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Parametrizing nilpotent orbits via Bruhat-Tits theory

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Abstract

Let k denote a field with nontrivial discrete valuation. We assume that k is complete with perfect residue field. Let G be the group of k-rational points of a reductive, linear algebraic group defined over k. Let $\mathfrak{g}$ denote the Lie algebra of G. Fix $r \in \mathbb{R}$. Subject to some restrictions, we show that the set of distinguished degenerate Moy-Prasad cosets of depth r (up to an equivalence relation) parametrizes the nilpotent orbits in $\mathfrak{g}$.

1. Introduction

In this paper we give a uniform parametrization of the nilpotent orbits in the Lie algebra of a p-adic reductive group. This classification, which was motivated by harmonic analysis considerations, matches nilpotent orbits with certain equivalence classes that arise naturally from Bruhat-Tits theory.

1.1. *Motivation.* In the early 1970s Harish-Chandra and Roger Howe studied the local behavior of the character of an irreducible smooth representation of a reductive p-adic group [13], [14]. For example, they established what is now called the Harish-Chandra-Howe local character expansion – in some unspecified neighborhood of the identity the character can be expressed as a linear combination of the Fourier transforms of nilpotent orbital integrals. At the heart of their proofs was a remarkable finiteness statement, referred to as “Howe’s conjecture” [15], about invariant distributions on the Lie algebra. In some stunning work of the 1990s, J.-L. Waldspurger proved a very precise version of Howe’s conjecture for “unramified classical groups” [28]. This sharpened finiteness statement allowed him to relate the range of validity for the Harish-Chandra-Howe local character expansion to the first occurrence of fixed-vectors with respect to congruence filtration subgroups [25].

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The fundamental work of Allen Moy and Gopal Prasad [20], [22] introduced new ways to use the structure theory of F. Bruhat and J. Tits [8], [9] to study questions in representation theory. One consequence of their work is that to each representation we can attach a number, called the depth of the representation. Roughly speaking, this number measures the first occurrence of fixed-vectors with respect to all the natural subgroup filtrations arising from Bruhat-Tits theory. The conjecture of Thomas Hales, Allen Moy, and Gopal Prasad [22, §1] seeks to strengthen the results of J.-L. Waldspurger by asking if the range of validity for the Harish-Chandra-Howe local character expansion is controlled by the depth of the representation; such a result would greatly enhance our understanding of characters. The parametrization of nilpotent orbits presented in this article is the cornerstone of my proof of their conjecture. The remainder of the proof appears in [2], [11], [12].

1.2. *The parametrization.* In a special situation ($r = 0$), the main result of this paper may be viewed as an affine analogue of Bala-Carter theory [4], [5]. Namely, it provides a classification of the nilpotent orbits in terms of equivalence classes of pairs $(G_F/G_F^+, X)$. Here F is a facet in the Bruhat-Tits building of our group, G_F is the associated parahoric subgroup with pro-unipotent radical G_F^+, and X is a distinguished element of the Lie algebra of G_F/G_F^+. (Recall that X is called distinguished provided that it is nilpotent and does not lie in a proper Levi subalgebra.)

In this article we prove this special case ($r = 0$) and take it one step further – we classify the nilpotent orbits in terms of Moy-Prasad cosets of an arbitrary fixed depth r (see below). We now discuss the parametrization scheme in detail.

Let k denote a field with nontrivial discrete valuation. We assume that k is complete with perfect residue field $\mathfrak{f}$. Let G denote the group of k-rational points of a reductive, linear algebraic group $\mathbf{G}$ defined over k and let $\mathfrak{g}$ denote its Lie algebra. We let G° denote the group of k-rational points of the identity component $\mathbf{G}^\circ$ of $\mathbf{G}$. Let $\mathcal{B}(G)$ denote the Bruhat-Tits building of G°. For each pair $(x, r) \in \mathcal{B}(G) \times \mathbb{R}$, Allen Moy and Gopal Prasad [20], [22] have defined the (Moy-Prasad) lattices $\mathfrak{g}_{x,r^+} \subset \mathfrak{g}_{x,r}$ of $\mathfrak{g}$. For $x \in \mathcal{B}(G)$, an element of $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$ is called a Moy-Prasad coset of depth r.

Suppose $r \in \mathbb{R}$. We partition $\mathcal{B}(G)$ into generalized r-facets – two points x and y in $\mathcal{B}(G)$ belong to the same generalized r-facet provided that $\mathfrak{g}_{x,r} = \mathfrak{g}_{y,r}$ and $\mathfrak{g}_{x,r^+} = \mathfrak{g}_{y,r^+}$. If F^* is a generalized r-facet and $x \in F^*$, then we define the $\mathfrak{f}$-vector space $V_{F^*} = \mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$. For example, if $r = 0$, then generalized 0-facets are facets in the usual sense, and if F is a facet of $\mathcal{B}(G)$, then V_F is $\mathrm{Lie}(G_F/G_F^+)$.

Let I_r denote the set of pairs (F^*, v) where F^* is a generalized r-facet and v is an element of V_{F^*}. The set I_r parametrizes the set of Moy-Prasad cosets of depth r. In Section 3.6 we define on I_r an equivalence relation, denoted $\sim$, which is a natural extension of the concept of associate [20], [22].

A pair $(F^*, v) \in I_r$ is degenerate if the coset it parametrizes contains a nilpotent element. Let I_r^n denote the subset of I_r consisting of degenerate pairs. With some restrictions on k and $\mathbf{G}$ (see Section 4.2), we generalize a result of Dan Barbasch and Allen Moy [6, §3]. We show that to each element (F^*, e) of I_r^n we can associate a unique nilpotent orbit $\mathcal{O}(F^*, e)$. This orbit is characterized by the fact that it is the nilpotent orbit of minimal dimension having nontrivial intersection with the coset corresponding to (F^*, e).

The set I_r^n is too large for our purposes. We therefore restrict our attention to the subset I_r^d of distinguished elements of I_r^n (see Section 5.5). For example, if $r = 0$, then $(F, e) \in I_0^n$ is distinguished if e is a distinguished element of $V_F = \mathrm{Lie}(G_F/G_F^+)$ in the sense discussed above.

We now state Theorem 5.6.1, the main result of this paper. Let $\mathcal{O}(0)$ denote the set of nilpotent orbits in $\mathfrak{g}$.

Theorem. *Assume that all of the hypotheses of Section* 4.2 *hold. There is a bijective correspondence between* $I_r^d/\sim$ *and* $\mathcal{O}(0)$ *given by the map which sends* (F^*, e) *to* $\mathcal{O}(F^*, e)$.

We remark that this result is false without some restrictions on k and $\mathbf{G}$. For example, if k is the field of Laurent series over the field with two elements, then for the group $\mathbf{SL}_2(k)$ the set $I_0^d/\sim$ has cardinality three, but $\mathcal{O}(0)$ has infinitely many elements. On the other hand, if we are not interested in a proof which works in a general setting, then we can get by with less severe restrictions. For example, we expect that the theorem is true for $\mathbf{GL}_n(k)$ with no restrictions on k; if $r = 0$, then this is easy to verify. If we assume that the residual characteristic of k is not two, then we expect that the result is valid for split classical groups.

In the special case when $r = 0$, the parametrization scheme discussed in this article is inherent (though neither stated nor proved) in a paper of Dan Barbasch and Allen Moy [6]. Magdy Assem pointed this out to Robert Kottwitz who, in turn, pointed it out to me. Also in the case when $r = 0$, J.-L. Waldspurger [27] develops a conjectural parametrization scheme similar to that given here but for unipotent orbits. He verifies his conjecture in a number of cases. Finally, if $r = 0$, k is the field of Laurent series over the complex numbers, and $\mathbf{G}$ is a connected, simple, adjoint, and k-split group, then the main result of Eric Sommers' paper [23] is equivalent to the main result of this paper; the proofs, however, are very different.

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2. Notation

2.1. *Basic notation.* Let k denote a field with nontrivial discrete valuation ν. We also denote by ν the unique extension of ν to any algebraic extension of k. We assume that k is complete and the residue field $\mathfrak{f}$ is perfect. Denote the ring of integers of k by R and fix a uniformizer ϖ.

Let K be a fixed maximal unramified extension of k. Let R_K denote the ring of integers of K and let $\mathfrak{F}$ denote the residue field of K. Note that $\mathfrak{F}$ is an algebraic closure of $\mathfrak{f}$.

If $\mathfrak{f}$ has positive characteristic, then we let p denote the characteristic of $\mathfrak{f}$. If $\mathfrak{f}$ has characteristic zero, then we let $p = \infty$. Suppose $n \in \mathbb{Z}$. If $p < \infty$, then $(n,p) = \gcd(n,p)$. If $p = \infty$, then $(n,p) = 1$.

Let $\mathbf{G}$ be a reductive, linear algebraic group defined over k. Let $\mathbf{G}^\circ$ denote the identity component of $\mathbf{G}$. Note the $\mathbf{G}^\circ$ is a connected, reductive, linear algebraic group which is defined over k. We let $G = \mathbf{G}(k)$ and $G^\circ = \mathbf{G}^\circ(k)$. We denote by $\boldsymbol{\mathfrak{g}}$ the Lie algebra of $\mathbf{G}$. We let $\mathfrak{g} = \boldsymbol{\mathfrak{g}}(k)$, the vector space of k-rational points of $\boldsymbol{\mathfrak{g}}$. Let $(X,Y) \mapsto [X,Y]$ denote the Lie algebra product for $\mathfrak{g}$.

We adopt the following conventions. We call a subgroup of $\mathbf{G}$ a *parabolic subgroup of* $\mathbf{G}$ provided that it is a parabolic subgroup of $\mathbf{G}^\circ$. Similar notation applies to tori and Levi subgroups.

Let L be the minimal Galois extension of K such that $\mathbf{G}^\circ$ is L-split. As in [20], we define $\ell = [L : K]$, and we normalize ν by requiring $\nu(L^\times) = \mathbb{Z}$.

If $g \in G$ and $X \in \mathfrak{g}$, then ${}^gX = \mathrm{Ad}(g)X$. If $X \in \mathfrak{g}$, then ${}^GX$ denotes the G-orbit of X in $\mathfrak{g}$. We let $\mathbf{X}_*^k(\mathbf{G})$ denote the set of one-parameter k-subgroups of $\mathbf{G}$.

