

Laplace Table Derivations

$$\bullet L(t^n) = \frac{n!}{s^{1+n}}$$

$$\bullet L(e^{at}) = \frac{1}{s-a}$$

$$\bullet L(\cos bt) = \frac{s}{s^2 + b^2}$$

$$\bullet L(\sin bt) = \frac{b}{s^2 + b^2}$$

$$\bullet L(H(t-a)) = \frac{e^{-as}}{s}$$

$$\bullet L(\delta(t-a)) = e^{-as}$$

$$\bullet L(\text{floor}(t/a)) = \frac{e^{-as}}{s(1 - e^{-as})}$$

$$\bullet L(\text{sqw}(t/a)) = \frac{1}{s} \tanh(as/2)$$

$$\bullet L(a \text{ trw}(t/a)) = \frac{1}{s^2} \tanh(as/2)$$

$$\bullet L(t^\alpha) = \frac{\Gamma(1 + \alpha)}{s^{1+\alpha}}$$

$$\bullet L(t^{-1/2}) = \sqrt{\frac{\pi}{s}}$$

Proof of $L(t^n) = n!/s^{1+n}$

Slide 1 of 3

The first step is to evaluate $L(f(t))$ for $f(t) = t^0$ [$n = 0$ case]. The function t^0 is written as **1**, but Laplace theory conventions require $f(t) = 0$ for $t < 0$, therefore $f(t)$ is technically the **unit step function**.

$$\begin{aligned}L(1) &= \int_0^{\infty} (1)e^{-st} dt \\ &= -(1/s)e^{-st} \Big|_{t=0}^{t=\infty} \\ &= 1/s\end{aligned}$$

Laplace integral of $f(t) = 1$.

Evaluate the integral.

Assumed $s > 0$ to evaluate $\lim_{t \rightarrow \infty} e^{-st}$.

Proof of $L(t^n) = n!/s^{1+n}$

Slide 2 of 3

The value of $L(f(t))$ for $f(t) = t$ can be obtained by s -differentiation of the relation $L(1) = 1/s$, as follows. Technically, $f(t) = 0$ for $t < 0$, then $f(t)$ is called the **ramp function**.

$$\begin{aligned}\frac{d}{ds}L(1) &= \frac{d}{ds} \int_0^\infty (1)e^{-st} dt \\ &= \int_0^\infty \frac{d}{ds} (e^{-st}) dt \\ &= \int_0^\infty (-t)e^{-st} dt \\ &= -L(t)\end{aligned}$$

Then

$$\begin{aligned}L(t) &= -\frac{d}{ds}L(1) \\ &= -\frac{d}{ds}(1/s) \\ &= 1/s^2\end{aligned}$$

Laplace integral for $f(t) = 1$.

$$\text{Used } \frac{d}{ds} \int_a^b F dt = \int_a^b \frac{dF}{ds} dt.$$

Calculus rule $(e^u)' = u'e^u$.

Definition of $L(t)$.

Rewrite last display.

Use $L(1) = 1/s$.

Differentiate.

Proof of $L(t^n) = n!/s^{1+n}$

Slide 3 of 3

This idea can be repeated to give

$$\begin{aligned}L(t^2) &= -\frac{d}{ds}L(t) \\ &= L(t^2) \\ &= \frac{2}{s^3}.\end{aligned}$$

The pattern is $L(t^n) = -\frac{d}{ds}L(t^{n-1})$, which implies the formula

$$L(t^n) = \frac{n!}{s^{1+n}}.$$

The proof is complete.

$$\text{Proof of } L(e^{at}) = \frac{1}{s-a}$$

The result follows from $L(1) = 1/s$, as follows.

$$\begin{aligned} L(e^{at}) &= \int_0^{\infty} e^{at} e^{-st} dt \\ &= \int_0^{\infty} e^{-(s-a)t} dt \\ &= \int_0^{\infty} e^{-St} dt \\ &= 1/S \\ &= 1/(s-a) \end{aligned}$$

Direct Laplace transform.

Use $e^A e^B = e^{A+B}$.

Substitute $S = s - a$.

Apply $L(1) = 1/s$.

