y_1	y_0	y_1
14	y_1	y_1	y_2	y_0	y_1	y_1
23	y_1	y_1	y_0	y_2	y_1	y_1
24	y_1	y_0	y_1	y_1	y_2	y_1
34	y_0	y_1	y_1	y_1	y_1	y_2

There is a natural $B(n) \times B(n)$ analog of the matrix $Y(n,i)$ defined above. For $A, B \in B(n)$, define the *type* of the pair (A, B) to be the triple $(|A|, |B|, |A \cap B|)$. For $n \in \mathbb{N}$ define

$$\mathcal{I}(n) = \{(i,j,t) \in \mathbb{N}^3 : t, i-t, j-t \geq 0 \text{ and } i-t+t+j-t = i+j-t \leq n\}.$$

It is easy to see that $\mathcal{I}(n)$ equals the set of all types $(|A|, |B|, |A \cap B|)$, where A, B range over $B(n)$. Clearly, $|\mathcal{I}(n)| = \binom{n+3}{3}$.

Let $\mathbf{x}(\mathbf{n}) = (x_{i,j,t} : (i,j,t) \in \mathcal{I}(n))$ be independent indeterminates corresponding to the different types and let $X(n)$ denote the $B(n) \times B(n)$ matrix whose entry in row A, column B is given by the indeterminate corresponding to the type of (A, B), i.e., $x_{|A|,|B|,|A \cap B|}$. We say that $X(n)$ is a *combinatorial matrix of type* n. For example, $X(3)$ is the following matrix:

	$\emptyset$	1	2	3	12	13	23	123
$\emptyset$	$x_{0,0,0}$	$x_{0,1,0}$	$x_{0,1,0}$	$x_{0,1,0}$	$x_{0,2,0}$	$x_{0,2,0}$	$x_{0,2,0}$	$x_{0,3,0}$
1	$x_{1,0,0}$	$x_{1,1,1}$	$x_{1,1,0}$	$x_{1,1,0}$	$x_{1,2,1}$	$x_{1,2,1}$	$x_{1,2,0}$	$x_{1,3,1}$
2	$x_{1,0,0}$	$x_{1,1,0}$	$x_{1,1,1}$	$x_{1,1,0}$	$x_{1,2,1}$	$x_{1,2,0}$	$x_{1,2,1}$	$x_{1,3,1}$
3	$x_{1,0,0}$	$x_{1,1,0}$	$x_{1,1,0}$	$x_{1,1,1}$	$x_{1,2,0}$	$x_{1,2,1}$	$x_{1,2,1}$	$x_{1,3,1}$
12	$x_{2,0,0}$	$x_{2,1,1}$	$x_{2,1,1}$	$x_{2,1,0}$	$x_{2,2,2}$	$x_{2,2,1}$	$x_{2,2,1}$	$x_{2,3,2}$
13	$x_{2,0,0}$	$x_{2,1,1}$	$x_{2,1,0}$	$x_{2,1,1}$	$x_{2,2,1}$	$x_{2,2,2}$	$x_{2,2,1}$	$x_{2,3,2}$
23	$x_{2,0,0}$	$x_{2,1,0}$	$x_{2,1,1}$	$x_{2,1,1}$	$x_{2,2,1}$	$x_{2,2,1}$	$x_{2,2,2}$	$x_{2,3,2}$
123	$x_{3,0,0}$	$x_{3,1,1}$	$x_{3,1,1}$	$x_{3,1,1}$	$x_{3,2,2}$	$x_{3,2,2}$	$x_{3,2,2}$	$x_{3,3,3}$

Note that $X(n)$ is not symmetric.

In Section 3 we define two further matrices $Y_q(n,i)$ and $X_q(n)$, the *q-analogs* of $Y(n,i)$ and $X(n)$ respectively (for a fixed prime power q). This paper is concerned with the explicit factorization into irreducible complex polynomials of the determinants of these four matrices, especially the determinant of $X_q(n)$, which is a new result.

In Section 2 we present the general theory of such determinants. We show that they arise as commuting algebra determinants of finite group actions on finite sets, a concept dual to the group determinant of Dedekind **[2]**.

In Section 3, quoting classical and recent results from the literature, we indicate the factorizations of determinants of $Y(n,i), Y_q(n,i), X(n), X_q(n)$ into complex irreducible polynomials. In all cases, it will turn out that the factors have integer coefficients.

2. Algebra determinant with respect to a basis. The theory of the group determinant (see **[2]**) has a simple extension to semisimple modules over an (associative) algebra with a distinguished basis. We now discuss this. All our algebras contain an identity element and algebra homomorphisms preserve the identity.

Let $\mathcal{A}$ be a finite dimensional complex algebra with distinguished basis $A = \{a_1, \ldots, a_n\}$. Let V be a finite dimensional complex vector space that is a (left) $\mathcal{A}$-module, the module structure being given by the homomorphism $\rho : \mathcal{A} \to \mathrm{End}(V)$, where $\mathrm{End}(V)$ denotes the algebra of linear operators on V. Let $x_1, \ldots, x_n$ be independent indeterminates.