An element $X \in \mathfrak{g}$ is *nilpotent* if and only if there exists $\lambda \in \mathbf{X}_*^k(\mathbf{G})$ such that $\lim_{t\to 0} {}^{\lambda(t)}X = 0$. Let $\mathcal{N}$ denote the set of nilpotent elements in $\mathfrak{g}$ and let $\mathcal{O}(0)$ denote the set of nilpotent G-orbits in $\mathfrak{g}$. It is more usual to say that an element is nilpotent if the Zariski closure of its G-orbit contains zero. Let $\mathcal{N}''$ denote the set of elements in $\mathfrak{g}$ that are nilpotent in this sense. We will let $\mathcal{N}'$ denote the set of elements in $\mathfrak{g}$ which contain zero in the p-adic closure of their G-orbit. It follows that $\mathcal{N} \subseteq \mathcal{N}' \subseteq \mathcal{N}''$. From [18] we have $\mathcal{N} = \mathcal{N}''$ if k is perfect. From [2] we have that if k is perfect or $\mathfrak{f}$ is finite, then $\mathcal{N} = \mathcal{N}'$.

Similarly, we say that $h \in G$ is *unipotent* provided that there exists $\lambda \in \mathbf{X}_*^k(\mathbf{G})$ such that $\lim_{t\to 0} \lambda(t)h(\lambda(t))^{-1} = 1$.

As in [24, §2.2.1] a subset H of G is *bounded* provided that for every k-regular function f on G, the set $\nu\Big(f(H)\Big)$ is bounded from below.

2.2. *Apartments, buildings, and associated notation.* Let $\mathcal{B}(G)$ denote the (enlarged) Bruhat-Tits building of G°; i.e., $\mathcal{B}(G)$ takes into account the center of G°. We identify $\mathcal{B}(G)$ with the $\mathrm{Gal}(K/k)$-fixed points of $\mathcal{B}(\mathbf{G}, K)$, the Bruhat-Tits building of $\mathbf{G}^\circ(K)$.

For $\Omega \subset \mathcal{B}(G)$, we let $\mathrm{stab}_G(\Omega)$ denote the stabilizer of Ω in G.

We let $\mathrm{dist}\colon \mathcal{B}(G) \times \mathcal{B}(G) \to \mathbb{R}_+$ denote a (nontrivial) G-invariant distance function as discussed in [24, §2.3]. For $x, y \in \mathcal{B}(G)$, let $[x, y]$ denote the geodesic in $\mathcal{B}(G)$ from x to y and let $(x, y]$ denote $[x, y] \smallsetminus \{x\}$.

For a k-Levi subgroup $\mathbf{M}$ of $\mathbf{G}$, we identify $\mathcal{B}(\mathbf{M}, k)$ in $\mathcal{B}(\mathbf{G}, k)$. There is not a canonical way to do this, but every natural embedding of $\mathcal{B}(\mathbf{M}, k)$ in $\mathcal{B}(\mathbf{G}, k)$ has the same image.

Given a maximal k-split torus $\mathbf{S}$ of $\mathbf{G}$ we have the torus $S = \mathbf{S}(k)$ in G and the corresponding apartment $\mathcal{A}(S) = \mathcal{A}(\mathbf{S}, k)$ in $\mathcal{B}(G)$. For $\Omega \subset \mathcal{A}(S)$, we let $A\Big(\Omega, \mathcal{A}(S)\Big)$ denote the smallest affine subspace of $\mathcal{A}(S)$ containing Ω.

We let $\Phi(S) = \Phi(\mathcal{A}) = \Phi(\mathbf{S}, k)$ denote the set of roots of $\mathbf{G}$ with respect to k and $\mathbf{S}$; we denote by $\Psi(S) = \Psi(\mathcal{A}) = \Psi(\mathbf{S}, k, \nu)$ the set of affine roots of $\mathbf{G}$ with respect to k, $\mathbf{S}$, and ν. If $\psi \in \Psi(\mathcal{A})$, then $\dot{\psi} \in \Phi(\mathcal{A})$ denotes the gradient of ψ.

For $\psi \in \Psi(\mathcal{A})$, let U_ψ and $U_\psi^+ := U_{\psi^+}$ denote the corresponding subgroups of the root group $U_{\dot{\psi}}$ (see [22, §2.4 and §3.1]).

For $x \in \mathcal{B}(G)$, we will denote the parahoric subgroup of G° attached to x by G_x, and we denote its pro-unipotent radical by G_x^+. Note that both G_x and G_x^+ depend only on the facet of $\mathcal{B}(G)$ to which x belongs. If F is a facet in $\mathcal{B}(G)$ and $x \in F$, then we define $G_F = G_x$ and $G_F^+ = G_x^+$.

Suppose $x \in \mathcal{B}(G)$. The quotient G_x/G_x^+ is the group of $\mathfrak{f}$-rational points of a connected reductive group G_x defined over $\mathfrak{f}$. We let Z_x denote the $\mathfrak{f}$-split torus in the center of G_x corresponding to the maximal k-split torus in the center of $\mathbf{G}$.

We denote the parahoric subgroup of $\mathbf{G}^\circ(K)$ corresponding to $x \in \mathcal{B}(\mathbf{G}, K)$ by $\mathbf{G}(K)_x$. We denote the pro-unipotent radical of $\mathbf{G}(K)_x$ by $\mathbf{G}(K)_x^+$. The subgroups $\mathbf{G}(K)_x$ and $\mathbf{G}(K)_x^+$ depend only on the facet of $\mathcal{B}(\mathbf{G}, K)$ to which x belongs. If F is a facet in $\mathcal{B}(\mathbf{G}, K)$ and $x \in F$, then we define $\mathbf{G}(K)_F = \mathbf{G}(K)_x$ and $\mathbf{G}(K)_F^+ = \mathbf{G}(K)_x^+$. For a facet F in $\mathcal{B}(\mathbf{G}, K)$, the quotient $\mathbf{G}(K)_F/\mathbf{G}(K)_F^+$ is the group of $\mathfrak{F}$-rational points of a connected, reductive $\mathfrak{F}$-group G_F.

2.3. *The Moy-Prasad filtrations of* $\mathfrak{g}$. When $\mathfrak{f}$ is finite, in [20], [22] Allen Moy and Gopal Prasad associate to a pair $(x, r) \in \mathcal{B}(G) \times \mathbb{R}$ a lattice $\mathfrak{g}_{x,r}$ in $\mathfrak{g}$. There is no difficulty in extending their definition to our setting (see [2]), and we will not repeat the definition here. However, we will need to know that $\mathfrak{g}_{x,r}$ has a nice decomposition (with respect to the field k).

Suppose that $\mathbf{S}$ is a maximal k-split torus of $\mathbf{G}$. Let $\mathbf{T}$ be a maximal K-split k-torus containing $\mathbf{S}$. We identify $\mathcal{A}(\mathbf{S}, k)$ with $\mathcal{A}(\mathbf{T}, K)^{\mathrm{Gal}(K/k)}$. For

$\phi \in \Psi\Big(\mathcal{A}(\mathbf{T}, K)\Big)$, we define as in [22, §3.2] the lattice $\mathfrak{u}_\phi$ in the root space $\mathfrak{u}_{\dot\phi}$ of $\mathfrak{g}(K)$. For $\psi \in \Psi\Big(\mathcal{A}(\mathbf{S}, k)\Big)$, define the lattice $\mathfrak{g}_\psi$ in the root space $\mathfrak{g}_{\dot\psi}$ of $\mathfrak{g}$ to be the $\mathrm{Gal}(K/k)$-fixed points of

$$\bigoplus_{\phi\in\Psi(\mathcal{A}(\mathbf{T},K));\ \phi|_{\mathcal{A}(\mathbf{S},k)}=\psi} \mathfrak{u}_\phi.$$

One can check that for $\psi, \psi' \in \Psi\Big(\mathcal{A}(\mathbf{S}, k)\Big)$ we have $\mathfrak{g}_\psi = \mathfrak{g}_{\psi'}$ if and only if $\psi = \psi'$. We also define the lattice $\mathfrak{g}^+_\psi$ in the root space $\mathfrak{g}_{\dot\psi}$ by

$$\mathfrak{g}^+_\psi = \bigcup \mathfrak{g}_{\psi'},$$

where the union is over those affine roots $\psi' \in \Psi\Big(\mathcal{A}(\mathbf{S}, k)\Big)$ such that $\dot\psi' = \dot\psi$ and $\psi'(x) > \psi(x)$ for some (hence any) $x \in \mathcal{A}(\mathbf{S}, k)$.

Let $\mathbf{m}$ denote the Lie algebra of the k-Levi subgroup $C_{\mathbf{G}^\circ}(\mathbf{S})$. Let $\mathfrak{m} = \mathbf{m}(k)$. For $x \in \mathcal{A}(\mathbf{S}, k)$, let $\mathfrak{m}_r = \mathfrak{m} \cap \mathfrak{g}_{x,r}$. The lattice $\mathfrak{m}_r \subset \mathfrak{m}$ is independent of the choice of $x \in \mathcal{A}(\mathbf{S}, k)$. If $x \in \mathcal{A}(\mathbf{S}, k)$, then

$$\mathfrak{g}_{x,r} = \mathfrak{m}_r \oplus \sum_{\psi\in\Psi(\mathcal{A}(\mathbf{S},k));\ \psi(x)\geq r} \mathfrak{g}_\psi.$$

We define $\mathfrak{g}_{x,r^+} := \cup_{s>r}\mathfrak{g}_{x,s}$.

For $x \in \mathcal{B}(\mathbf{G}, K)$ and $s \in \mathbb{R}$, we denote by $\mathfrak{g}(K)_{x,s}$ the Moy-Prasad filtration lattice of $\mathfrak{g}(K)$ associated to x and s. If x is $\mathrm{Gal}(K/k)$-invariant, then $\mathfrak{g}_{x,s} = (\mathfrak{g}(K)_{x,s})^{\mathrm{Gal}(K/k)}$.

For $(x, r) \in \mathcal{B}(G) \times \mathbb{R}_{\geq 0}$, Moy and Prasad also define subgroups $G_{x,r} \subset G_x$ (see also [2]).

3. Generalized r-facets and associated objects

Fix $r \in \mathbb{R}$. None of the statements in this section depend on the structure of $\mathfrak{g}$ as a Lie algebra. Consequently, all statements remain true when the roles of $\mathfrak{g}$ and $\mathfrak{g}^*$ are interchanged.