Back-substitute $S = s - a$.

$$\text{Proof of } L(\cos bt) = \frac{s}{s^2 + b^2} \text{ and } L(\sin bt) = \frac{b}{s^2 + b^2}$$

Slide 1 of 2

Use will be made of Euler's formula

$$e^{i\theta} = \cos \theta + i \sin \theta,$$

usually first introduced in trigonometry. In this formula, θ is a real number in radians and $i = \sqrt{-1}$ is the complex unit.

$$e^{ibt} e^{-st} = (\cos bt)e^{-st} + i(\sin bt)e^{-st}$$

$$\int_0^{\infty} e^{-ibt} e^{-st} dt = \int_0^{\infty} (\cos bt)e^{-st} dt + i \int_0^{\infty} (\sin bt)e^{-st} dt$$

$$\frac{1}{s - ib} = \int_0^{\infty} (\cos bt)e^{-st} dt + i \int_0^{\infty} (\sin bt)e^{-st} dt$$

Substitute $\theta = bt$ into Euler's formula and multiply by e^{-st} .

Integrate $t = 0$ to $t = \infty$. Then use properties of integrals.

Evaluate the left hand side using $L(e^{at}) = 1/(s - a)$, $a = ib$.

Proof of $L(\cos bt) = \frac{s}{s^2 + b^2}$ and $L(\sin bt) = \frac{b}{s^2 + b^2}$

Slide 2 of 2

$$\frac{1}{s - ib} = L(\cos bt) + iL(\sin bt)$$

$$\frac{s + ib}{s^2 + b^2} = L(\cos bt) + iL(\sin bt)$$

$$\frac{s}{s^2 + b^2} = L(\cos bt)$$

$$\frac{b}{s^2 + b^2} = L(\sin bt)$$

Direct Laplace transform definition.

Use complex rule $1/z = \bar{z}/|z|^2$, $z = A + iB$, $\bar{z} = A - iB$, $|z| = \sqrt{A^2 + B^2}$.

Extract the real part.

Extract the imaginary part.

Proof of $L(H(t - a)) = e^{-as}/s$

$$\begin{aligned}L(H(t - a)) &= \int_0^{\infty} H(t - a)e^{-st} dt \\&= \int_a^{\infty} (1)e^{-st} dt \\&= \int_0^{\infty} (1)e^{-s(x+a)} dx \\&= e^{-as} \int_0^{\infty} (1)e^{-sx} dx \\&= e^{-as}(1/s)\end{aligned}$$

Direct Laplace transform. Assume $a \geq 0$.

Because $H(t - a) = 0$ for $0 \leq t < a$.

Change variables $t = x + a$.

Constant e^{-as} moves outside integral.

Apply $L(1) = 1/s$.

Proof of $L(\delta(t - a)) = e^{-as}$

Slide 1 of 3

The *definition* of the delta function is a formal one, in which every occurrence of symbol $\delta(t - a)dt$ under an integrand is replaced by $dH(t - a)$. The differential symbol $dH(t - a)$ is taken in the sense of the Riemann-Stieltjes integral. This integral is defined in Rudin's *Real analysis* for monotonic integrators $\alpha(x)$ as the limit

$$\int_a^b f(x)d\alpha(x) = \lim_{N \rightarrow \infty} \sum_{n=1}^N f(x_n)(\alpha(x_n) - \alpha(x_{n-1}))$$

where $x_0 = a$, $x_N = b$ and $x_0 < x_1 < \dots < x_N$ forms a partition of $[a, b]$ whose mesh approaches zero as $N \rightarrow \infty$.

The steps in computing the Laplace integral of the delta function appear below. Admittedly, the proof requires advanced calculus skills and a certain level of mathematical maturity. The reward is a fuller understanding of the Dirac symbol $\delta(x)$.

Proof of $L(\delta(t - a)) = e^{-as}$

Slide 2 of 3

$$L(\delta(t - a)) = \int_0^{\infty} e^{-st} \delta(t - a) dt$$

$$= \int_0^{\infty} e^{-st} dH(t - a)$$

$$= \lim_{M \rightarrow \infty} \int_0^M e^{-st} dH(t - a)$$

$$= e^{-sa}$$

Laplace integral, $a > 0$ assumed.

Replace $\delta(t - a)dt$ by $dH(t - a)$.

Definition of improper integral.

Explained below.