Given the above data, we can define the linear form with operator coefficients $\sum_{i=1}^{n} x_i \rho(a_i)$ and take its determinant, which is a homogeneous polynomial of degree $\dim(V)$ in $\mathbb{C}[x_1, \ldots, x_n]$:

$$D_{(\mathcal{A},A)}(V) = \det\left(\sum_{i=1}^{n} x_i \rho(a_i)\right).$$

We call $D_{(\mathcal{A},A)}(V)$ the *algebra determinant of the pair* $(\mathcal{A}, V)$ *with respect to the basis* A *of* $\mathcal{A}$.

Example 2.1. *(the discussion of this Example continues until the beginning of Example 2.2 below.)* Let G be a finite group acting on the finite set S and let $V = V(S)$ denote the complex vector space with S as basis. The action of G on S gives rise to a permutation representation of G on V. For $g \in G$ and $v \in V$, the action of g on v yields the element $g \cdot v$ of V, which we also denote by gv or $g(v)$. We think of the elements of V as column vectors with components indexed by S. We represent elements of $\mathrm{End}(V)$, in the standard basis S, as $S \times S$ matrices. For $r, c \in S$, the entry in row r, column c of a matrix M will be denoted $M(r, c)$. The matrix representing $f \in \mathrm{End}(V)$ is denoted M_f.

Set

$$\mathcal{A} = \{M_f : f \in \mathrm{End}_G(V)\}.$$

So $\mathcal{A}$, the *commuting algebra* of the action of G on S, is a semisimple algebra of $S \times S$ matrices acting on V.

Let $f : V \to V$ be linear and $g \in G$. Then, for $c \in S$, we have

$$f(gc) = \sum_{r \in S} M_f(r, gc) r \text{ and } g(f(c)) = \sum_{r \in S} M_f(r, c) gr.$$

It follows that f is G-linear if and only if

$$M_f(r, c) = M_f(gr, gc), \text{ for all } r, c \in S,\ g \in G, \tag{2.1}$$

i.e., M_f is constant on the orbits of the action of G on $S \times S$.

Let $\mathcal{O} = \{O_1, \ldots, O_p\}$ be the orbits of G acting on $S \times S$. Given $r, c \in S$, the *type* of the pair (r, c) is the unique integer i with $(r, c) \in O_i$. For $i = 1, \ldots, p$, let M_i be the $S \times S$ characteristic matrix of the ordered pairs of type i, i.e, the orbit O_i,

$$M_i((r, c)) = \begin{cases} 1 & \text{if } (r, c) \in O_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $A = \{M_1, \ldots, M_p\}$ is a basis of $\mathcal{A}$. We call A the *standard basis* of the commuting algebra $\mathcal{A}$.

We can associate two $S \times S$ determinants with the action of G on S.

Let $x_g, g \in G$ be independent indeterminates. Consider the group algebra $\mathbb{C}G$ with distinguished basis G and consider the $\mathbb{C}G$-module V. The *group determinant* of (G, S) is defined by $D_{(\mathbb{C}G,G)}(V) \in \mathbb{C}[x_g : g \in G]$. If we define the $S \times S$ *generic group action matrix* N by

$$N(r, c) = \sum_g x_g, \tag{2.2}$$

where the sum is over all $g \in G$ with $gc = r$, then it is easily seen that $D_{(\mathbb{C}G,G)}(V) = \det(N)$.

Let $y_1, \ldots, y_p$ be independent indeterminates. Consider the algebra $\mathcal{A}$ with distinguished basis A and now consider V as an $\mathcal{A}$-module. The *commuting algebra determinant* of (G, S) is defined by $D_{(\mathcal{A},A)}(V) \in \mathbb{C}[y_1, \ldots, y_p]$. If we define the $S \times S$ *generic commuting algebra matrix* N' by

$$N'(r, c) = y_i, \tag{2.3}$$

where $(r, c) \in O_i$, then it is easily seen that $D_{(\mathcal{A},A)}(V) = \det(N')$. Note that $N' = y_1 M_1 + \cdots + y_p M_p$.

Example 2.2. Consider the action of G on itself by left multiplication. We use the notation of Example 2.1.

Now $(g_1, h_1), (g_2, h_2) \in G \times G$ are in the same G-orbit if and only if there exists a $g \in G$ such that $gg_1 = g_2$ and $gh_1 = h_2$. That is, $g_2 g_1^{-1} = h_2 h_1^{-1}$ or $g_1^{-1} h_1 = g_2^{-1} h_2$. So we can consider the G-orbits of $G \times G$ to be parametrized by the elements of G, the element $f \in G$ parametrizing the orbit $\{(g, h) : g^{-1}h = f\}$.