3.1. *r-facets.* Fix a maximal k-split torus $\mathbf{S}$ of $\mathbf{G}$. Let $\mathcal{A} = \mathcal{A}(\mathbf{S}, k)$ be the corresponding apartment in $\mathcal{B}(G)$. For each $\psi \in \Psi(\mathcal{A})$, let

$$H_{\psi-r} := \{x \in \mathcal{A} \,|\, \psi(x) = r\}.$$

This defines a facet structure on $\mathcal{A}$; a nonempty subset $F_\mathcal{A} \subset \mathcal{A}$ is called an *r-facet of* $\mathcal{A}$ provided that there exists a finite subset $S \subset \Psi(\mathcal{A})$ such that

$$F_\mathcal{A} \subset H_S := \bigcap_{\psi\in S} H_{\psi-r},$$

and $F_{\mathcal{A}}$ is a connected component (in H_S) of

$$H_S \smallsetminus \bigcup_{\psi \in \Psi(\mathcal{A}) \smallsetminus S} (H_S \cap H_{\psi - r}).$$

If $F_{\mathcal{A}}$ is an r-facet of $\mathcal{A}$, then we define the dimension of $F_{\mathcal{A}}$ by

$$\dim F_{\mathcal{A}} := \dim A(F_{\mathcal{A}}, \mathcal{A}).$$

If $F_{\mathcal{A}}$ is an r-facet of $\mathcal{A}$ of maximal dimension, then $F_{\mathcal{A}}$ is called an *r-alcove* of $\mathcal{A}$.

Example 3.1.1.

$$\{r\text{-alcoves of } \mathcal{A}\} = \{\text{connected components of } \mathcal{A} \smallsetminus \textstyle\bigcup_{\psi \in \Psi(\mathcal{A})} H_{\psi - r}\}.$$

Remark 3.1.2. $F_{\mathcal{A}}$ is an r-facet of $\mathcal{A}$ if and only if $F_{\mathcal{A}}$ is a $(-r)$-facet of $\mathcal{A}$.

If $F_{\mathcal{A}}$ is an r-facet of $\mathcal{A}$ and $x, y \in F_{\mathcal{A}}$, then $\mathfrak{g}_{x,r} = \mathfrak{g}_{y,r}$ and $\mathfrak{g}_{x,r^+} = \mathfrak{g}_{y,r^+}$. Therefore, the following definitions make sense.

Definition 3.1.3. Let $F_{\mathcal{A}}$ be an r-facet of $\mathcal{A}$. Fix $x \in F_{\mathcal{A}}$.

$$\mathfrak{g}_{F_{\mathcal{A}}} := \mathfrak{g}_{x,r}$$

and

$$\mathfrak{g}^+_{F_{\mathcal{A}}} := \mathfrak{g}_{x,r^+}.$$

Sometimes, in order to avoid confusion, we denote $\mathfrak{g}_{F_{\mathcal{A}}}$ by $\mathfrak{g}_{F_{\mathcal{A}},r}$ and $\mathfrak{g}^+_{F_{\mathcal{A}}}$ by $\mathfrak{g}_{F_{\mathcal{A}},r^+}$.

LEMMA 3.1.4. *Let $F_{\mathcal{A}}$ be an r-facet of $\mathcal{A}$. A point $x \in \mathcal{A}$ lies in $F_{\mathcal{A}}$ if and only if $\mathfrak{g}_{x,r} = \mathfrak{g}_{F_{\mathcal{A}}}$ and $\mathfrak{g}_{x,r^+} = \mathfrak{g}^+_{F_{\mathcal{A}}}$.*

Thanks to a suggestion of Jiu-Kang Yu, the proof below is far more elegant than the original.

Proof. The r-facet in $\mathcal{A}$ to which x belongs is completely determined by the three sets

$$\{\psi \in \Psi(\mathcal{A}) \,|\, \psi(x) > r\} \qquad \{\psi \in \Psi(\mathcal{A}) \,|\, \psi(x) = r\} \qquad \{\psi \in \Psi(\mathcal{A}) \,|\, \psi(x) < r\}.$$

These three sets are, in turn, completely determined by $\mathfrak{g}_{x,r}$ and $\mathfrak{g}_{x,r^+}$. □

LEMMA 3.1.5. *If $y \in \mathcal{A}$, then the union of all r-facets of $\mathcal{A}$ which contain y in their closure is an open neighborhood of y in $\mathcal{A}$.*

3.2. *Generalized r-facets.*

Definition 3.2.1. For $x \in \mathcal{B}(G)$, define

$$F^*(x) := \{y \in \mathcal{B}(G) \,|\, \mathfrak{g}_{x,r} = \mathfrak{g}_{y,r} \text{ and } \mathfrak{g}_{x,r^+} = \mathfrak{g}_{y,r^+}\}.$$

Definition 3.2.2.

$$\mathcal{F}(r) := \{F^*(x) \mid x \in \mathcal{B}(G)\}.$$

Definition 3.2.3. An element of $\mathcal{F}(r)$ is called a *generalized r-facet*.

Remark 3.2.4. 1. If $x \in \mathcal{B}(G)$, then for all $y \in F^*(x)$ we have $F^*(x) = F^*(y)$.

2. Suppose that $x, y \in \mathcal{B}(G)$. We write $x \sim y$ if and only if $F^*(x) = F^*(y)$. Then
$$\mathcal{B}(G) = \coprod_{x \in \mathcal{B}(G)/\sim} F^*(x) = \coprod_{F^* \in \mathcal{F}(r)} F^*.$$

3. For $x \in \mathcal{B}(G)$ and $g \in G$ we have $gF^*(x) = F^*(gx)$.

4. If $F^* \in \mathcal{F}(r)$ and $\mathcal{A}$ is an apartment of $\mathcal{B}(G)$ such that $F_{\mathcal{A}} = \mathcal{A} \cap F^* \neq \emptyset$, then it follows from Lemma 3.1.4 that $F_{\mathcal{A}}$ is an r-facet of $\mathcal{A}$.

5. If $F^* \in \mathcal{F}(r)$, then F^* is a nonempty and convex subset of $\mathcal{B}(G)$.

LEMMA 3.2.5. *$F^* \in \mathcal{F}(r)$ if and only if $F^* \in \mathcal{F}(-r)$.*

Proof. This follows from Remarks 3.1.2 and 3.2.4 (4). □

LEMMA 3.2.6. *If $x \in \mathcal{B}(G)$ and $\mathcal{A}$ is an apartment in $\mathcal{B}(G)$ such that $F_{\mathcal{A}} = F^*(x) \cap \mathcal{A} \neq \emptyset$, then for all $y \in F_{\mathcal{A}}$ we have*
$$F^*(x) = G_y F_{\mathcal{A}}.$$

Proof. Fix $y \in F_{\mathcal{A}}$.

"$\subset$": Suppose $z \in F^*(x)$. Then there exists an $h \in G_y$ such that $hz \in \mathcal{A}$. Note that
$$\mathfrak{g}_{hz,r} = {}^h\mathfrak{g}_{z,r} = {}^h\mathfrak{g}_{x,r} = {}^h\mathfrak{g}_{y,r} = \mathfrak{g}_{hy,r} = \mathfrak{g}_{y,r} = \mathfrak{g}_{x,r}$$
and similarly $\mathfrak{g}_{hz,r^+} = \mathfrak{g}_{x,r^+}$. Thus $hz \in \mathcal{A} \cap F^*(x) = F_{\mathcal{A}}$, and so $z \in G_y F_{\mathcal{A}}$.

"$\supset$": Suppose $z \in F_{\mathcal{A}}$ and $h \in G_y$. We have
$$\mathfrak{g}_{hz,r} = {}^h\mathfrak{g}_{z,r} = {}^h\mathfrak{g}_{y,r} = \mathfrak{g}_{hy,r} = \mathfrak{g}_{y,r} = \mathfrak{g}_{x,r}$$
and similarly $\mathfrak{g}_{hz,r^+} = \mathfrak{g}_{x,r^+}$. Thus $hz \in F^*(x)$. □

COROLLARY 3.2.7. *If $F^* \in \mathcal{F}(r)$, then the image of F^* in $\mathcal{B}^{\text{red}}(G)$, the reduced Bruhat-Tits building, is bounded.*

LEMMA 3.2.8. *For $x \in \mathcal{B}(G)$ we have*
$$N_G(\mathfrak{g}_{x,r}) \cap N_G(\mathfrak{g}_{x,r^+}) = \operatorname{stab}_G\Big(F^*(x)\Big).$$

Proof. Let $F^* = F^*(x)$.

We have $\operatorname{stab}_G(F^*) \subset N_G(\mathfrak{g}_{x,r}) \cap N_G(\mathfrak{g}_{x,r^+})$.

Suppose $n \in N_G(\mathfrak{g}_{x,r}) \cap N_G(\mathfrak{g}_{x,r^+})$. Fix $z \in F^*$. Let $\mathcal{A}$ be an apartment of $\mathcal{B}(G)$ containing x and nz. Let $F_{\mathcal{A}} = \mathcal{A} \cap F^*$ ($\neq \emptyset$). If $y \in \mathcal{A}$ such that $\mathfrak{g}_{y,r} = \mathfrak{g}_{x,r}$ and $\mathfrak{g}_{y,r^+} = \mathfrak{g}_{x,r^+}$, then from Lemma 3.1.4 we have $y \in F_{\mathcal{A}}$. Thus, since $\mathfrak{g}_{nz,r} = {}^n\mathfrak{g}_{z,r} = {}^n\mathfrak{g}_{x,r} = \mathfrak{g}_{x,r}$ and similarly $\mathfrak{g}_{nz,r^+} = \mathfrak{g}_{x,r^+}$, we have $nz \in F_{\mathcal{A}} \subset F^*$. Since z was arbitrary, we have $nF^* \subset F^*$. □

LEMMA 3.2.9. *If $F^* \in \mathcal{F}(r)$ and $\mathcal{A}$ is an apartment in $\mathcal{B}(G)$ such that $F_{\mathcal{A}} = F^* \cap \mathcal{A} \neq \emptyset$, then*

$$\overline{F_{\mathcal{A}}} = \overline{F^*} \cap \mathcal{A}.$$

Proof. Suppose $F^* \in \mathcal{F}(r)$ and $\mathcal{A}$ is an apartment in $\mathcal{B}(G)$ such that $F_{\mathcal{A}} = F^* \cap \mathcal{A} \neq \emptyset$. It is enough to show that $\overline{F^*} \cap \mathcal{A} \subset \overline{F_{\mathcal{A}}}$.

Suppose $x \in \overline{F^*} \cap \mathcal{A}$. Let $\{x_n\}$ be a sequence in F^* which converges to x. Fix $y \in F_{\mathcal{A}}$.

