Torsion subgroups of Small Cancellation Groups

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Motivation

Conjecture [5, Conjecture 1.5] Every action of a finitely generated torsion group on a finite-dimensional CAT(0) complex is elliptic.

Meta-Conjecture [3] Every locally elliptic action of a finitely generated group on a finite-dimensional nonpositively curved complex is elliptic. In particular, every action of a finitely generated torsion group on such a complex is elliptic.

A **locally elliptic** action is an action in which the orbit of each element is bounded. If the orbit of the whole group is bounded we say that the action is **elliptic**.

Some examples of nonpositively curved complexes:
- CAT(0);
- Helly;
- Small Cancellation ($C(6)$, $C(4)$–$T(4)$, $C(3)$–$T(6)$);
- systolic;
- quadric.

In the small cancellation setting having a bounded orbit is equivalent to having a fixed point. Therefore 'locally elliptic' can be thought of as 'every group element fixes a point' and, likewise, 'elliptic' should mean 'having a global fixed point'.

The following is a strongly related question that motivates our work.

Question [6] Can a group acting properly and cocompactly on a CAT(0) space have an infinite torsion subgroup?

Small Cancellation

Let X be a simply connected combinatorial 2-complex. A *piece* in X is a non-trivial path P such that there exist two distinct 2-cells $F_1, F_2 \in X$ with $P \subset F_1 \cap F_2$.

Definition. A complex X satisfies the $C(p)$ condition if there is no 2-cell in X whose boundary can be covered by fewer than p pieces.

A complex X satisfies the $T(q)$ condition if there is no disc diagram D in X with an internal vertex v of valence $2 < \delta(v) < q$.

Proposition. Let X be a simply connected $C(4)$–$T(4)$ small cancellation complex and let $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$ be a set of pairwise intersecting 2-cells of X. Then the intersection of all cells from $\mathcal{F}$ is either a piece or a single vertex.

Definition. A group presentation satisfies the $C(p)$ or $T(q)$ condition if its Cayley complex does.

Bibliography

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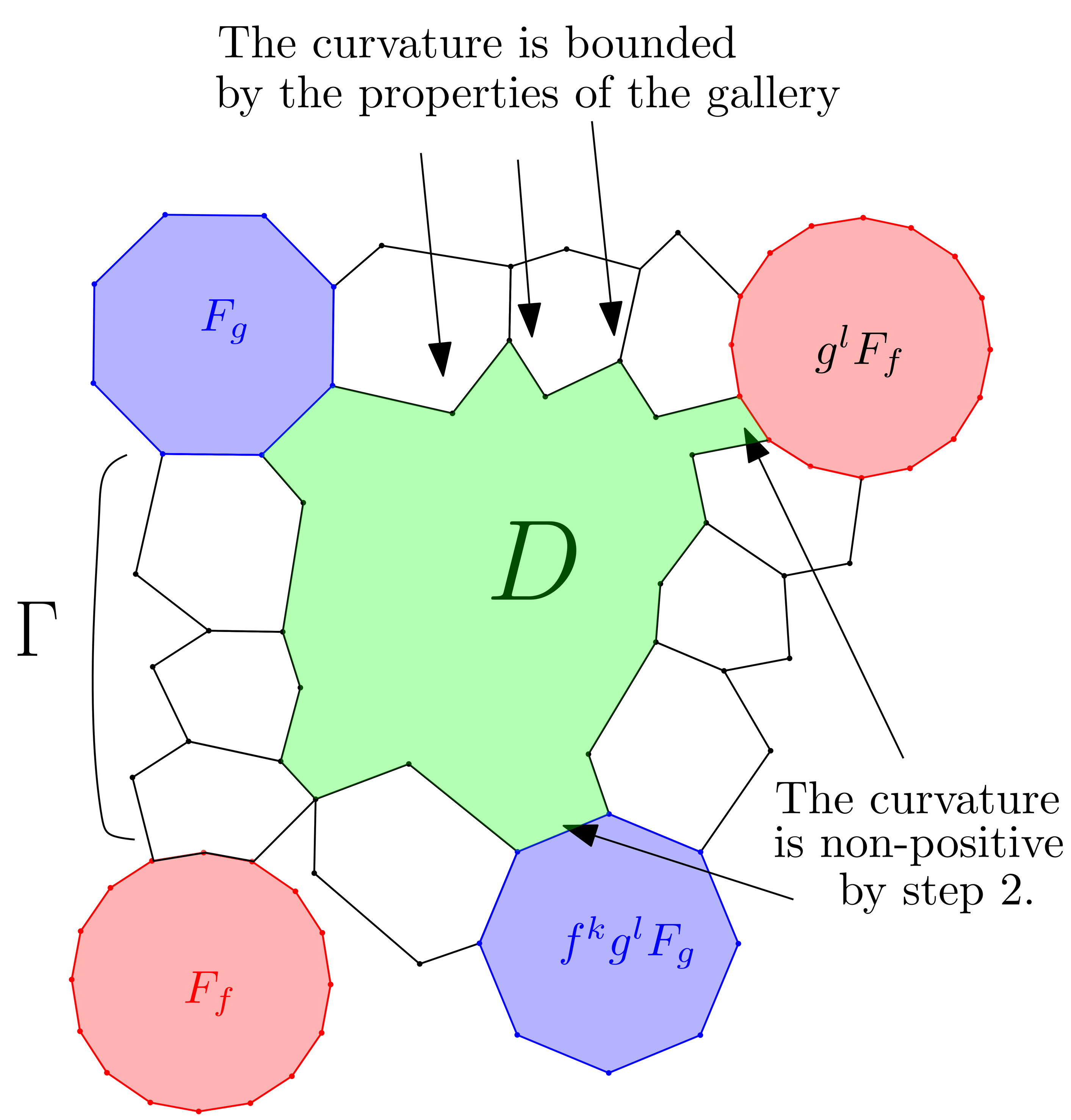
Main results [1, 2]