It now follows from (2.2) that the generic group action matrix is the $G \times G$ matrix with (g, h) entry $x_{gh^{-1}}$ and it follows from (2.3) that the generic commuting algebra matrix is the $G \times G$ matrix with (g, h) entry $x_{g^{-1}h}$. Since they differ only by a common permutation of the rows and columns ($g \to g^{-1}$) it follows that the two determinants are the same.

The next result collects together some basic properties of the algebra determinant.

Theorem 2.3. *Let $\mathcal{A}$ be a finite dimensional complex algebra with distinguished basis $A = \{a_1, \ldots, a_n\}$. Let V, W be finite dimensional complex vector spaces and let $\rho : \mathcal{A} \to End(V)$, $\tau : \mathcal{A} \to End(W)$ be algebra homomorphisms.*

(i) If $\mathcal{A}$ is abelian then $D_{(\mathcal{A},A)}(V)$ factors into linear terms.

(ii) If V and W are isomorphic $\mathcal{A}$-modules then $D_{(\mathcal{A},A)}(V) = D_{(\mathcal{A},A)}(W)$.

(iii) If $V = V_1 \oplus V_2$ is a direct sum of two $\mathcal{A}$-submodules V_1 and V_2 then

$$D_{(\mathcal{A},A)}(V) = D_{(\mathcal{A},A)}(V_1)\, D_{(\mathcal{A},A)}(V_2).$$

(iv) If V is an irreducible $\mathcal{A}$-module then $D_{(\mathcal{A},A)}(V)$ is an irreducible polynomial in $\mathbb{C}[x_1, \ldots, x_n]$ of degree $\dim(V)$.

(v) If V, W are nonisomorphic irreducible $\mathcal{A}$-modules then $D_{(\mathcal{A},A)}(V)$ and $D_{(\mathcal{A},A)}(W)$ are not proportional in $\mathbb{C}[x_1, \ldots, x_n]$.

(vi) Assume V is a semisimple $\mathcal{A}$-module with $V_1, V_2, \ldots, V_t$ the nonisomorphic irreducible $\mathcal{A}$-modules occuring in V with respective multiplicities $m_1, m_2, \ldots, m_t$. Then

$$D_{(\mathcal{A},A)}(V) = \prod_{i=1}^{t} D_{(\mathcal{A},A)}(V_i)^{m_i},$$

is the factorization of $D_{(\mathcal{A},A)}(V)$ into powers of distinct irreducibles in $\mathbb{C}[x_1, \ldots, x_n]$.

Proof. (i) Since $\mathcal{A}$ is abelian there exists a basis of V such that the matrix of the operator $\rho(a)$, for any $a \in \mathcal{A}$, is upper triangular with respect to this basis. The result follows.

(ii) and (iii) are clear.

(iv) Fix a basis of V of cardinality k and write down the matrix $N = N(x_1, \ldots, x_n)$ of $\sum_{i=1}^{n} x_i \rho(a_i)$ with respect to this basis. The entries of N are linear polynomials in

$\mathbb{C}[x_1, \ldots, x_n]$. Denote the entry in row i, column j, $1 \le i, j \le k$, of N by

$$\sum_{p=1}^{n} \mu_{i,j,p}\, x_p,$$

where $\mu_{i,j,p}$ are scalars. Since V is irreducible it follows from Burnside's theorem (Corollary 4.1.7 in **[7]**) that, for every $k \times k$ complex matrix M, there exist scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ such that $N(\alpha_1, \ldots, \alpha_n) = M$. Thus the k^2 entries of N are linearly independent polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ of degree 1. Extend them to a basis

$$\left\{ \sum_{p=1}^{n} \mu_{i,j,p}\, x_p : 1 \le i, j \le k \right\} \cup \left\{ \sum_{p=1}^{n} \mu_{m,p}\, x_p : 1 \le m \le l \right\},$$

where $l = n - k^2$, of the vector space of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ of degree 1.

Take independent indeterminates $\{y_{i,j} : 1 \le i, j \le k\} \cup \{z_m : 1 \le m \le l\}$ and consider the ring homomorphism

$$\mathbb{C}[y_{i,j}, z_m : 1 \le i, j \le k,\ 1 \le m \le l] \to \mathbb{C}[x_1, \ldots, x_n]$$

given by $y_{i,j} \mapsto \sum_{p=1}^{n} \mu_{i,j,p}\, x_p$ and $z_m \mapsto \sum_{p=1}^{n} \mu_{m,p}\, x_p$. This map is an isomorphism on the vector space of degree 1 polynomials on both sides and thus takes irreducible polynomials to irreducible polynomials. Since the determinant of the $k \times k$ matrix $(y_{i,j})_{1 \le i,j \le k}$ is an irreducible polynomial in $\mathbb{C}[y_{i,j} : 1 \le i, j \le k]$ (and hence in $\mathbb{C}[y_{i,j}, z_m : 1 \le i, j \le k,\ 1 \le m \le l]$) the result follows.