By choosing a subsequence of $\{x_n\}$, we may assume that for each $n \in \mathbb{N}$ there exists a zero-alcove C_n such that x_n and x both live in $\overline{C_n}$. We may also assume that $\operatorname{dist}(x_n, x) < 1/n$ for all $n \in \mathbb{N}$. For the remainder of this paragraph, fix $n \in \mathbb{N}$. Let $\mathcal{A}_n$ be an apartment in $\mathcal{B}(G)$ containing both C_n and y. Since x and y both lie in $\mathcal{A}_n \cap \mathcal{A}$, there exists $g_n \in G$ such that g_n fixes both x and y and $g_n\mathcal{A}_n = \mathcal{A}$. Since g_n fixes x, we have

$$\operatorname{dist}(g_n x_n, x) = \operatorname{dist}(x_n, x) < 1/n.$$

Since g_n fixes y, it follows from Lemma 3.2.8 that $g_n \in \operatorname{stab}_G(F^*)$. Since $g_n\mathcal{A}_n = \mathcal{A}$ and $g_n \in \operatorname{stab}_G(F^*)$, we have $g_n x_n \in F^* \cap \mathcal{A} = F_{\mathcal{A}}$.

Consequently, the sequence $\{g_n x_n\}$ in $F_{\mathcal{A}}$ converges to x. Thus $x \in \overline{F_{\mathcal{A}}}$. □

Definition 3.2.10. For $F^* \in \mathcal{F}(r)$ and $\delta > 0$, define

$$F^*(\delta) := \{x \in \overline{F^*} \mid \operatorname{dist}(x,z) \geq \delta \text{ for all } z \in \overline{F^*} \setminus F^*\}.$$

LEMMA 3.2.11. *Suppose $F^* \in \mathcal{F}(r)$ and $\delta > 0$. We have that $F^*(\delta)$ is a convex, closed, and* $\operatorname{stab}_G(F^*)$*-invariant subset of $\mathcal{B}(G)$. Moreover, $F^*(\delta)$ is a nonempty subset of F^* if and only if there exists an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ such that the subset of $F_{\mathcal{A}} = F^* \cap \mathcal{A}$ defined by*

$$F_{\mathcal{A}}(\delta) = \{x \in F_{\mathcal{A}} \mid \operatorname{dist}(x,z) \geq \delta \text{ for all } z \in \overline{F_{\mathcal{A}}} \setminus F_{\mathcal{A}}\}$$

is nonempty.

Proof. $F^*(\delta)$ is a closed and $\operatorname{stab}_G(F^*)$-invariant subset of $\mathcal{B}(G)$. We now consider the last statement of the lemma.

For all apartments $\mathcal{A}$ of $\mathcal{B}(G)$ we have $F^*(\delta) \cap \mathcal{A} \subset F_{\mathcal{A}}(\delta)$. Thus, if $F^*(\delta)$ is nonempty, then there exists an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ such that $F_{\mathcal{A}}(\delta) \neq \emptyset$.

We will show that if there is an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ such that $F_{\mathcal{A}}(\delta) \neq \emptyset$, then

$$G_y F_{\mathcal{A}}(\delta) = F^*(\delta) \tag{1}$$

for all $y \in F_{\mathcal{A}}(\delta)$. This implies that if $F_{\mathcal{A}}(\delta) \neq \emptyset$, then $F^*(\delta) \neq \emptyset$.

Suppose $\mathcal{A}$ is an apartment in $\mathcal{B}(G)$ such that $\emptyset \neq F_{\mathcal{A}}(\delta) \subset F_{\mathcal{A}} = F^* \cap \mathcal{A}$. Fix $w \in F_{\mathcal{A}}(\delta)$.

We first show that $G_w F_{\mathcal{A}}(\delta) \subset F^*(\delta)$. Since $G_w \leq \mathrm{stab}_G(F^*)$, we have that $F^*(\delta)$ is G_w-invariant. Thus, it will be enough to show that $F_{\mathcal{A}}(\delta) \subset F^*(\delta)$. Fix $x \in F_{\mathcal{A}}(\delta)$. Suppose $z \in \overline{F^*} \smallsetminus F^*$. Choose an apartment $\mathcal{A}_z$ such that x and z both belong to $\mathcal{A}_z$. There exists a $g \in G_x$ such that $g\mathcal{A}_z = \mathcal{A}$. Since $g \in G_x$, it follows from Lemma 3.2.8 that $g \in \mathrm{stab}_G(F^*)$. From Lemma 3.2.9 we have $gz \in (\overline{F^*} \smallsetminus F^*) \cap \mathcal{A} = \overline{F_{\mathcal{A}}} \smallsetminus F_{\mathcal{A}}$. Thus

$$\delta \leq \mathrm{dist}(x, gz) = \mathrm{dist}(x, z).$$

Since z was arbitrary, we have $x \in F^*(\delta)$.

We now show that $G_w F_{\mathcal{A}}(\delta) \supset F^*(\delta)$. Fix $z \in F^*(\delta)$. From Lemma 3.2.6 there exists $k \in G_w$ such that $kz \in F_{\mathcal{A}}$. Since $k \in G_w \subset \mathrm{stab}_G(F^*)$, we have $kz \in F_{\mathcal{A}} \cap kF^*(\delta) = F_{\mathcal{A}} \cap F^*(\delta) \subset F_{\mathcal{A}}(\delta)$. Thus, equation (1) is valid.

It remains to see that $F^*(\delta)$ is convex. If $F^*(\delta)$ is empty, there is nothing to prove. So suppose $F^*(\delta)$ is nonempty. Then there exists an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ such that $F_{\mathcal{A}}(\delta)$ is nonempty. Suppose $x, z \in F^*(\delta)$. Fix $w \in F_{\mathcal{A}}(\delta)$. From (1) there exists $k \in G_w$ such that $kx \in F_{\mathcal{A}}(\delta)$. Since $k \in G_w \leq \mathrm{stab}_G(F^*)$, we have $kz \in F^*(\delta)$. Thus, another application of (1) shows that there exists $k_1 \in G_{kx}$ such that $k_1 kz \in F_{\mathcal{A}}(\delta)$. As $F_{\mathcal{A}}(\delta)$ is convex, we have

$$[k_1 kx, k_1 kz] \subset F_{\mathcal{A}}(\delta) \subset F^*(\delta).$$

Since $F^*(\delta)$ is $\mathrm{stab}_G(F^*)$-invariant, we have $[x, z] \subset F^*(\delta)$. □

Definition 3.2.12. For $F^* \in \mathcal{F}(r)$, define

$$C(F^*) := \left\{ y \in F^* \mid \begin{array}{c} \text{for all apartments } \mathcal{A} \text{ of } \mathcal{B}(G) \text{ for which} \\ \mathcal{A} \cap F^* \neq \emptyset \text{ we have } y \in \mathcal{A}. \end{array} \right\}$$

COROLLARY 3.2.13. *If $F^* \in \mathcal{F}(r)$, then $C(F^*) \neq \emptyset$.*

Proof. Without loss of generality, we suppose that $\mathbf{G}^\circ$ is semisimple. Let $N = \mathrm{stab}_G(F^*)$. From Corollary 3.2.7 we have that F^* is bounded in $\mathcal{B}(G) = \mathcal{B}^{\mathrm{red}}(G)$. Thus it follows from [24, §2.2.1] that N is a bounded subgroup of G.

If F^* consists of a single point, there is nothing to prove. So we suppose that F^* is not a point. Let $\mathcal{A}$ be an apartment in $\mathcal{B}(G)$ such that $F_{\mathcal{A}} = \mathcal{A} \cap F^* \neq \emptyset$. It follows from Lemma 3.2.6 that $\dim F_{\mathcal{A}} > 0$. Thus, there exists

$\delta > 0$ such that the set

$$\{x \in F_{\mathcal{A}} \mid \operatorname{dist}(x,z) \geq \delta \text{ for all } z \in \overline{F_{\mathcal{A}}} \smallsetminus F_{\mathcal{A}}\}$$

is nonempty. From Lemma 3.2.11 there exists $\delta > 0$ such that $F^*(\delta)$ is a nonempty, convex, closed, N-stable subset of $\mathcal{B}(G)$. Consequently, there exists a $y \in F^*(\delta) \subset F^*$ such that $ny = y$ for all $n \in N$ [8, Proposition 3.2.4].

We now show that $y \in C(F^*)$. Suppose $\mathcal{A}'$ is an apartment of $\mathcal{B}(G)$ such that $F_{\mathcal{A}'} = \mathcal{A}' \cap F^* \neq \emptyset$. Choose $z \in F_{\mathcal{A}'}$. From Lemma 3.2.6 we have $G_z F_{\mathcal{A}'} = F^*$. However, $G_z \subset N$ from Lemma 3.2.8. Thus $G_z y = y$. Consequently, we must have $y \in F_{\mathcal{A}'} \subset \mathcal{A}'$. □

COROLLARY 3.2.14. *If $\mathcal{A}_i$ $(i = 1, 2)$ are two apartments of $\mathcal{B}(G)$ and $F^* \in \mathcal{F}(r)$ such that $F_{\mathcal{A}_i} = F^* \cap \mathcal{A}_i \neq \emptyset$, then* $\dim A(F_{\mathcal{A}_1}, \mathcal{A}_1) = \dim A(F_{\mathcal{A}_2}, \mathcal{A}_2)$.

LEMMA 3.2.15. *If $F_i^* \in \mathcal{F}(r)$ $(i = 1, 2)$ such that $F_1^* \cap \overline{F_2^*} \neq \emptyset$, then $F_1^* \subset \overline{F_2^*}$.*

Proof. Fix $y_i \in C(F_i^*)$. We first show that $y_1 \in F_1^* \cap \overline{F_2^*}$. Let $z \in F_1^* \cap \overline{F_2^*}$. Choose an apartment $\mathcal{A}$ containing z and y_2. Let $F_{i,\mathcal{A}} = F_i^* \cap \mathcal{A}$. Since $F_{1,\mathcal{A}} \neq \emptyset$, we have $y_1 \in F_{1,\mathcal{A}}$. We also have $z \in \overline{F_2^*} \cap \mathcal{A} = \overline{F_{2,\mathcal{A}}}$ from Lemma 3.2.9. Now $z \in F_{1,\mathcal{A}} \cap \overline{F_{2,\mathcal{A}}}$ so $F_{1,\mathcal{A}} \subset \overline{F_{2,\mathcal{A}}}$ since these are both r-facets of $\mathcal{A}$. Thus $y_1 \in \overline{F_2^*}$.

