Real Forms

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1 Introduction

This is a brief survey of the theory of real forms of Lie groups.

We will always start with a connected, complex reductive group $G = G(\mathbb{C})$. We are interested in real groups $G(\mathbb{R})$ whose complexification is $G(\mathbb{C})$.

Example 1.1 Suppose $G(\mathbb{C}) = SL(2,\mathbb{C})$. Then we can take $G(\mathbb{R}) = SL(2,\mathbb{R})$ (the split form). This means: $G(\mathbb{R})$ is the fixed points of the anti-holomorphic involution $\sigma_s(g) = \overline{g}$. On the Lie algebra level, $\mathfrak{sl}(2,\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}(2,\mathbb{C})$.

We can also take $G(\mathbb{R}) = SU(2)$ (the compact form). This is the fixed points of the antiholomorphic involution $\sigma_c(g) = {}^t\overline{g}^{-1}$.

Example 1.2 Suppose $G = GL(n,\mathbb{C})$. The split form is $GL(n,\mathbb{R})$, and the compact form is $U(n)$. We also have the groups $U(p,q)$, preserving a symmetric Hermitian form of signature (p,q): $\{g \in GL(n,\mathbb{C}) \mid gJ_{p,q}{}^t\overline{g} = J_{p,q}$ where $J_{p,q} = \mathrm{diag}(I_p, -I_q)$.

Are there any others? Only 1 (or 0): if n is even, there is the group $GL(n/2,\mathbb{H})$, also known as $U^*(n)$, where $\mathbb{H}$ is the quaternions.

Theorem 1.3 (Cartan) *Fix a compact real form σ_c of $G(\mathbb{C})$. That is σ_c is anti-holomorphic, and $G(\mathbb{R})^{\sigma_c}$ is compact. This exists, and is unique up to conjugation by $G(\mathbb{C})$.*

Now suppose θ is a holomorphic involution of G. Then some conjugate of θ commutes with σ_c. After replacing θ by such a conjugate, define $\sigma = \theta \circ \sigma_c$. Then σ is an antiholomorphic involution of G. This defines a bijection:

$$\{\theta \mid holomorphic, \theta^2 = 1\}/G \leftrightarrow \{\sigma \mid anti-holomorphic, \sigma^2 = 1\}/G$$

Definition 1.4 *A real form of G is a G-conjugacy class of holomorphic involutions.*

By the Theorem this is equivalent to the usual definition of real form, which is a G-conjugacy class of anti-holomorphic involutions. We say the holomorphic involution θ is the *Cartan involution* of the real form.

Example 1.5 The Cartan involution is the identity if and only if $G(\mathbb{R})$ is compact. It is the Chevalley involution (i.e. inverse on a Cartan subgroup) if and only if $G(\mathbb{R})$ is split.

The Cartan involution of $U(p,q)$ is conjugation by $\mathrm{diag}(I_p, -I_q)$.

2 Compact Inner Class

There is an exact sequence

$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$$

where $\mathrm{Int}(G) = G/Z(G) = G_{ad}$ is the adjoint group, or group of inner automorphism, $\mathrm{Aut}(G)$ is all (holomorphic) automorphisms, and $\mathrm{Out}(G)$ is defined to be $\mathrm{Aut}(G)/\mathrm{Int}(G)$. If $\tau \in \mathrm{Aut}(G)$ is an involution then so is its image in $\mathrm{Out}(G)$.

If $g \in G$ let $\mathrm{int}(g)\mathrm{Int}(G)$ be the corresponding inner automorphism: $\mathrm{int}(g)(y) = gyg^{-1}$.

Definition 2.1 *We say two involutions θ, θ' of G are in the same inner class, or inner to each other, if they have the same image in $Out(G)$. In other words*

$$\theta' = \theta \circ int(g)$$

for some $g \in G$.

Exercise 2.2 *Show that the involution $\theta(g) = {}^tg^{-1}$ of $SL(n,\mathbb{C})$ is inner if and only if $n = 2$.*

We'll focus on the compact inner class of real forms.

Definition 2.3 *A strong real form of G, in the compact inner class, is a G-conjugacy class of elements g such that $g^2 \in Z(G)$.*

If $x \in G$ is a strong real form, $\theta_x = \mathrm{int}(x)$ is an involution of G.