Theorem 1 Let X be a simply connected $C(6)$, $C(4)$–$T(4)$ or $C(3)$–$T(6)$ small cancellation complex. Let G be a group acting on X by automorphisms s.t. the action on the 1-skeleton of X is free. If the G-action is locally elliptic, then it is elliptic.

Theorem 2 Torsion subgroups of groups defined by $C(6)$, $C(4)$–$T(4)$ or $C(3)$–$T(6)$ small cancellation presentations are finite.

Automatic continuity for small cancellation groups

Corollary [1, 2] Let G be a group defined by a $C(6)$, $C(4)$–$T(4)$, or $C(3)$–$T(6)$ small cancellation presentation. Then any group homomorphism $\varphi\colon L \to G$ from a locally compact group L is continuous or there exists a normal open subgroup $N \subseteq L$ such that $\varphi(N)$ is a torsion group.



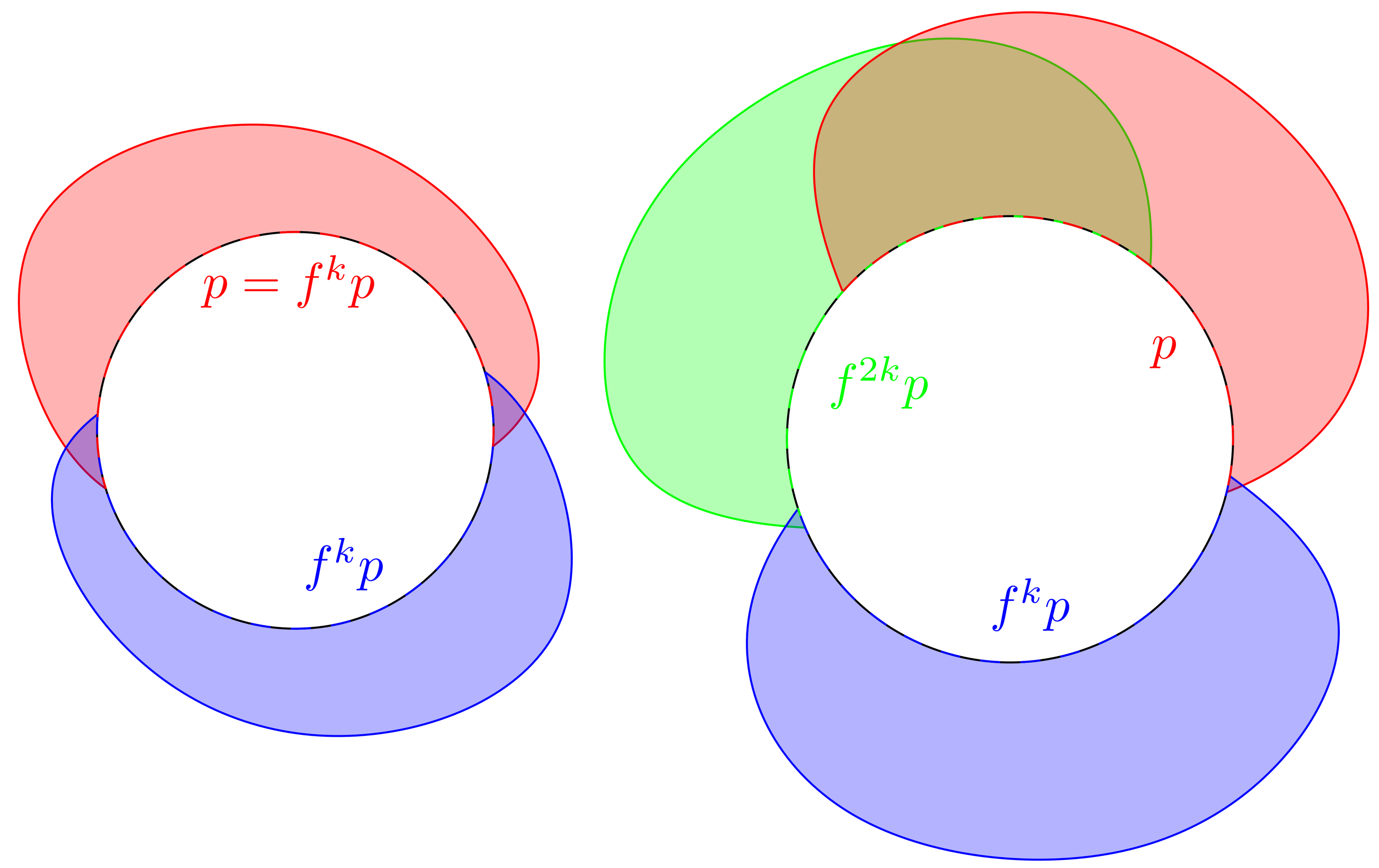
Idea of the proof of Theorem 1, ($C(4)$–$T(4)$ case)

