MATH 327: FIFTH PROBLEM SET

Due Monday, May 2

We have shown that in a local ring, permutations of regular sequences are regular. Here is a conceptual statement that encodes this invariance differently.

1. Prove that if R is local, M is finitely generated, and $I = (x_1, \cdots, x_n)$ is proper and contains a regular sequence for M of length n, then $\{x_1, \cdots, x_n\}$ is itself a regular sequence for M.

2. Prove that if $\{x_1, \cdots, x_n\}$ is a regular sequence in R, then so is $\{x_1^{r_1}, \cdots, x_n^{r_n}\}$ for any n positive integers r_i. [17.5, page 442, Eisenbud]

The following two problems are relatively hard, and optional. If you do 3, don't just copy out of Eisenbud. Try to understand it better.

3*. Show that if an ideal I in R (not assumed to be local) can be generated by a regular sequence, then it can be generated by some regular sequence any permutation of which is again regular. [This is 17.6, p. 442, in Eisenbud and a sketch proof is given on p.740.]

There should be easy way of doing the following problem, but I don't myself know one. I do know that the claim is true.

4*. Show that an integrally closed local domain R of dimension 2 must be CM.

5. Let R be the localization of $k[x^3, x^2y, xy^2, y^3] \subset k[x, y]$ at the maximal ideal $\mathfrak{m} = (x^3, x^2y, xy^2, y^3)$. Show that R is Cohen-Macaulay.

6. Let R be the localization of $k[x^4, x^3y, xy^3, y^4] \subset k[x, y]$ at the maximal ideal $\mathfrak{m} = (x^4, x^3y, xy^3, y^4)$. Show that R is not Cohen-Macaulay.

(Problems 5 and 6 are taken from Eisenbud, 18.7 and 18.8, page 469, where some discussion of them may be found. Eisenbud calls height "codimension", and he notes that $ht(\mathfrak{m}) = 2$ in both cases.)

An ideal $I \subset R$ is "perfect" if $\text{depth}_I(R) = pd(R/I)$.

7. Show that if R is CM and I is perfect, then $S = R/I$ is CM.

8. Show that an ideal generated by a regular sequence is perfect.

9. Give an example of a perfect ideal not generated by a regular sequence.

By Problem 9, Problem 7 is a generalization of the statement that a quotient of a CM local ring by an ideal generated by a regular sequence is CM.