Lemma 2.4 *The map $x \to \theta_x$ is a surjection from the set of strong real forms to the set of real forms of G (in the compact inner class).*

This is immediate.

For simplicity suppose G is semisimple. Fix a Cartan subgroup H, let $\mathfrak{h} = \mathrm{lie}(H) \simeq \mathbb{C}^n$, and consider the exponentional map $\exp : \mathfrak{h} \to H$.

Definition 2.5

$$X_*^\vee = \frac{1}{2\pi i}\ker(\exp)$$
$$P^\vee = \frac{1}{2\pi i}\exp^{-1}(Z(G))$$

This is equivalent to the algebraic groups definition of the lattices. In particular

$$\exp : X_* \backslash P^\vee \simeq Z(G)$$

and also

$$R^\vee \subset X_* \subset P^\vee$$

where $X_*(H)$ is the co-character lattice

Exercise 2.6 *Fix a Cartan subgroup H of G. Show that the set of strong real forms of G, in the compact inner class, is in bijection with*

$$(\frac{1}{2}P^\vee \backslash X_*)/W.$$

Exercise 2.7 *Let $G = Sp(2n, \mathbb{C})$. Show that every strong real form of G is conjugate to either $diag(I_p, -I_q, I_p, -I_q)$ $(p + q = n)$ or $diag(iI, -iI)$.*

If $x = diag(I_p, -I_q, I_p, -I_q)$ $(p + q = n)$ show that $G^{\theta_x} \simeq Sp(2p, \mathbb{C}) \times Sp(2q, \mathbb{C})$. On the other hand if $g = diag(iI, -iI)$ show that $G^{\theta_x} \simeq GL(n, \mathbb{C})$.

If θ is the Cartan involution of a real form, then $K(\mathbb{C}) = G(\mathbb{C})^\theta$ is a complex group. Also θ commutes with the corresponding antiholomorphic σ, so θ acts on $G(\mathbb{R})$, and $G(\mathbb{R})^\theta$ is a *maximal compact subgroup of* $G(\mathbb{R})$. In other words we have the following picture:

$$
\begin{array}{ccccc}
 & & G(\mathbb{C}) & & \\
 & \swarrow^{\theta} & & \searrow^{\sigma} & \\
K(\mathbb{C}) = G(\mathbb{C})^{\theta} & & & & G(\mathbb{R}) = G(\mathbb{C})^{\sigma} \\
 & \searrow_{\sigma} & & \swarrow_{\theta} & \\
 & & K(\mathbb{R}) = G(\mathbb{R})^{\theta} = K(\mathbb{C})^{\sigma} & &
\end{array}
$$

Here is the example of $U(p,q)$:

$$
\begin{array}{ccccc}
 & & GL(n,\mathbb{C}) & & \\
 & \swarrow^{\theta} & & \searrow^{\sigma} & \\
GL(p,\mathbb{C}) \times GL(q,\mathbb{C}) & & & & U(p,q) \\
 & \searrow_{\sigma} & & \swarrow_{\theta} & \\
 & & U(p) \times U(q) & &
\end{array}
$$

Exercise 2.8 *The groups in the previous exercise are $Sp(p,q)$ and $Sp(2n,\mathbb{R})$. Draw the corresponding pictures for these groups.*

Exercise 2.9 *Show that the real forms of $SO(2n+1,\mathbb{C})$ are parametrized by $p+q=2n+1$. (These are the groups $SO(p,q)$. Note that G is adjoint, so strong real forms and real forms are the same.)*

Exercise 2.10 *What are the strong real forms of $SO(2n,\mathbb{C})$?*

In general the Cartan involutions, and real forms, are given by their *Kac diagram*. Here are tables of all real forms (including the noncompact inner class) of all simple groups.