Suppose $w \in F_1^*$. Let $\mathcal{A}'$ be an apartment containing w and y_2. Let $F_{i,\mathcal{A}'} = F_i^* \cap \mathcal{A}'$. From the previous paragraph we have $y_1 \in F_{1,\mathcal{A}'}$ and $y_1 \in \overline{F_2^*} \cap \mathcal{A}' = \overline{F_{2,\mathcal{A}'}}$. Since $F_{2,\mathcal{A}'}$ and $F_{1,\mathcal{A}'}$ are both r-facets of $\mathcal{A}'$, we have $w \in F_{1,\mathcal{A}'} \subset \overline{F_{2,\mathcal{A}'}}$. □

Thanks to Corollary 3.2.14 the following definition makes sense.

Definition 3.2.16. Suppose $F^* \in \mathcal{F}(r)$. Let $\mathcal{A}$ be an apartment in $\mathcal{B}(G)$ such that $\mathcal{A} \cap F^* \neq \emptyset$. We define

$$\dim F^* := \dim A(F^* \cap \mathcal{A}, \mathcal{A}).$$

Moreover, it follows from Lemma 3.2.15 that $\overline{F^*}$ is the disjoint union of F^* and generalized r-facets which meet $\overline{F^*}$ and have dimension strictly smaller than that of F^*.

LEMMA 3.2.17. *If $F_i^* \in \mathcal{F}(r)$ $(i = 1, 2)$ such that $F_1^* \neq F_2^*$ and $F_1^* \subset \overline{F_2^*}$, then for fixed $y_i \in C(F_i^*)$ there exists an $x_2 \in F_2^*$ such that*

1. $G_{x_2} \subset G_{y_1}$ *and*
2. $x_2 \in (y_1, y_2]$.

Proof. Choose an apartment $\mathcal{A}$ containing y_1 and y_2. Let $F_{i,\mathcal{A}} = \mathcal{A} \cap F_i^*$. We have $F_{1,\mathcal{A}} = F_1^* \cap \mathcal{A} \subset \overline{F_2^*} \cap \mathcal{A} = \overline{F_{2,\mathcal{A}}}$. Let

$$\mathcal{F}(y_1, 0) = \{H \in \mathcal{F}(0) \,|\, H \subset \mathcal{A} \text{ and } y_1 \in \overline{H}\}.$$

Note that $\bigcup_{H \in \mathcal{F}(y_1,0)} H$ is an open neighborhood of y_1 in $\mathcal{A}$. Consequently, there exists an $H \in \mathcal{F}(y_1, 0)$ such that $H \cap (y_1, y_2] \neq \emptyset$.

Choose $x_2 \in H \cap (y_1, y_2] \subset F_{2,\mathcal{A}}$. We have $G_{x_2} = G_H \subset G_{y_1}$. □

Definition 3.2.18. Suppose $F^* \in \mathcal{F}(r)$. Fix $x \in F^*$. We define

$$\mathfrak{g}_{F^*} := \mathfrak{g}_{x,r}$$

and

$$\mathfrak{g}^+_{F^*} := \mathfrak{g}_{x,r^+}.$$

Sometimes, to avoid confusion, we denote $\mathfrak{g}_{F^*}$ by $\mathfrak{g}_{F^*,r}$ and $\mathfrak{g}^+_{F^*}$ by $\mathfrak{g}_{F^*,r^+}$. We now present a corollary to the proof of Lemma 3.2.15.

Corollary 3.2.19. *Suppose $F_i^* \in \mathcal{F}(r)$ for $i = 1, 2$. If $F_1^* \subset \overline{F_2^*}$, then*

$$\mathfrak{g}^+_{F_1^*} \subset \mathfrak{g}^+_{F_2^*} \subset \mathfrak{g}_{F_2^*} \subset \mathfrak{g}_{F_1^*}.$$

Proof. Choose $y_i \in C(F_i^*)$. Let $\mathcal{A}$ be an apartment in $\mathcal{B}(G)$ containing y_1 and y_2. Let $F_{i,\mathcal{A}} = F_i^* \cap \mathcal{A}$. From the proof of Lemma 3.2.15, we have $F_{1,\mathcal{A}} \subset \overline{F_{2,\mathcal{A}}}$. We then have $\mathfrak{g}^+_{F_{1,\mathcal{A}}} \subset \mathfrak{g}^+_{F_{2,\mathcal{A}}} \subset \mathfrak{g}_{F_{2,\mathcal{A}}} \subset \mathfrak{g}_{F_{1,\mathcal{A}}}$. □

Lemma 3.2.20. *If $y \in \mathcal{B}(G)$, then the union of all generalized r-facets that contain y in their closure is an open neighborhood of y in $\mathcal{B}(G)$.*

Proof. Fix $y \in \mathcal{B}(G)$ and an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ which contains y. Let $H_{\mathcal{A}}$ denote the union of all r-facets of $\mathcal{A}$ which contain y in their closure. From Lemma 3.1.5 the set $H_{\mathcal{A}}$ is an open neighborhood of y in $\mathcal{A}$. Fix $\varepsilon > 0$ so that if $x \in \mathcal{A}$ and $\mathrm{dist}(x, y) < \varepsilon$, then $x \in H_{\mathcal{A}}$.

Let H denote the union of all generalized r-facets that contain y in their closure. We will show that the ball in $\mathcal{B}(G)$ of radius ε centered around y is contained in H. Fix $z \in \mathcal{B}(G)$ such that $\mathrm{dist}(z, y) < \varepsilon$. There exists $g \in G_y$ such that $gz \in \mathcal{A}$. Since $\mathrm{dist}(gz, y) = \mathrm{dist}(gz, gy) = \mathrm{dist}(z, y) < \varepsilon$, we have $gz \in H_{\mathcal{A}}$. Since $gz \in H_{\mathcal{A}}$, there exists $F^* \in \mathcal{F}(r)$ such that $y \in \overline{F^* \cap \mathcal{A}}$ and $gz \in F^* \cap \mathcal{A}$. Thus we have $gz \in F^*$ and $y \in \overline{F^*}$. Since $y = g^{-1}y \in g^{-1}\overline{F^*} = \overline{g^{-1}F^*}$, we conclude that $z \in g^{-1}F^* \subset H$. □

3.3. *Associativity.* In this subsection we introduce an equivalence relation on the elements of $\mathcal{F}(r)$ which is a generalization of the concept of "associate" found in [20], [22] (see Remark 3.3.5).

Definition 3.3.1. Suppose $F^* \in \mathcal{F}(r)$ and $\mathcal{A}$ is an apartment in $\mathcal{B}(G)$. We define $A(\mathcal{A}, F^*) := A(F^* \cap \mathcal{A}, \mathcal{A})$.

Definition 3.3.2. Two generalized r-facets F_1^* and F_2^* are *strongly r-associated* if for all apartments $\mathcal{A}$ such that $F_1^* \cap \mathcal{A} \neq \emptyset$ and $F_2^* \cap \mathcal{A} \neq \emptyset$, we have

$$A(\mathcal{A}, F_1^*) = A(\mathcal{A}, F_2^*).$$

LEMMA 3.3.3. *Two generalized r-facets $F_1^*, F_2^* \in \mathcal{F}(r)$ are strongly r-associated if and only if there exists an apartment $\mathcal{A}$ such that $\emptyset \neq A(\mathcal{A}, F_1^*) = A(\mathcal{A}, F_2^*)$.*

Proof. "$\Rightarrow$": This follows from the definition.

"$\Leftarrow$": Choose $x_i \in C(F_i^*)$ for $i = 1, 2$. Recall that for an apartment $\mathcal{A}'$ of $\mathcal{B}(G)$ we have $\mathcal{A}' \cap F_i^* \neq \emptyset$ if and only if $x_i \in \mathcal{A}'$. Suppose $\mathcal{A}' \cap F_1^* \neq \emptyset$ and $\mathcal{A}' \cap F_2^* \neq \emptyset$. There exists a $g \in G$ such that g fixes $\mathcal{A} \cap \mathcal{A}'$ point-wise and $g\mathcal{A} = \mathcal{A}'$. Thus $gx_1 = x_1$ and $gx_2 = x_2$. This implies that $g \in \mathrm{stab}_G(F_1^*) \cap \mathrm{stab}_G(F_2^*)$ and

$$\begin{aligned} A(\mathcal{A}', F_1^*) &= A(g\mathcal{A}, gF_1^*) = gA(\mathcal{A}, F_1^*) \\ &= gA(\mathcal{A}, F_2^*) = A(g\mathcal{A}, gF_2^*) \\ &= A(\mathcal{A}', F_2^*). \end{aligned}$$

□

Definition 3.3.4. Two generalized r-facets F_1^* and F_2^* are *r-associated* if there exists a $g \in G$ such that F_1^* and gF_2^* are strong r-associates.

Remark 3.3.5. If $F_1^*, F_2^* \in \mathcal{F}(0)$ are 0-associated, then the parahoric subgroups $G_{F_1^*,0}$ and $G_{F_2^*,0}$ are associate in the sense of [22].

Example 3.3.6. In Figure 1, we have represented a 0-alcove in the building of $\mathbf{SL}_3(k)$ (resp., $\mathbf{G}_2(k)$). The edges identified with hatch marks are 0-associates; none of the remaining pictured 0-facets are 0-associated.

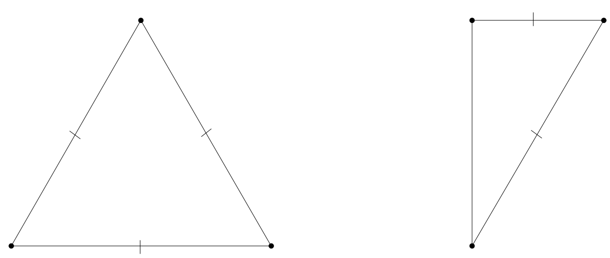

Figure 1. Associates in 0-alcoves for $\mathbf{SL}_3(k)$ (resp., $\mathbf{G}_2(k)$).

LEMMA 3.3.7. *r-associativity is an equivalence relation on $\mathcal{F}(r)$.*

Proof. For two generalized r-facets F_1^* and F_2^*, we write $F_1^* \sim F_2^*$ if and only if F_1^* and F_2^* are r-associated. The relation is reflexive and symmetric. We now show that it is transitive.