(v) We first show that the trace function of the representation of $\mathcal{A}$ on V can be recovered from $D_{(\mathcal{A},A)}(V)$ (to show this we do not need the fact that V is irreducible).

Set $D_{(\mathcal{A},A)}(V) = f(x_1, \ldots, x_n)$ and $\rho(a_i) = M_i,\ i = 1, \ldots, n$. There are scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ such that $\alpha_1 a_1 + \cdots + \alpha_n a_n = 1$ and thus $\alpha_1 M_1 + \cdots + \alpha_n M_n = I$.

Fix $i \in \{1, \ldots, n\}$. Now, trace of M_i is the coefficient of x_i in

$$\det(I + x_i M_i) = f(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + x_i, \alpha_{i+1}, \ldots, \alpha_n). \tag{2.4}$$

Thus the traces of $\rho(a_i)$, for $i = 1, \ldots, n$, and hence the traces of $\rho(a)$, for all $a \in \mathcal{A}$ can be recovered from $D_{(\mathcal{A},A)}(V)$. It follows from (2.4) that the trace functions of V and W are proportional if $D_{(\mathcal{A},A)}(V)$ and $D_{(\mathcal{A},A)}(W)$ are proportional. This contradicts the fact that trace functions of nonisomorphic irreducible $\mathcal{A}$-modules are linearly independent (Corollary 4.1.18 in **[7]**).

The result follows.

(vi) This follows from parts (ii) to (v). □

Our next two lemmas give more detail in the abelian situation.

LEMMA 2.4. *Let $\mathcal{A}$ be a finite dimensional complex algebra with distinguished basis A. Let V be finite dimensional complex vector space that is a semisimple $\mathcal{A}$-module, the module structure given by the homomorphism $\rho : \mathcal{A} \to End(V)$. Then $D_{(\mathcal{A},A)}(V)$ factors into linear terms if and only if $\rho(\mathcal{A})$ is abelian.*

Proof. The if part follows from Theorem 2.3 (i). For the only if part, since the degree of the algebra determinant is the dimension of the underlying vector space on which the algebra acts it follows from Theorem 2.3 (iv) that V is a direct sum of irreducible $\mathcal{A}$-modules of dimension 1. Thus there is a common eigenbasis for all operators in $\rho(a), a \in \mathcal{A}$, and hence $\rho(\mathcal{A})$ is abelian. □

LEMMA 2.5. *Let a finite group G act on the finite set S. Preserve the notation of Example 2.1.*

(i) Assume that the generic commuting algebra matrix N' is symmetric, i.e., each of $M_1, \ldots, M_p$ is symmetric. Then $\mathcal{A}$ is abelian and the commuting algebra determinant of (G, S) factors into linear terms.

(ii) $\mathcal{A}$ is abelian iff the number of distinct irreducibles in the decomposition of $V(S)$ as a $\mathbb{C}G$-module is p.

(iii) Assume that $\mathcal{A}$ is abelian. The $M_1, \ldots, M_p$ are all symmetric if and only if the eigenvalues of M_i are real, for $i = 1, \ldots, p$.

Proof. (i) Follows from Theorem 2.3 (i) and the fact that a complex algebra of square matrices that has a basis of symmetric matrices is abelian.

(ii) Let there be t distinct irreducibles in the decomposition of $V(S)$ as a $\mathbb{C}G$-module with multiplicities $m_1, \ldots, m_t$. Now $\mathcal{A}$ is a direct sum of matrix algebras of sizes $m_i,\ i = 1, \ldots, t$. Thus

$$p = \dim(\mathcal{A}) = \sum_{i=1}^{t} m_i^2.$$

Since $\mathcal{A}$ is abelian iff $m_i = 1$ for all i, the result follows.

(iii) The only if part is clear. We now prove the if part. Introduce an inner product structure on the complex vector space $V = V(S)$ by declaring S to be an orthonormal basis (we think of V as column vectors with components indexed by S). Note that this inner product is G-invariant. Since $\mathcal{A}$ is abelian it follows that V is a multiplicity free $\mathbb{C}G$-module. Thus the decomposition

$$V = V_1 \oplus \cdots \oplus V_p$$

into irreducible $\mathbb{C}G$-submodules is canonical. Moreover, this decomposition is orthogonal. The $\mathbb{C}G$-submodules $V_1, \ldots, V_p$ are the common eigenspaces of $M_1, \ldots, M_p$. Choose orthonormal bases B_i for each V_i, $i = 1, \ldots, p$. Then $B = B_1 \cup \cdots \cup B_t$ is an orthonormal basis of V. Form a unitary matrix U with columns B. Then $M_i = UD_iU^*, i = 1, \ldots, p$, with D_i real diagonal for all i. It follows that M_i is symmetric. □

3. Combinatorial Examples. In this section we give four combinatorial examples of commuting algebra determinants and, quoting results from classical and recent literature, we indicate their factorizations into irreducible complex polynomials.