Table 6

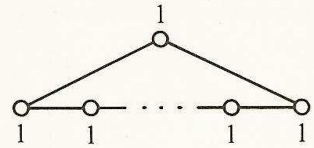
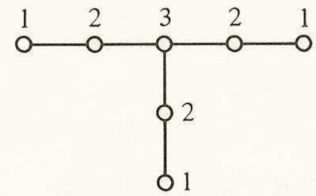
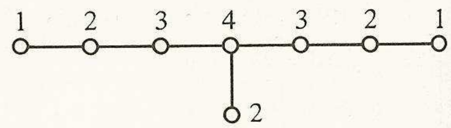
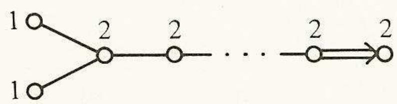
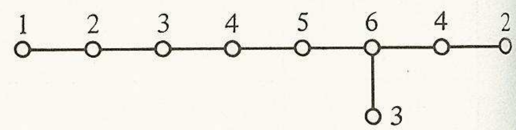
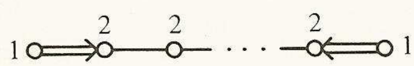
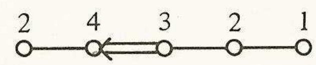
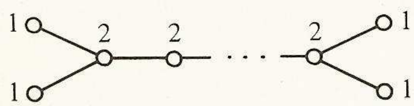
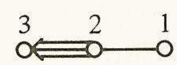
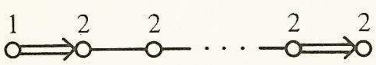
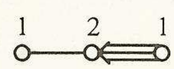
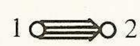
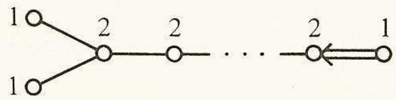
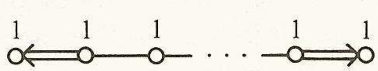
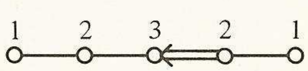
Type	Affine diagram	Type	Affine diagram
$A_l^{(1)}$ $(l \geqslant 2)$		$E_6^{(1)}$	
$A_1^{(1)}$		$E_7^{(1)}$	
$B_l^{(1)}$ $(l \geqslant 3)$		$E_8^{(1)}$	
$C_l^{(1)}$ $(l \geqslant 2)$		$F_4^{(1)}$	
$D_l^{(1)}$ $(l \geqslant 4)$		$G_2^{(1)}$	
$A_{2l}^{(2)}$ $(l \geqslant 2)$		$D_4^{(3)}$	
$A_2^{(2)}$			
$A_{2l-1}^{(2)}$ $(l \geqslant 3)$			
$D_{l+1}^{(2)}$ $(l \geqslant 2)$			
$E_6^{(2)}$			

Table 7. Involutive Automorphisms of Complex Simple Lie Algebras. In the table there are listed all the Kac diagram of all order 2 automorphisms θ of complex simple Lie algebras $\mathfrak{g}$ (up to conjugacy in the group Aut $\mathfrak{g}$). Since all the nonzero numerical labels of Kac diagrams of automorphisms of order 2 equal 1/2, it suffices to distinguish the vertices of the corresponding affine Dynkin diagram endowed with nonzero numerical labels. Therefore the numerical labels are omitted and the vertices with nonzero labels are black, the other being white. The vertices of an affine Dynkin diagram $L_n^{(k)}$ are numbered so that if $\Psi = \{\alpha_0, \alpha_1, \ldots, \alpha_l\}$ is the corresponding numbered admissible system of vectors then $\Pi^\tau = \{(\alpha_0, 1/k), (\alpha_1, 0), \ldots, (\alpha_l, 0)\}$ is the system of simple roots of the pair $(\mathfrak{g}, \tau)$, where $\tau = \eta(\theta) \in \operatorname{Aut} \Pi$, and $\Pi_0 = \{\alpha_1, \ldots, \alpha_l\}$ is the system of simple roots of $\mathfrak{g}^\theta$ numbered as in Table 1. There are also indicated: the type of $\mathfrak{g}^\theta$ and the real form of $\mathfrak{g}$ corresponding to θ. The automorphisms θ are divided into the following three types (see Problem 5.1.38): type I—the inner automorphisms with a semisimple $\mathfrak{g}^\theta$, type II—the inner automorphisms with a nonsemisimple $\mathfrak{g}^\theta$, type III—the outer automorphisms.