Proof of $L(\delta(t - a)) = e^{-as}$

Slide 3 of 3

To explain the last step, apply the definition of the Riemann-Stieltjes integral:

$$\int_0^M e^{-st} dH(t - a) = \lim_{N \rightarrow \infty} \sum_{n=0}^{N-1} e^{-st_n} (H(t_n - a) - H(t_{n-1} - a))$$

where $0 = t_0 < t_1 < \dots < t_N = M$ is a partition of $[0, M]$ whose mesh $\max_{1 \leq n \leq N} (t_n - t_{n-1})$ approaches zero as $N \rightarrow \infty$. Given a partition, if $t_{n-1} < a \leq t_n$, then $H(t_n - a) - H(t_{n-1} - a) = 1$, otherwise this factor is zero. Therefore, the sum reduces to a single term e^{-st_n} . This term approaches e^{-sa} as $N \rightarrow \infty$, because t_n must approach a .

Proof of $L(\text{floor}(t/a)) = \frac{e^{-as}}{s(1 - e^{-as})}$

Slide 1 of 3

The library function **floor** present in computer languages C and Fortran is defined by **floor**(x) = greatest whole integer $\leq x$, e.g., **floor**(5.2) = 5 and **floor**(-1.9) = -2. The computation of the Laplace integral of **floor**(t) requires ideas from infinite series, as follows.

$$\begin{aligned} F(s) &= \int_0^{\infty} \text{floor}(t) e^{-st} dt \\ &= \sum_{n=0}^{\infty} \int_n^{n+1} (n) e^{-st} dt \\ &= \sum_{n=0}^{\infty} \frac{n}{s} (e^{-ns} - e^{-ns-s}) \\ &= \frac{1 - e^{-s}}{s} \sum_{n=0}^{\infty} n e^{-sn} \end{aligned}$$

Laplace integral definition.

On $n \leq t < n + 1$,
floor(t) = n .

Evaluate each integral.

Common factor removed.

Proof of $L(\text{floor}(t/a)) = \frac{e^{-as}}{s(1 - e^{-as})}$

Slide 2 of 3

$$= \frac{x(1-x)}{s} \sum_{n=0}^{\infty} nx^{n-1}$$

Define $x = e^{-s}$.

$$= \frac{x(1-x)}{s} \frac{d}{dx} \sum_{n=0}^{\infty} x^n$$

Term-by-term differentiation.

$$= \frac{x(1-x)}{s} \frac{d}{dx} \frac{1}{1-x}$$

Geometric series sum.

$$= \frac{x}{s(1-x)}$$

Compute the derivative, simplify.

$$= \frac{e^{-s}}{s(1 - e^{-s})}$$

Substitute $x = e^{-s}$.

$$\text{Proof of } L(\mathbf{floor}(t/a)) = \frac{e^{-as}}{s(1 - e^{-as})}$$

Slide 3 of 3

To evaluate the Laplace integral of $\mathbf{floor}(t/a)$, a change of variables is made.

$$\begin{aligned} L(\mathbf{floor}(t/a)) &= \int_0^{\infty} \mathbf{floor}(t/a) e^{-st} dt \\ &= a \int_0^{\infty} \mathbf{floor}(r) e^{-asr} dr \\ &= aF(as) \end{aligned}$$

$$= \frac{e^{-as}}{s(1 - e^{-as})}$$

Laplace integral definition.

Change variables $t = ar$.

Apply the formula for $F(s)$.

Simplify.

Proof of $L(\text{sqw}(t/a)) = \frac{1}{s} \tanh(as/2)$

Slide 1 of 3

The square wave defined by $\text{sqw}(x) = (-1)^{\text{floor}(x)}$ is periodic of period **2** and piecewise-defined. Let $P = \int_0^2 \text{sqw}(t)e^{-st} dt$.

$$\begin{aligned} P &= \int_0^1 \text{sqw}(t)e^{-st} dt + \int_1^2 \text{sqw}(t)e^{-st} dt \\ &= \int_0^1 e^{-st} dt - \int_1^2 e^{-st} dt \end{aligned}$$

$$\begin{aligned} &= \frac{1}{s}(1 - e^{-s}) + \frac{1}{s}(e^{-2s} - e^{-s}) \\ &= \frac{1}{s}(1 - e^{-s})^2 \end{aligned}$$

Apply $\int_a^b = \int_a^c + \int_c^b$.

Use $\text{sqw}(x) = 1$ on $0 \leq x < 1$ and $\text{sqw}(x) = -1$ on $1 \leq x < 2$.