Our first two examples follow from classical combinatorial results. The symmetric group S_n acts on $B(n)_i, 0 \le i \le n/2$. It is easily seen that $(A, B), (A', B') \in B(n)_i \times B(n)_i$ are in the same S_n-orbit iff they have the same type. It follows that $Y(n, i)$ is the generic commuting algebra matrix for the S_n-action on $B(n)_i$. There is a q-analog of $Y(n, i)$ which we now define.

Fix a prime power q and let $\mathbb{F}_q$ denote the finite field with q elements. Let $B_q(n)$ denote the set of all subspaces of $\mathbb{F}_q^n$, the n-dimensional vector space (of all column vectors with n components) over $\mathbb{F}_q$. For $0 \le i \le n$, let $B_q(n)_i$ denote the set of all i-dimensional subspaces of $\mathbb{F}_q^n$. The number of k-dimensional subspaces in $B_q(n)$ is the q-binomial coefficient $\left[{n \atop k}\right]_q$ (we take $\left[{n \atop k}\right]_q$ to be 0 if $n < 0$ or $k < 0$ (or both)) and the total number of subspaces is the *Galois number*

$$G_q(n) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

For $0 \le i \le n/2$, let $Y_q(n, i)$ denote the $B_q(n)_i \times B_q(n)_i$ matrix whose entry in row U, column V, where $U, V \in B_q(n)_i$, is given by $y_{\dim(U \cap V)}$. We see that $Y_q(n, i)$ is the generic commuting algebra matrix of the $GL(n, \mathbb{F}_q)$-action on $B_q(n)_i$.

We now discuss the factorization of $\det(Y(n, i))$ and $\det(Y_q(n, i))$. Let $\mathcal{A}$ and $\mathcal{B}$ denote, respectively, the commuting algebras of the actions of S_n on $B(n)_i$ and $GL(n, \mathbb{F}_q)$ on $B_q(n)_i$. Since there are $i + 1$ distinct types and $Y(n, i), Y_q(n, i)$ are symmetric, their determinants factor into linear terms and the standard bases of $\mathcal{A}, \mathcal{B}$ are both commuting families of $i + 1$ real symmetric matrices. It follows that, as an $\mathcal{A}$-module (respectively, $\mathcal{B}$-module), $V(B(n)_i)$ (respectively, $V(B_q(n)_i)$) is a direct sum of $i + 1$ common eigenspaces. The dimensions of these eigenspaces of $V(B(n)_i)$ (respectively, $V(B_q(n)_i)$) are $\binom{n}{k} - \binom{n}{k-1}$ (respectively, $\left[{n \atop k}\right]_q - \left[{n \atop k-1}\right]_q$), $k = 0, \ldots, i$. The eigenvalues of the standard basis elements of $\mathcal{A}$ and $\mathcal{B}$ on these eigenspaces are also known. These classical results are due to Delsarte **[3, 4]** and they determine the factorizations of $\det(Y(n, i))$ and $\det(Y_q(n, i))$.

For $i, k, t \in \{0, 1, \ldots, n\}$ define the following integers

$$\gamma_{i,k}^{n,t} = \sum_{u=0}^{n} (-1)^{u-t} \binom{u}{t} \binom{i-k}{u-k} \binom{n-k-u}{i-u},$$

$$\gamma_{i,k}^{n,t}(q) = \sum_{u=0}^{n} (-1)^{u-t}\, q^{\binom{u-t}{2}+k(i-u)} \begin{bmatrix} u \\ t \end{bmatrix}_q \begin{bmatrix} i-k \\ u-k \end{bmatrix}_q \begin{bmatrix} n-k-u \\ i-u \end{bmatrix}_q.$$

To see that $\gamma_{i,k}^{n,t}(q)$ is an integer note that if $i < u$ then $\begin{bmatrix} n-k-u \\ i-u \end{bmatrix}_q = 0$.

We now state the factorizations that follow from Delsarte's results.

THEOREM 3.1. *We have the following factorizations into powers of distinct irreducible polynomials in* $\mathbb{C}[y_0, \ldots, y_i]$.

$$\det(Y(n,i)) = \prod_{k=0}^{i} \left[\sum_{t=0}^{i} \gamma_{i,k}^{n,t}\; y_t \right]^{\binom{n}{k}-\binom{n}{k-1}}, \tag{3.1}$$

$$\det(Y_q(n,i)) = \prod_{k=0}^{i} \left[\sum_{t=0}^{i} \gamma_{i,k}^{n,t}(q)\; y_t \right]^{\begin{bmatrix} n \\ k \end{bmatrix}_q - \begin{bmatrix} n \\ k-1 \end{bmatrix}_q}. \tag{3.2}$$

Before discussing the next two examples we make a useful observation.