Table 7

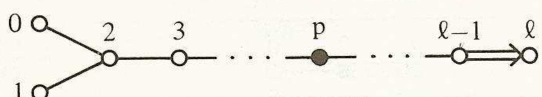
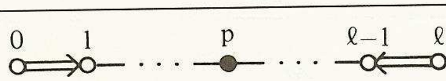
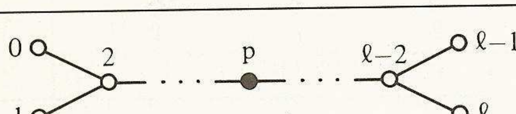
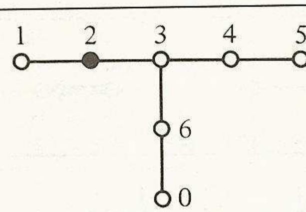
Type I				
$\mathfrak{g}$	Type of affine diagram	Kac diagram of θ	Type of $\mathfrak{g}^\theta$	Real form
$\mathfrak{so}_{2l+1}(\mathbb{C})$ ($l \geqslant 3$)	$B_l^{(1)}$	 ($2 \leqslant p \leqslant \ell$)	$D_p \oplus B_{l-p}$	$\mathfrak{so}_{2p, 2(l-p)+1}$
$\mathfrak{sp}_{2l}(\mathbb{C})$ ($l \geqslant 2$)	$C_l^{(1)}$	 ($1 \leqslant p \leqslant [\ell/2]$)	$C_p \oplus C_{l-p}$	$\mathfrak{sp}_{p, l-p}$
$\mathfrak{so}_{2l}(\mathbb{C})$ ($l \geqslant 4$)	$D_l^{(1)}$	 ($2 \leqslant p \leqslant [\ell/2]$)	$D_p \oplus D_{l-p}$	$\mathfrak{so}_{2p, 2(l-p)}$
E_6	$E_6^{(1)}$		$A_1 \oplus A_5$	EII

Table 7 (cont.)

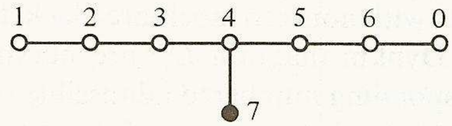
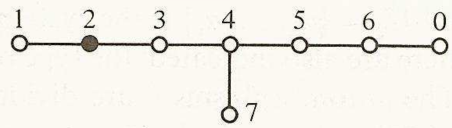
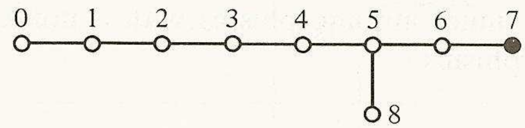
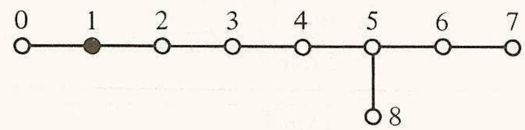
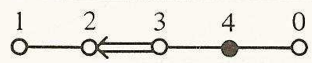
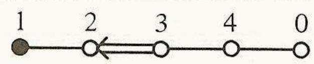
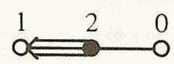
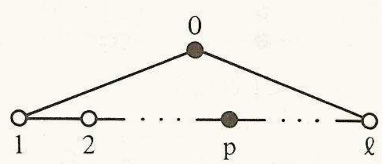
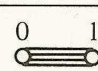
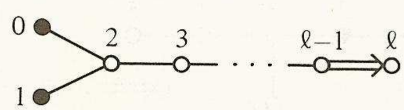
$\mathfrak{g}$	Type of affine diagram	Kac diagram of θ	Type of $\mathfrak{g}^{\theta}$	Real form
Type I				
E_7	$E_7^{(1)}$		A_7	EV
			$A_1 \oplus D_6$	EVI
E_8	$E_8^{(1)}$		D_8	$EVIII$
			$A_1 \oplus E_7$	EIX
F_4	$F_7^{(1)}$		$C_3 \oplus A_1$	FI
			B_4	FII
G_2	$G_2^{(1)}$		$A_1 \oplus A_1$	G
Type II				
$\mathfrak{sl}_{l+1}(\mathbb{C})$ ($l \geqslant 2$)	$A_l^{(1)}$	 ($1 \leqslant p \leqslant [(\ell+1)/2]$)	$A_{p-1} \oplus A_{l-p} \oplus \mathbb{C}$	$\mathfrak{su}_{p,l+1-p}$
$\mathfrak{sl}_2(\mathbb{C})$	$A_1^{(1)}$		$\mathbb{C}$	$\mathfrak{su}_{1,1}$
$\mathfrak{so}_{2l+1}(\mathbb{C})$ ($l \geqslant 3$)	$B_l^{(1)}$		$B_{l-1} \oplus \mathbb{C}$	$\mathfrak{so}_{2,2l-1}$
$\mathfrak{sp}_{2l}(\mathbb{C})$ ($l \geqslant 2$)	$C_l^{(1)}$		$A_{l-1} \oplus \mathbb{C}$	$\mathfrak{sp}_{2l}(\mathbb{R})$

Table 7 (cont.)

