MATH 327: FOURTH PROBLEM SET

Due Monday, April 25

1. Let $R = k[x, y, s, t]/(xs - yt)$ and $S = R/(x, y) \cong k[s, t]$. Let $P = (s, t) \subset R$ and let Q be its image in S. Show that $ht(P) = 1$ but $ht(Q) = 2$.

2. let $R = k[x, y, z]/(xz - z)$ and consider the set $\{x, xy - y\}$. Notice that the ideal $(x, xy - y)$ in R is (x, y). Show that $\{x, xy - y\}$ is a regular sequence but $\{xy - y, x\}$ is not a regular sequence: permutations of regular sequences need not be regular.

3. Let $\{a, b\}$ be non-zero elements in an integral domain R.

(a) Show that the ideal $(ax - b)$ of $R[x]$ is prime if and only if $H_1(K(a, b)) = 0$ (which holds if $\{a, b\}$ is a regular sequence).
(b) When $(ax-b)$ is prime, show that $R[x]/(ax-b)$ is isomorphic to the subring $R(b/a)$ of the quotient field of R.

Recall that an associated prime of an R module M is a prime ideal that is the annhilator of a nonzero element of M. An associated prime P of an ideal I is defined (confusingly!) to be an associated prime of the R-module R/I, so that there is an $x \in R - I$ such that $P = \{r | rx \in I\}$. Notes on these notions are posted.

All given rings R are local (and Noetherian) in the rest of the problems.

4. Let $\{a_1, \ldots, a_n\}$ be elements of the maximal ideal of R.

(a) Show that $\{a_1, \ldots, a_n\}$ is a regular sequence if and only if a_i is not in any associated prime ideal of $(a_1, \ldots, a_{i-1})$ for $1 \leq i \leq n$.
(b) Show that $\{a_1, \ldots, a_n\}$ is part of a system of parameters of R such that each $(a_1, \ldots, a_i)$ is of height i if and only if a_i is not in any minimal prime ideal of $R/(a_1, \ldots, a_{i-1})$ for $1 \leq i \leq n$.

5. Let M be a finitely generated R-module, P a prime ideal.

(a) Show that $\text{depth}_P(M) \leq \text{depth}_{PR_P}(M_P)$.
(b) Show that the inequality can be strict. Hint: Consider $R = k[x, y, z]$, $M = (xy, y^2, yz)$, and $P = (x, y)$. Note that this is not an esoteric example.

6. Let $\dim(R) = 0$. Show the following.

(a) R is a Cohen Macaulay (CM) ring.
(b) R is regular if and only if R is a field.

7 and 8. Let $\dim(R) = 1$.

(a) If R has no nilpotent elements, show that R is CM.
(b) Construct an R such that $\dim(R) = 1$ but R is not CM.
(c) If R is regular, then R is a DVR.
(d) For a field k, show that the subring of the DVR $k[[x]]$ generated by x^2 and x^3 is CM of dimension 1 but is not regular.

9. Let $R = S/I$, where S is a regular local ring and I is generated by a regular sequence in S. Show that any localization of R is also a quotient of a regular local ring by an ideal generated by a regular sequence.

A local ring R is called a "complete intersection" if its completion at its maximal ideal is the quotient of a regular local ring by an ideal generated by a regular sequence. The rings of problem 9 are examples.

10. Let $\mathbf{a} = (a_1, \cdots, a_n)$ be any sequence of elements in R, let $S = R[x_1, \cdots, x_n]$ and let $f\colon S \longrightarrow R$ be the ring homomorphism that sends x_i to a_i. Let M be an R-module and regard M as an S-module by pullback along f, $sm = f(s)m$. Regard R as the quotient S-module $S/(x_1, \cdots, x_n)$. Prove that

$$H_*(K(\mathbf{a}) \otimes_R M) \cong \operatorname{Tor}^S_*(R, M).$$