Evaluate each integral.

Collect terms.

Proof of $L(\text{sqw}(t/a)) = \frac{1}{s} \tanh(as/2)$

Slide 2 of 3 – Compute $L(\text{sqw}(t))$

$$L(\text{sqw}(t)) = \frac{\int_0^2 \text{sqw}(t) e^{-st} dt}{1 - e^{-2s}}$$

Periodic function formula.

$$= \frac{1}{s} (1 - e^{-s})^2 \frac{1}{1 - e^{-2s}}$$

Use the computation of P above.

$$= \frac{1 - e^{-s}}{s(1 + e^{-s})}$$

Factor

$$1 - e^{-2s} = (1 - e^{-s})(1 + e^{-s})$$

$$= \frac{1 e^{s/2} - e^{-s/2}}{s e^{s/2} + e^{-s/2}}$$

Multiply the fraction by $e^{s/2}/e^{s/2}$.

$$= \frac{1 \sinh(s/2)}{s \cosh(s/2)}$$

Use $\sinh u = (e^u - e^{-u})/2$,
 $\cosh u = (e^u + e^{-u})/2$.

$$= \frac{1}{s} \tanh(s/2)$$

Use $\tanh u = \sinh u / \cosh u$.

Proof of $L(\mathbf{sqw}(t/a)) = \frac{1}{s} \tanh(as/2)$

Slide 3 of 3

To complete the computation of $L(\mathbf{sqw}(t/a))$, a change of variables is made:

$$\begin{aligned} L(\mathbf{sqw}(t/a)) &= \int_0^\infty \mathbf{sqw}(t/a) e^{-st} dt \\ &= \int_0^\infty \mathbf{sqw}(r) e^{-asr} (a) dr \\ &= \frac{a}{as} \tanh(as/2) \\ &= \frac{1}{s} \tanh(as/2) \end{aligned}$$

Direct transform.

Change variables $r = t/a$.

See $L(\mathbf{sqw}(t))$ above.

$$\text{Proof of } L(a \text{ trw}(t/a)) = \frac{1}{s^2} \tanh(as/2)$$

The triangular wave is defined by $\text{trw}(t) = \int_0^t \text{sqw}(x) dx$.

$$L(a \text{ trw}(t/a)) = \frac{f(0) + L(f'(t))}{s}$$

$$= \frac{1}{s} L(\text{sqw}(t/a))$$

$$= \frac{1}{s^2} \tanh(as/2)$$

Let $f(t) = a \text{ trw}(t/a)$. Use
 $L(f'(t)) = sL(f(t)) - f(0)$.

Use $f(0) = 0$, then use
 $(a \int_0^{t/a} \text{sqw}(x) dx)' = \text{sqw}(t/a)$.

Table entry for **sqw**.

Proof of $L(t^\alpha) = \frac{\Gamma(1 + \alpha)}{s^{1+\alpha}}$

$$\begin{aligned} L(t^\alpha) &= \int_0^\infty t^\alpha e^{-st} dt \\ &= \int_0^\infty (u/s)^\alpha e^{-u} du / s \\ &= \frac{1}{s^{1+\alpha}} \int_0^\infty u^\alpha e^{-u} du \\ &= \frac{1}{s^{1+\alpha}} \Gamma(1 + \alpha). \end{aligned}$$

Definition of Laplace integral.

Change variables $u = st$, $du = sdt$.

Because $s = \text{constant}$ for u -integration.

Because $\Gamma(x) \equiv \int_0^\infty u^{x-1} e^{-u} du$.

Gamma Function

The *generalized factorial function* $\Gamma(x)$ is defined for $x > 0$ and it agrees with the classical factorial $n! = (1)(2) \cdots (n)$ in case $x = n + 1$ is an integer. In literature, $\alpha!$ means $\Gamma(1 + \alpha)$. For more details about the Gamma function, see Abramowitz and Stegun or `maple` documentation.

Proof of $L(t^{-1/2}) = \sqrt{\frac{\pi}{s}}$

$$\begin{aligned} L(t^{-1/2}) &= \frac{\Gamma(1 + (-1/2))}{s^{1-1/2}} \\ &= \frac{\sqrt{\pi}}{\sqrt{s}} \end{aligned}$$

Apply the previous formula.

Use $\Gamma(1/2) = \sqrt{\pi}$.