Suppose $F_1^*, F_2^*, F_3^* \in \mathcal{F}(r)$ such that $F_1^* \sim F_2^*$ and $F_2^* \sim F_3^*$. There exist $g_2, g_3 \in G$ and apartments $\mathcal{A}_{12}, \mathcal{A}_{23}$ in $\mathcal{B}(G)$ such that

$$A(\mathcal{A}_{12}, F_1^*) = A(\mathcal{A}_{12}, g_2 F_2^*) \neq \emptyset$$

and

$$A(\mathcal{A}_{23}, F_2^*) = A(\mathcal{A}_{23}, g_3 F_3^*) \neq \emptyset.$$

Let $z \in C(F_2^*)$. Then $z \in g_2^{-1}\mathcal{A}_{12} \cap \mathcal{A}_{23}$ and so there exists $h \in G_z \subset \mathrm{stab}_G(F_2^*)$ such that $hg_2^{-1}\mathcal{A}_{12} = \mathcal{A}_{23}$. We have

$$\begin{aligned} A(\mathcal{A}_{12}, F_1^*) &= A(\mathcal{A}_{12}, g_2 F_2^*) = g_2 A(g_2^{-1}\mathcal{A}_{12}, F_2^*) \\ &= g_2 h^{-1} A(\mathcal{A}_{23}, F_2^*) = g_2 h^{-1} A(\mathcal{A}_{23}, g_3 F_3^*) \\ &= A(\mathcal{A}_{12}, g_2 h^{-1} g_3 F_3^*). \end{aligned}$$

□

Remark 3.3.8. $\mathcal{F}(r)/\sim$ is finite.

3.4. *Some finite-dimensional vector spaces.*

Definition 3.4.1. For $x \in \mathcal{B}(G)$ denote the finite-dimensional $\mathfrak{f}$-vector space $\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+}$ by $V_{x,r}$.

Definition 3.4.2. If $F^* \in \mathcal{F}(r)$ and $x \in F^*$, then $V_{F^*} := V_{x,r}$.

Definition 3.4.3. If $\mathcal{A}$ is an apartment in $\mathcal{B}(G)$, $F_{\mathcal{A}}$ is an r-facet of $\mathcal{A}$, and $x \in F_{\mathcal{A}}$, then $V_{F_{\mathcal{A}}} := V_{x,r}$.

3.5. *A natural identification.* In this subsection we show that if $F_1^*, F_2^* \in \mathcal{F}(r)$ are strongly r-associated, then we can naturally identify $V_{F_1^*}$ with $V_{F_2^*}$. Moreover, we show that these two spaces have the same orbit structure under this identification.

LEMMA 3.5.1. *If $F_1^*, F_2^* \in \mathcal{F}(r)$ are strongly r-associated, then the natural map*

$$\mathfrak{g}_{F_1^*} \cap \mathfrak{g}_{F_2^*} \to V_{F_i^*}$$

is surjective with kernel $\mathfrak{g}^+_{F_1^*} \cap \mathfrak{g}_{F_2^*} = \mathfrak{g}_{F_1^*} \cap \mathfrak{g}^+_{F_2^*} = \mathfrak{g}^+_{F_1^*} \cap \mathfrak{g}^+_{F_2^*}$.

Proof. Choose an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ for which $F_{i,\mathcal{A}} = F_i^* \cap \mathcal{A} \neq \emptyset$ for $i = 1, 2$. If $\psi \in \Psi(\mathcal{A})$ such that $\psi|_{F_{i,\mathcal{A}}} = r$, then $A(\mathcal{A}, F_i^*) \subset H_{\psi - r} \subset \mathcal{A}$. Thus, since $F_{1,\mathcal{A}}$ and $F_{2,\mathcal{A}}$ are open in $A(\mathcal{A}, F_2^*) = A(\mathcal{A}, F_1^*)$, we have

$$\psi|_{F_{1,\mathcal{A}}} = r \text{ if and only if } \psi|_{F_{2,\mathcal{A}}} = r$$

for all $\psi \in \Psi(\mathcal{A})$. The lemma follows. □

Remark 3.5.2. From Lemma 3.5.1, we obtain a bijective identification of $V_{F_1^*}$ with $V_{F_2^*}$. We write

$$V_{F_1^*} \stackrel{i}{=} V_{F_2^*}$$

for this identification. More generally, we will use the "$\stackrel{i}{=}$" notation whenever two objects are to be identified via this natural bijection.

Definition 3.5.3. If $F^* \in \mathcal{F}(r)$ and $x \in F^*$, then the image of G_x in $\mathrm{Aut}_{\mathfrak{f}}(V_{F^*})$ is denoted by $N_x(F^*)$.

LEMMA 3.5.4. *Suppose $F_i^* \in \mathcal{F}(r)$ and $x_i \in F_i^*$ for $i = 1, 2$. If F_1^* and F_2^* are strongly r-associated, then $N_{x_i}(F_i^*)$ is the image of $G_{x_1} \cap G_{x_2}$ in $\mathrm{Aut}_{\mathfrak{f}}(V_{F_i^*})$ for $i = 1, 2$. Moreover*

$$N_{x_1}(F_1^*) \stackrel{i}{=} N_{x_2}(F_2^*)$$

under the identification of $V_{F_1^*}$ with $V_{F_2^*}$ introduced above.

Proof. We have that $V_{F_i^*}$ is the image of $\mathfrak{g}_{x_1,r} \cap \mathfrak{g}_{x_2,r}$ in $V_{F_i^*} (= \mathfrak{g}_{x_i,r}/\mathfrak{g}_{x_i,r^+})$ for $i = 1, 2$. Let $\mathcal{A}$ be an apartment in $\mathcal{B}(G)$ containing x_1 and x_2. Suppose $\psi \in \Psi(\mathcal{A})$ such that $\psi(x_1) = 0$ and the image of U_ψ in $\mathrm{Aut}_{\mathfrak{f}}(V_{F_1^*})$ is nontrivial. Since the image of U_ψ is nontrivial, there exist $X \in \mathfrak{g}_{x_1,r} \cap \mathfrak{g}_{x_2,r}$ and $g \in U_\psi$ such that ${}^gX \neq X \bmod (\mathfrak{g}_{x_1,r^+} \cap \mathfrak{g}_{x_2,r^+})$.

We now show that $\psi(x_2) = 0$.

If $\psi(x_2) > 0$, then ${}^gX = X \bmod \mathfrak{g}_{x_2,r^+}$. Since ${}^gX - X \in \mathfrak{g}_{x_1,r}$, from Lemma 3.5.1 we have ${}^gX = X \bmod (\mathfrak{g}_{x_1,r^+} \cap \mathfrak{g}_{x_2,r^+})$. We therefore conclude that $\psi(x_2) \leq 0$.

If $\psi(x_2) < 0$, then we let v denote the vector $(x_2 - x_1)$. For all $\varepsilon \in \mathbb{R}$ we have $x_1 + \varepsilon \cdot v \in \mathcal{A}$. Consider the function

$$f_v : \mathbb{R} \to \mathbb{R}$$

which sends ε to $\psi(x_1 + \varepsilon \cdot v)$. Note that $f_v(0) = 0$ and $f_v(1) < 0$. Since ψ is affine, we have that $f_v(\varepsilon) > 0$ for all $\varepsilon < 0$. Since $F_1^* \cap \mathcal{A}$ is open in $A(\mathcal{A}, F_1^*) = A(\mathcal{A}, F_2^*)$ and $x_1 + \mathbb{R} \cdot v$ is an affine subspace of $A(\mathcal{A}, F_1^*)$, there exists an $\varepsilon < 0$ such that $x_1 + \varepsilon \cdot v \in F_1^* \cap \mathcal{A}$. Thus for some $\varepsilon < 0$ we have $x_1 + \varepsilon \cdot v \in F_1^*$ and $U_\psi \subset G^+_{x_1+\varepsilon\cdot v}$. Consequently, $g \in U_\psi$ acts trivially on $\mathfrak{g}_{F_1^*} \bmod \mathfrak{g}^+_{F_1^*}$.

We therefore conclude that $\psi(x_2) = 0$. Thus, if $g \in G_{x_1}$ has nontrivial image in $\mathrm{Aut}_{\mathfrak{f}}(V_{F_1^*})$, then it follows that we may assume that $g \in G_{x_1} \cap G_{x_2}$. The remainder of the lemma now follows. ∎

From Lemma 3.5.4 we have that if $F^* \in \mathcal{F}(r)$ and $x, y \in F^*$, then $N_x(F^*) = N_y(F^*)$. Therefore, the following definition makes sense.

Definition 3.5.5. If $F^* \in \mathcal{F}(r)$ and $x \in F^*$, then define $N(F^*) \subset \mathrm{Aut}_{\mathfrak{f}}(V_{F^*})$ by

$$N(F^*) := N_x(F^*).$$

We can now restate Lemma 3.5.4.

COROLLARY 3.5.6. *If $F_1^*, F_2^* \in \mathcal{F}(r)$ are strongly r-associated, then $N(F_1^*) \stackrel{i}{=} N(F_2^*)$.*

3.6. *An equivalence relation on depth r cosets.* In this subsection we introduce the set I_r and an equivalence relation on I_r. The set I_r parametrizes the set of all cosets of the form $X + \mathfrak{g}_{x,r^+}$ where $x \in \mathcal{B}(G)$ and $X \in \mathfrak{g}_{x,r}$.

Definition 3.6.1.

$$I_r := \{(F^*, v) \,|\, F^* \in \mathcal{F}(r) \text{ and } v \in V_{F^*}\}.$$

We now introduce a relation on I_r. Roughly speaking, two elements (F_1^*, v_1) and (F_2^*, v_2) of I_r are identified if (1) F_1^* and F_2^* are r-associated, and (2) v_1 can then be identified with a twist of v_2 (under the natural identification of the previous subsection).

Definition 3.6.2. For (F_1^*, v_1) and (F_2^*, v_2) in I_r we write $(F_1^*, v_1) \sim (F_2^*, v_2)$ if and only if there exist a $g \in G$ and an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ such that

1. $\emptyset \neq A(\mathcal{A}, F_1^*) = A(\mathcal{A}, gF_2^*)$ and
2. ${}^g v_2 \stackrel{i}{=} v_1$ in $V_{gF_2^*} \stackrel{i}{=} V_{F_1^*}$.

Here ${}^g v_2$ has the obvious interpretation: if $X_2 \in \mathfrak{g}_{F_2^*}$ is any lift of v_2, then ${}^g v_2$ denotes the image of ${}^g X_2$ in $V_{gF_2^*}$.

LEMMA 3.6.3. *The relation defined in Definition* 3.6.2 *is an equivalence relation on I_r.*

Proof. The relation is reflexive. We will show the relation is transitive; once we do this, one can prove that the relation is symmetric in a similar fashion.