Let G be a finite group acting on the finite set S with generic group action matrix N, generic commuting algebra matrix N', and $\mathcal{A}$ the commuting algebra. Suppose that there are t distinct irreducibles occuring in the $\mathbb{C}G$-module $V(S)$ with dimensions $d_1, \ldots, d_t$ and respective multiplicities $m_1, \ldots, m_t$. Write the isotypical decomposition of $V(S)$ (as a $\mathbb{C}G$-module) as

$$V(S) = V_1 \oplus \cdots \oplus V_t, \tag{3.3}$$

with $\dim(V_i) = m_i d_i$ for all i.

It follows from the double centralizer theorem (see **[7, 10]**) that (3.3) is also the isotypical decomposition of $V(S)$ as an $\mathcal{A}$-module. However, the dimensions of the t distinct $\mathcal{A}$ irreducibles are now $m_1, \ldots, m_t$ and the corresponding multiplicities $d_1, \ldots, d_t$. Thus, there is a bijection between the irreducible factors of $\det(N)$ and $\det(N')$ such that the pair (degree, multiplicity) of an irreducible factor of $\det(N)$ is equal to the pair (multiplicity, degree) of the corresponding irreducible factor of $\det(N')$. In our next two examples the number of distinct irreducibles occuring in $V(S)$ as a $\mathbb{C}G$-module, their dimensions and multiplicity are known and therefore, these numbers are also known for $V(S)$ as an $\mathcal{A}$-module. We will only quote the numbers for the commuting algebra $\mathcal{A}$.

We now come to the two main examples of this paper, the nonabelian analogs of $Y(n,i)$ and $Y_q(n,i)$. Consider the S_n-action on $B(n)$. It is easily seen that $(A,B),(A',B') \in B(n)\times B(n)$ are in the same S_n orbit iff they have the same type. It follows that $X(n)$ is the generic commuting algebra matrix of the S_n-action on $B(n)$ and $\det(X(n))$ is a homogeneous polynomial of degree 2^n in $\mathbb{C}[\mathbf{x}(\mathbf{n})]$. We now define a q-analog of $X(n)$.

Let $X_q(n)$ denote the $B_q(n)\times B_q(n)$ matrix whose entry in row U, column V, where $U,V\in B_q(n)$, is given by $x_{\dim(U),\dim(V),\dim(U\cap V)}$. We see that $X_q(n)$ is the generic commuting algebra matrix of the $GL(n,\mathbb{F}_q)$-action on $B_q(n)$ and that $\det(X_q(n))$ is a homogeneous polynomial in $\mathbb{C}[\mathbf{x}(\mathbf{n})]$ of degree $G_q(n)$.

We now discuss the factorizations of $\det(X(n))$ and $\det(X_q(n))$. Let $\mathcal{A}$ and $\mathcal{B}$ denote, respectively, the commuting algebras for the actions of S_n on $V(B(n))$ and $GL(n,\mathbb{F}_q)$ on $V(B_q(n))$. Let us write down the standard bases of $\mathcal{A}$ and $\mathcal{B}$.

For $0\le i,j,t\le n$ let $M_{i,j,t}$ be the $B(n)\times B(n)$ matrix given by

$$M_{i,j,t}(X,Y)=\begin{cases}1 & \text{if } |X|=i,\ |Y|=j,\ |X\cap Y|=t,\\ 0 & \text{otherwise.}\end{cases}$$

For $0\le i,j,t\le n$ let $M_{i,j,t}(q)$ be the $B_q(n)\times B_q(n)$ matrix given by

$$M_{i,j,t}(q)(X,Y)=\begin{cases}1 & \text{if } \dim(X)=i,\ \dim(Y)=j,\ \dim(X\cap Y)=t,\\ 0 & \text{otherwise.}\end{cases}$$

It follows that $A=\{M_{i,j,t}\mid (i,j,t)\in\mathcal{I}(n)\}$ and $B=\{M_{i,j,t}(q)\mid (i,j,t)\in\mathcal{I}(n)\}$ are the standard bases of $\mathcal{A}$ and $\mathcal{B}$ respectively.

The following facts are well known (see **[3, 4]**):

(i) As an $\mathcal{A}$-module, $V(B(n))$ has $1+\lfloor n/2\rfloor$ distinct irreducibles occuring in it and their dimensions and multiplicity are known and are as follows. We can fix nonisomorphic irreducible $\mathcal{A}$-submodules $W_0,W_1,\ldots,W_{\lfloor n/2\rfloor}$ of $V(B(n))$ so that

$$\text{dimension of } W_k = n-2k+1,\quad \text{multiplicity of } W_k=\binom{n}{k}-\binom{n}{k-1},$$

for $k=0,1,\ldots,\lfloor n/2\rfloor$.