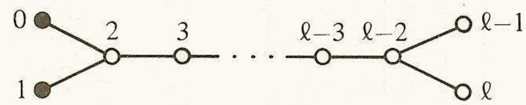
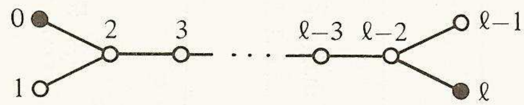
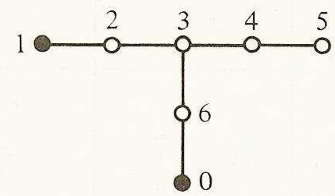
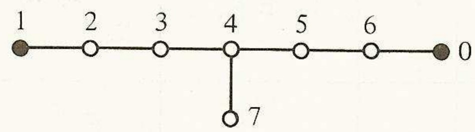
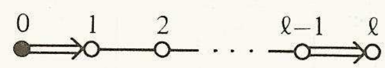
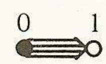
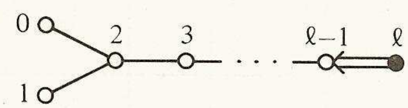
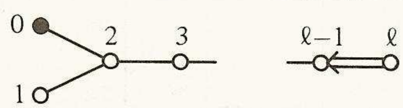
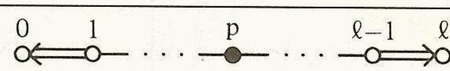
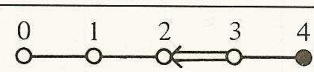
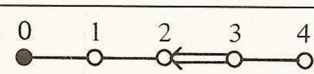
$\mathfrak{g}$	Type of affine diagram	Kac diagram of θ	Type of $\mathfrak{g}^\theta$	Real form
		Type II		
$\mathfrak{so}_{2l}(\mathbb{C})$ $(l \geqslant 4)$	$D_l^{(1)}$		$D_{l-1} \oplus \mathbb{C}$	$\mathfrak{so}_{2,2l-2}$
			$A_{l-1} \oplus \mathbb{C}$	$\mathfrak{u}_l^*(\mathbb{H})$
E_6	$E_6^{(1)}$		$D_5 \oplus \mathbb{C}$	$EIII$
E_7	$E_7^{(1)}$		$E_6 \oplus \mathbb{C}$	$EVII$
		Type III		
$\mathfrak{sl}_{2l+1}(\mathbb{C})$ $(l \geqslant 2)$	$A_{2l}^{(2)}$		B_l	$\mathfrak{sl}_{2l+1}(\mathbb{R})$
$\mathfrak{sl}_3(\mathbb{C})$	$A_2^{(2)}$		A_1	$\mathfrak{sl}_3(\mathbb{R})$
$\mathfrak{sl}_{2l}(\mathbb{C})$ $(l \geqslant 3)$	$A_{2l-1}^{(2)}$		D_l	$\mathfrak{sl}_{2l}(\mathbb{R})$
			C_l	$\mathfrak{sl}_l(\mathbb{H})$
$\mathfrak{so}_{2l+2}(\mathbb{C})$ $(l \geqslant 2)$	$D_{l+1}^{(2)}$	 $(0 \leqslant p \leqslant [\ell/2])$	$B_p \oplus B_{l-p}$	$\mathfrak{so}_{2p+1,2(l-p)+1}$
E_6	$E_6^{(2)}$		C_4	EI
			F_4	EIV

Table 8. Matrix Realizations of Classical Real Lie Algebras. In the table are given matrix realizations of real forms $\mathfrak{g}$ of classical complex Lie algebras, their Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and the maximal $\mathbb{R}$-diagonalizable subalgebras $\mathfrak{a} \subset \mathfrak{g}$. The matrices are real for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$, $\mathfrak{so}_{p,q}$, $\mathfrak{sp}_n(\mathbb{R})$ and complex otherwise.