We now show that the relation is transitive. Suppose that $(F_1^*, v_1), (F_2^*, v_2), (F_3^*, v_3) \in I_r$ such that $(F_1^*, v_1) \sim (F_2^*, v_2)$ and $(F_2^*, v_2) \sim (F_3^*, v_3)$. Then there exist $g_2, g_3 \in G$ and apartments $\mathcal{A}_{12}, \mathcal{A}_{23}$ of $\mathcal{B}(G)$ such that

$$\emptyset \neq A(\mathcal{A}_{12}, F_1^*) = A(\mathcal{A}_{12}, g_2 F_2^*)$$
$$\emptyset \neq A(\mathcal{A}_{23}, F_2^*) = A(\mathcal{A}_{23}, g_3 F_3^*),$$

and

$$ {}^{g_2}v_2 \overset{i}{=} v_1 \quad \text{in} \quad V_{g_2F_2^*} \overset{i}{=} V_{F_1^*} $$

$$ {}^{g_3}v_3 \overset{i}{=} v_2 \quad \text{in} \quad V_{g_3F_3^*} \overset{i}{=} V_{F_2^*}. $$

We now wish to show that $(F_1^*, v_1) \sim (F_3^*, v_3)$. We claim that in Definition 3.6.2 the role of the pair $(g,\mathcal{A})$ will be played by $(h''h^{-1}g_2g_3,\ \mathcal{A}_{12})$ where $h \in G_{g_2x_2}$ and $h'' \in G_{h^{-1}g_2g_3x_3} \cap G_{h^{-1}g_2x_2}$ will be specified below.

Fix $x_i \in C(F_i^*)$. There exists an element $h \in G_{g_2x_2}$ such that $h\mathcal{A}_{12} = g_2\mathcal{A}_{23}$. As in the proof of Lemma 3.3.7 we have

$$
\begin{aligned}
\emptyset \neq A(\mathcal{A}_{12}, F_1^*) &= A(\mathcal{A}_{12}, g_2F_2^*) = h^{-1}A(h\mathcal{A}_{12}, hg_2F_2^*) \\
&= h^{-1}g_2A(\mathcal{A}_{23}, F_2^*) = h^{-1}g_2A(\mathcal{A}_{23}, g_3F_3^*) \\
&= A(\mathcal{A}_{12}, h^{-1}g_2g_3F_3^*).
\end{aligned}
$$

Arguing as in the proof of Lemma 3.5.1 we have that $\mathfrak{g}_{F_1^*} \cap \mathfrak{g}_{g_2F_2^*} \cap \mathfrak{g}_{h^{-1}g_2g_3F_3^*}$ surjects, under the natural map, onto $V_{F_1^*}$ (resp., $V_{g_2F_2^*}$, resp., $V_{h^{-1}g_2g_3F_3^*}$). Choose $X \in \mathfrak{g}_{F_1^*} \cap \mathfrak{g}_{g_2F_2^*} \cap \mathfrak{g}_{h^{-1}g_2g_3F_3^*}$ such that the image of X in $V_{F_1^*}$ is v_1.

We have that the image of X in $V_{g_2F_2^*}$ is ${}^{g_2}v_2$. Thus the image of ${}^{g_2^{-1}}X$ in $V_{F_2^*}$ is v_2. Since $g_2^{-1}hg_2 \in G_{x_2}$, this implies that the image of ${}^{g_2^{-1}h}X = {}^{(g_2^{-1}hg_2)g_2^{-1}}X$ in $V_{F_2^*}$ is ${}^{g_2^{-1}hg_2}v_2$. Note that ${}^{g_2^{-1}h}X \in \mathfrak{g}_{F_2^*} \cap \mathfrak{g}_{g_3F_3^*}$. Recall from Corollary 3.5.6 that $N(F_2^*) \overset{i}{=} N(g_3F_3^*)$. Thus, from Lemma 3.5.4 there exists an $h' \in G_{g_3x_3} \cap G_{x_2}$ such that

$$ {}^{g_2^{-1}hg_2}v_2 \overset{i}{=} {}^{h'g_3}v_3 \quad \text{in} \quad V_{F_2^*} \overset{i}{=} V_{g_3F_3^*}. $$

Thus, the image of X in $V_{h^{-1}g_2g_3F_3^*}$ is

$$ {}^{h^{-1}g_2h'g_3}v_3 = {}^{h^{-1}(g_2h'g_2^{-1})g_2g_3}v_3 = {}^{h''h^{-1}g_2g_3}v_3 $$

where $h'' \in h^{-1}g_2(G_{g_3x_3} \cap G_{x_2})g_2^{-1}h \subset G_{h^{-1}g_2g_3x_3}$. We have shown that

$$ \emptyset \neq A(\mathcal{A}_{12}, F_1^*) = A(\mathcal{A}_{12}, h^{-1}g_2g_3F_3^*) = A(\mathcal{A}_{12}, h''h^{-1}g_2g_3F_3^*) $$

and

$$ v_1 \overset{i}{=} {}^{h''h^{-1}g_2g_3}v_3 \quad \text{in} \quad V_{F_1^*} \overset{i}{=} V_{h''h^{-1}g_2g_3F_3^*}. $$

So the relation is transitive. □

Remark 3.6.4. If $\mathfrak{f}$ is finite, then $I_r/\!\sim$ is finite.

4. Jacobson-Morosov triples over $\mathfrak{f}$ and k

Fix $r \in \mathbb{R}$. Much of the material in this section may be thought of as a generalization of the material in [6, §3].

In Section 4.3, we start with an $x \in \mathcal{B}(G)$ and an $\mathfrak{sl}_2(\mathfrak{f})$-triple in $V_{x,-r} \times V_{x,0} \times V_{x,r}$. From this data we manufacture an $\mathfrak{sl}_2(k)$-triple in $\mathfrak{g}$ which descends to our $\mathfrak{sl}_2(\mathfrak{f})$-triple.

In Section 4.5 we perform this process in reverse. That is, we start with an $\mathfrak{sl}_2(k)$-triple in $\mathfrak{g}$ and produce an $x \in \mathcal{B}(G)$ such that our given $\mathfrak{sl}_2(k)$-triple descends to an $\mathfrak{sl}_2(\mathfrak{f})$-triple in $V_{x,-r} \times V_{x,0} \times V_{x,r}$.

4.1. *Degenerate cosets.*

Definition 4.1.1. Suppose $F^* \in \mathcal{F}(r)$. An element $e \in V_{F^*}$ is *degenerate* if and only if there exists a lift $E \in \mathfrak{g}_{F^*}$ of e such that $E \in \mathcal{N}$.

LEMMA 4.1.2 (Moy and Prasad). *Fix $F^* \in \mathcal{F}(r)$. An element $e \in V_{F^*}$ is degenerate if and only if zero is in the Zariski closure of $^{\mathsf{G}_x}e$ for all $x \in F^*$.*

Proof. "$\Rightarrow$": Fix $x \in F^*$. Suppose $E \in \mathfrak{g}_{x,r} \cap \mathcal{N}$ is a lift of e. The desired conclusion follows from [22, Proposition 4.3].

"$\Leftarrow$": This may also be derived from [22]. We offer a slightly different proof.

We need to produce an $E \in \mathcal{N} \cap \mathfrak{g}_{F^*}$ such that E is a lift of e.

Fix $x \in F^*$. Let $\mathbf{S}$ be a maximal k-split torus of $\mathbf{G}$ such that $x \in \mathcal{A}\Big(\mathbf{S}(k)\Big)$. From [18] there exists a one-parameter subgroup $\bar{\nu} \in \mathbf{X}_*^{\mathfrak{f}}(\mathsf{G}_x)$ such that $\lim_{t\to 0} {}^{\bar{\nu}(t)}e = 0$. Let S be the maximal $\mathfrak{f}$-split torus of G_x corresponding to $\mathbf{S}$. Since maximal $\mathfrak{f}$-split tori are $\mathsf{G}_x(\mathfrak{f})$-conjugate, there exist $\bar{\mu} \in \mathbf{X}_*(\mathsf{S})$ and $\bar{g} \in \mathsf{G}_x(\mathfrak{f})$ such that $\lim_{t\to 0} {}^{\bar{\mu}(t)\bar{g}}e = 0$. Let $\mu \in \mathbf{X}_*(\mathbf{S})$ be the lift of $\bar{\mu}$ and let $g \in G_x$ be a lift of $\bar{g}$. Let $E' \in \mathfrak{g}_{x,r} = \mathfrak{g}_{F^*}$ be any lift of e. We have ${}^g(E' + \mathfrak{g}_{x,r^+}) = {}^g(E' + \mathfrak{g}_{F^*}^+) \subset \mathfrak{g}_{x+\varepsilon\cdot\mu,r^+}$ for all ε sufficiently small and positive. Consequently, from [2] we have $(E' + \mathfrak{g}_{F^*}^+) \cap \mathcal{N} \neq \emptyset$. Choose E in this intersection. $\square$

4.2. *Some hypotheses.* The statements below list properties which I require; no attempt has been made to produce a minimal list of hypotheses. If we assume that p is larger than some constant which can be determined by examining the absolute root datum of $\mathbf{G}^\circ$, then all of the hypotheses are valid. In particular, if $\mathfrak{f}$ has characteristic zero, then the following hypotheses always hold. Where appropriate, I have identified references where a discussion about the conditions under which the hypothesis is valid may be found.

We begin by defining a finite-dimensional $\mathfrak{f}$-Lie algebra $\overline{\mathfrak{g}}_x$. Since we have fixed a uniformizer ϖ for k, for $s \in \mathbb{R}$ and $j \in \mathbb{Z}$ we have a natural identification of $V_{x,s}$ with $V_{x,s+j\cdot\ell}$. With respect to this identification, we define

$$\overline{\mathfrak{g}}_x := \bigoplus_{s \in \mathbb{R}/\ell\cdot\mathbb{Z}} V_{x,s}.$$

Note that $\dim_{\mathfrak{f}} \overline{\mathfrak{g}}_x = \dim_k \mathfrak{g}$. We define a product operation on $\overline{\mathfrak{g}}_x$ in the following manner. If $\overline{X}_s \in V_{x,s}$ and $\overline{X}_t \in V_{x,t}$, then we define $[\overline{X}_s, \overline{X}_t]$ to be the image of $[X_s, X_t] \in \mathfrak{g}_{x,(s+t)}$ in $V_{x,(s+t)}$ where $X_s \in \mathfrak{g}_{x,s}$ and $X_t \in \mathfrak{g}_{x,t}$ are any lifts of $\overline{X}_t$ and $\overline{X}_s$, respectively. Linearly extend this operation to an operation on $\overline{\mathfrak{g}}_x$. With this product $\overline{\mathfrak{g}}_x$ is an $\mathfrak{f}$-Lie algebra. For $v \in \overline{\mathfrak{g}}_x$, define $\mathrm{ad}(v) \in \mathrm{End}_{\mathfrak{f}}\left(\overline{\mathfrak{g}}_x\right)$ by $\mathrm{ad}(v)w = [v, w]$ for all $w \in \overline{\mathfrak{g}}_x$.