(ii) As a $\mathcal{B}$-module, $V(B_q(n))$ has $1+\lfloor n/2\rfloor$ distinct irreducibles occuring in it and their dimensions and multiplicity are known and are as follows. We can fix nonisomorphic irreducible $\mathcal{B}$-submodules $U_0,U_1,\ldots,U_{\lfloor n/2\rfloor}$ of $V(B_q(n))$ so that

$$\text{dimension of } U_k = n-2k+1,\quad \text{multiplicity of } U_k=\begin{bmatrix}n\\k\end{bmatrix}_q-\begin{bmatrix}n\\k-1\end{bmatrix}_q,$$

for $k = 0, 1, \ldots, \lfloor n/2 \rfloor$.

We now need to calculate $D_{(\mathcal{A},A)}(W_k)$ and $D_{(\mathcal{B},B)}(U_k)$, for $k = 0, 1, \ldots, \lfloor n/2 \rfloor$. This in turn requires that we find suitable bases of W_k, U_k with respect to which we can explicitly write down the matrices representing the action of $M_{i,j,t}$ and $M_{i,j,t}(q)$, for $(i,j,t) \in \mathcal{I}(n)$. For W_k, this was done in Dunkl **[6]** and Schrijver **[11]** and for U_k this was done in two recent papers of Bachoc, Passuello, and Vallentin **[1]** and the second author **[12]**. For the U_k case, the approach in **[1]** is motivated by the work of Dunkl **[5]** on q-Hahn polynomials (also see Marco and Parcet **[9]** for a closely related paper) while the approach in **[12]** is purely combinatorial and leads to a formulation that directly reduces to the formulation in **[11]** for the W_k case in the $q \to 1$ limit. We shall use the formulations in **[11, 12]**.

For $i, j, k, t \in \{0, 1, \ldots, n\}$ define the following integers

$$\gamma_{i,j,k}^{n,t} = \sum_{u=0}^{n} (-1)^{u-t} \binom{u}{t} \binom{i-k}{u-k} \binom{n-k-u}{j-u},$$

$$\gamma_{i,j,k}^{n,t}(q) = \sum_{u=0}^{n} (-1)^{u-t}\, q^{\binom{u-t}{2}+k(j-u)} \begin{bmatrix} u \\ t \end{bmatrix}_q \begin{bmatrix} i-k \\ u-k \end{bmatrix}_q \begin{bmatrix} n-k-u \\ j-u \end{bmatrix}_q.$$

Note that $\gamma_{i,k}^{n,t} = \gamma_{i,i,k}^{n,t}$ and $\gamma_{i,k}^{n,t}(q) = \gamma_{i,i,k}^{n,t}(q)$.

For $0 \le k \le \lfloor n/2 \rfloor$ and $k \le i, j \le n-k$, define $E_{i,j,k}$ to be the $n-2k+1 \times n-2k+1$ matrix, with rows and columns indexed by $\{k, k+1, \ldots, n-k\}$, and with entry in row i and column j equal to 1 and all other entries 0.

The following results are proved in **[11, 12]** (the $\mathcal{A}$-module case in **[11]** and the $\mathcal{B}$-module case in **[12]**).

Theorem 3.2. *Let $0 \le k \le \lfloor n/2 \rfloor$ and $(i,j,t) \in \mathcal{I}(n)$. Consider the irreducible $\mathcal{A}$-submodule W_k of $V(B(n))$ and the irreducible $\mathcal{B}$-submodule U_k of $V(B_q(n))$.*

(i) If $i, j \notin \{k, \ldots, n-k\}$ then the action of $M_{i,j,t}, M_{i,j,t}(q)$ on W_k, U_k (respectively) is 0.

(ii) Suppose $k \le i, j \le n-k$. There is a basis of W_k such that the matrix $M_{i,j,t,k}$ of the action of $M_{i,j,t}$ on W_k with respect to this basis is given as follows. It will be convenient to index the rows and columns of $M_{i,j,t,k}$ by the set $\{k, \ldots, n-k\}$. We have

$$M_{i,j,t,k} = \gamma_{i,j,k}^{n,t}\, E_{i,j,k}.$$

(iii) Suppose $k \le i, j \le n-k$. There is a basis of U_k such that the matrix $M_{i,j,t,k}(q)$ of the action of $M_{i,j,t}(q)$ on U_k with respect to this basis is given as follows. It will

be convenient to index the rows and columns of $M_{i,j,t,k}(q)$ *by the set* $\{k, \ldots, n-k\}$. *We have*

$$M_{i,j,t,k}(q) = \ \gamma_{i,j,k}^{n,t}(q)\ E_{i,j,k}.$$

Remark The bases for W_k, U_k given in **[11, 12]** are not quite the same as those given in parts (ii) and (iii) of Theorem 3.2. However, these bases differ by a simple scaling. Let us make this precise. We denote by $M'_{i,j,t,k}$ (respectively, $M_{i,j,t,k}(q)'$) the matrix of the action of $M_{i,j,t}$ (respectively, $M_{i,j,t}(q)$) on W_k (respectively, U_k) with respect to the basis of W_k (respectively, U_k) given in **[11]** (respectively, **[12]**).