The proof consists of the three following steps:

1. Each group element f fixes exactly one cell F_f in X.

2. $\forall f \in G \setminus \{id\}$, $\exists k \forall F' \neq F_f \quad F' \cap F_f \neq \emptyset \Rightarrow f^k F' \cap F' = \emptyset$.

Since G acts freely on the 1-skeleton, the element f acts on F_f by a rotation of finite order m. Suppose (after replacing f by some of its powers if necessary) that f is a rotation by $2\pi/m$. We claim that $k = m/2$ for even m and $k = (m-1)/2$ for odd m is as required.
If $f^k F' \cap F' \neq \emptyset$, then for some piece p the intersection $p \cap f^k p$ is non-empty. Then we have three pieces $p, f^k p, f^{2k} p$ covering the whole boundary of F_f (see picture below). A contradiction with the condition $C(4)$.



3. Let $f, g \in G, F_f \neq F_g$. For k, l as in step 2 the element $f^k g^l$ has infinite order.

Since X is connected, we can find a shortest 'gallery' Γ between F_f and F_g i.e. set $\Gamma := \{F_f = F_0, F_1, \dots, F_{n-1}, F_n = F_g\}$ such that $F_i \cap F_{i+1} \neq \emptyset$.
If $f^k g^l$ has finite order, then $\beta := \bigcup (f^k g^l)^i (\Gamma \cup f^k \Gamma)$ is closed and there exists a closed path α that consists only of boundary edges of cells from β.
By the Lyndon–van Kampen lemma there exists a minimal area disc diagram D with boundary path α. By the combinatorial Gauss–Bonnet lemma such D has a 'significant positive curvature' at the boundary (picture on the left).
The choice of k, l guarantees that the curvature at the fixed cells is nonpositive. Since Γ is a shortest gallery, the curvature along it and its translates is bounded from above, a contradiction.

Theorem 1 $\Rightarrow$ Theorem 2

Theorem [4] Let G be a group with a small cancellation presentation $\langle X, R \rangle$. If $x \in G$ is an element of finite order s then $\exists z, q$ such that $z^q = r \in R$ and $s | q$ and x is conjugate to $z^{q/s}$.

Proof of Theorem 2 Let H be a torsion subgroup of G. The action of H on the 1-skeleton of the Cayley complex of G is free. By the Theorem above the H-action is locally elliptic. By Theorem 1 the action of H has a global fixed point. Since the 2-cells are finite and the action is free on the 1-skeleton, H has to be finite.