Define a $n-2k+1 \times n-2k+1$ diagonal matrix Z, with rows and columns indexed by $\{k, k+1, \ldots, n-k\}$, and with entry in row j and column j, $k \le j \le n-k$, given by $\binom{n-2k}{j-k}^{\frac{1}{2}}$. Define a $n-2k+1 \times n-2k+1$ diagonal matrix $Z(q)$, with rows and columns indexed by $\{k, k+1, \ldots, n-k\}$, and with entry in row j and column j, $k \le j \le n-k$, given by $q^{\frac{k(j-k)}{2}} \begin{bmatrix} n-2k \\ j-k \end{bmatrix}_q^{\frac{1}{2}}$. Then it may be easily checked that $M_{i,j,k,t} = Z^{-1} M'_{i,j,k,t} Z$ and $M_{i,j,k,t}(q) = Z(q)^{-1} M_{i,j,k,t}(q)' Z(q)$.

The main reason for scaling the bases from **[11, 12]** is so that our matrices $M_{i,j,t,k}$ and $M_{i,j,k,t}(q)$ have integer entries.

Let $0 \le k \le n/2$. Define a $n-2k+1 \times n-2k+1$ matrix $M(k,n)$, with rows and columns indexed by $\{k, k+1, \ldots, n-k\}$, and with entry in row i and column j given by the following linear polynomial with integer coefficients

$$\sum_{t=0}^{n} \gamma_{i,j,k}^{n,t}\ x_{i,j,t}, \quad k \le i, j \le n-k,$$

where we take $x_{i,j,t} = 0$ whenever $(i,j,t) \notin \mathcal{I}(n)$. Define the homogeneous polynomial $d(k, \mathbf{x}(\mathbf{n})) \in \mathbb{C}[\mathbf{x}(\mathbf{n})]$, with integral coefficients, of degree $n-2k+1$ by $d(k, \mathbf{x}(\mathbf{n})) = \det(M(k,n))$. Being of different degrees $d(0, \mathbf{x}(\mathbf{n})), \ldots, d(\lfloor \frac{n}{2} \rfloor, \mathbf{x}(\mathbf{n}))$ are pairwise nonproportional.

Let $0 \le k \le n/2$. Define a $n-2k+1 \times n-2k+1$ matrix $M_q(k,n)$, with rows and columns indexed by $\{k, k+1, \ldots, n-k\}$, and with entry in row i and column j given by the following linear polynomial with integer coefficients

$$\sum_{t=0}^{n} \gamma_{i,j,k}^{n,t}(q)\ x_{i,j,t}, \quad k \le i, j \le n-k,$$

where we take $x_{i,j,t} = 0$ whenever $(i,j,t) \notin \mathcal{I}(n)$. Define the homogeneous polynomial $d_q(k, \mathbf{x}(\mathbf{n})) \in \mathbb{C}[\mathbf{x}(\mathbf{n})]$, with integral coefficients, of degree $n-2k+1$ by $d_q(k, \mathbf{x}(\mathbf{n})) = \det(M_q(k,n))$. Being of different degrees $d_q(0, \mathbf{x}(\mathbf{n})), \ldots, d_q(\lfloor \frac{n}{2} \rfloor, \mathbf{x}(\mathbf{n}))$ are pairwise nonproportional.

The following result now follows from Theorems 2.3 and 3.2.

THEOREM 3.3. *(i) The polynomials* $d(0,\mathbf{x}(\mathbf{n})), d(1,\mathbf{x}(\mathbf{n})), \ldots, d(\lfloor \frac{n}{2} \rfloor, \mathbf{x}(\mathbf{n}))$ *are irreducible in* $\mathbb{C}[\mathbf{x}(\mathbf{n})]$*, pairwise nonproportional and we have*

$$\det(X(n)) = \prod_{k=0}^{\lfloor \frac{n}{2} \rfloor} d(k,\mathbf{x}(\mathbf{n}))^{\binom{n}{k}-\binom{n}{k-1}}.$$

(ii) The polynomials $d_q(0,\mathbf{x}(\mathbf{n})), d_q(1,\mathbf{x}(\mathbf{n})), \ldots, d_q(\lfloor \frac{n}{2} \rfloor, \mathbf{x}(\mathbf{n}))$ *are irreducible in* $\mathbb{C}[\mathbf{x}(\mathbf{n})]$*, pairwise nonproportional and we have*

$$\det(X_q(n)) = \prod_{k=0}^{\lfloor \frac{n}{2} \rfloor} d_q(k,\mathbf{x}(\mathbf{n}))^{\left[{n \atop k}\right]_q - \left[{n \atop k-1}\right]_q}.$$

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