For more information about Hypothesis 4.2.1, see Appendix A.

Hypothesis 4.2.1. *Suppose $x \in \mathcal{B}(G)$. If $X \in \mathcal{N} \cap (\mathfrak{g}_{x,r} \smallsetminus \mathfrak{g}_{x,r^+})$, then there exist $H \in \mathfrak{g}_{x,0}$ and $Y \in \mathfrak{g}_{x,-r}$ such that*

$$\begin{aligned} [H, X] &= 2X \ \mathit{mod}\ \mathfrak{g}_{x,r^+} \\ [H, Y] &= -2Y \ \mathit{mod}\ \mathfrak{g}_{x,(-r)^+} \\ [X, Y] &= H \ \mathit{mod}\ \mathfrak{g}_{x,0^+}. \end{aligned}$$

If (f, h, e) denotes the image of (Y, H, X) in $V_{x,-r} \times V_{x,0} \times V_{x,r} \subset \overline{\mathfrak{g}}_x$, then (f, h, e) is an $\mathfrak{sl}_2(\mathfrak{f})$-triple, and $\overline{\mathfrak{g}}_x$ decomposes into a direct sum of irreducible (f, h, e)-modules of highest weight at most $(p-3)$. Moreover, there exists $\bar{\lambda} \in X_*^{\mathfrak{f}}(\mathsf{G}_x)$, uniquely determined up to an element of $\mathbf{X}_*(\mathsf{Z}_x)$ whose differential is zero, such that the following two conditions hold.

1. *The image of $d\bar{\lambda}$ in $\mathrm{Lie}(\mathsf{G}_x)$ coincides with the one-dimensional subspace spanned by h.*

2. *Suppose $i \in \mathbb{Z}$. For $v \in \overline{\mathfrak{g}}_x$*

$$\text{if } {}^{\bar{\lambda}(t)}v = t^i v, \text{ then } |i| \leq (p-3) \text{ and } \mathrm{ad}(h)v = iv.$$

Definition 4.2.2. In the notation of Hypothesis 4.2.1, we say that $\bar{\lambda} \in \mathbf{X}_*^{\mathfrak{f}}(\mathsf{G}_x)$ is *adapted* to the $\mathfrak{sl}_2(\mathfrak{f})$-triple obtained from the image of (Y, H, X) in $V_{x,-r} \times V_{x,0} \times V_{x,r}$.

Hypothesis 4.2.3. *If $X \in \mathcal{N}$, then there exists $m \in \mathbb{N}$ with $m \leq (p-2)$ such that* $\mathrm{ad}(X)^m = 0$.

For more background on the next hypothesis see, for example, [10, §5.5].

HYPOTHESIS 4.2.4. *Choose $m \in \mathbb{N}$ such that $\mathrm{ad}(X)^m = 0$ for all $X \in \mathcal{N}$. Suppose either that k has characteristic zero or that the characteristic of k is greater than m. There exists a unique G-invariant map $\exp_t\colon \mathcal{N} \to \mathcal{U}$ defined over k such that for all $X \in \mathcal{N}$ the adjoint action of $\exp_t(X)$ on $\mathfrak{g}$ is given by*

$$\sum_{i=0}^{m} \frac{\left(\mathrm{ad}(X)\right)^i}{i!}.$$

For more information about Hypothesis 4.2.5 see [10, §5.5].

HYPOTHESIS 4.2.5. *Suppose Hypothesis 4.2.4 is valid. Suppose $X \in \mathcal{N}$. There exists an $\mathfrak{sl}_2(k)$-triple completing X. For any $\mathfrak{sl}_2(k)$-triple (Y, H, X) completing X there is a group homomorphism $\varphi\colon \mathbf{SL}_2 \to \mathbf{G}$ defined over k such that $d\varphi\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = X$, $d\varphi\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) = Y$, $d\varphi\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = H$, and for all $t \in k$*

1. $\varphi\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) = \exp_t(tX)$ *and*
2. $\varphi\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right) = \exp_t(tY)$.

Finally, any two $\mathfrak{sl}_2(k)$-triples completing X are conjugate by an element of $C_G(X)$.

Remark 4.2.6. We note that the map φ occurring in Hypothesis 4.2.5 is uniquely determined by $d\varphi$.

For more information about Hypothesis 4.2.7, see [1, §1.6].

HYPOTHESIS 4.2.7. *Suppose $x \in \mathcal{B}(G)$. For all $s \in \mathbb{R}_{>0}$ and for all $t \in \mathbb{R}$ there exists a map $\phi_x\colon \mathfrak{g}_{x,s} \to G_{x,s}$ such that for $V \in \mathfrak{g}_{x,s}$ and $W \in \mathfrak{g}_{x,t}$ we have*

$$^{\phi_x(V)}W = W + [V, W] \mod \mathfrak{g}_{x,(s+t)^+}.$$

4.3. *From Jacobson-Morosov triples over $\mathfrak{f}$ to Jacobson-Morosov triples over k.* Fix $x \in \mathcal{B}(G)$. Suppose that $(f, h, e) \in V_{x,-r} \times V_{x,0} \times V_{x,r} \subset \bar{\mathfrak{g}}_x$ is a (nontrivial) $\mathfrak{sl}_2(\mathfrak{f})$-triple with adapted $\bar{\mu} \in \mathbf{X}_*^{\mathfrak{f}}(\mathsf{G}_x)$. We now show that, subject to some conditions on k and $\mathbf{G}$, there exist $Y \in \mathfrak{g}_{x,-r}$, $H \in \mathfrak{g}_{x,0}$ and $X \in \mathfrak{g}_{x,r}$ such that (Y, H, X) is an $\mathfrak{sl}_2(k)$-triple in $\mathfrak{g}$ and (Y, H, X) is a lift of (f, h, e) in the obvious sense. We follow [6, §§3.8–3.9] where the proof is carried out for certain $\mathbf{G}$ when $r = 0$.

Let $\mathbf{S}$ be a maximal k-split torus of $\mathbf{G}$ such that $x \in \mathcal{A}(\mathbf{S}, k)$. Let S be the maximal $\mathfrak{f}$-split torus of G_x corresponding to $\mathbf{S}$. Since maximal $\mathfrak{f}$-split tori are $\mathsf{G}_x(\mathfrak{f})$-conjugate, there exist $\bar{\lambda} \in \mathbf{X}_*(\mathsf{S})$ and $\bar{g} \in \mathsf{G}_x(\mathfrak{f})$ such that $\bar{\lambda} = {}^{\bar{g}}\bar{\mu}$. Let $\lambda \in \mathbf{X}_*(\mathbf{S})$ be the lift of $\bar{\lambda}$ and replace (f, h, e) with $({}^{\bar{g}}f, {}^{\bar{g}}h, {}^{\bar{g}}e)$.

For $i \in \mathbb{Z}$, define

$$\mathfrak{g}(i) := \{Z \in \mathfrak{g} \mid {}^{\lambda(t)}Z = t^i \cdot Z\} \text{ and } \bar{\mathfrak{g}}_x(i) := \{v \in \bar{\mathfrak{g}}_x \mid {}^{\bar{\lambda}(t)}v = t^i \cdot v\}.$$

For $i \in \mathbb{Z}$ and $s \in \mathbb{R}$ define

$$\mathfrak{g}_{x,s}(i) := \{Z \in \mathfrak{g}_{x,s} \mid {}^{\lambda(t)}Z = t^i \cdot Z\} \text{ and } V_{x,s}(i) := \{v \in V_{x,s} \mid {}^{\bar{\lambda}(t)}v = t^i \cdot v\}.$$

Because $x \in \mathcal{A}(\mathbf{S}, k)$, $\lambda \in \mathbf{X}_*(\mathbf{S})$, and $\bar{\lambda} \in \mathbf{X}_*(\mathsf{S})$ we have

$$\mathfrak{g}_{x,s} = \bigoplus_i \mathfrak{g}_{x,s}(i) \text{ and } V_{x,s} = \bigoplus_i V_{x,s}(i)$$

for $s \in \mathbb{R}$ and $i \in \mathbb{Z}$.

Lemma 4.3.1. *Suppose that Hypothesis* 4.2.1 *holds. If $X \in \mathfrak{g}_{x,r}(2)$ is any lift of e, then for all $s \in \mathbb{R}$, the map*

$$\mathrm{ad}(X)^2 \colon \mathfrak{g}_{x,s-r}(-2) \to \mathfrak{g}_{x,s+r}(2)$$

is an isomorphism.

Proof. Fix $X \in \mathfrak{g}_{x,r}(2)$ which is a lift of e. Note that X is nilpotent. Since $\mathrm{ad}(X)^2$ takes $\mathfrak{g}(-2)$ to $\mathfrak{g}(2)$ and k is complete, it will be sufficient to show that for all $t \in \mathbb{R}$, the map

$$\mathrm{ad}(e)^2 \colon V_{x,t-r}(-2) \to V_{x,t+r}(2)$$

is an isomorphism.

From Hypothesis 4.2.1 we have that the space $\overline{\mathfrak{g}}_x$ is a direct sum of irreducible (f, h, e)-modules. Consequently, it follows from $\mathfrak{sl}_2(\mathfrak{f})$-representation theory that the map

$$\mathrm{ad}(e)^2 \colon \overline{\mathfrak{g}}_x(-2) \to \overline{\mathfrak{g}}_x(2)$$

is an isomorphism. The result follows. □

Corollary 4.3.2. *Suppose that Hypotheses* 4.2.1 *and* 4.2.3 *hold. If $X \in \mathfrak{g}_{x,r}(2)$ is a lift of e, then there exist lifts $Y \in \mathfrak{g}_{x,-r}$ of f and $H \in \mathfrak{g}_{x,0}$ of h such that (Y, H, X) is an $\mathfrak{sl}_2(k)$-triple in $\mathfrak{g}$.*

Proof. Fix $X \in \mathfrak{g}_{x,r}(2)$ which is a lift of e. Since $\mathrm{ad}(X)^2 \colon \mathfrak{g}_{x,-r}(-2) \to \mathfrak{g}_{x,r}(2)$ is a surjection there exists $Y \in \mathfrak{g}_{x,-r}(-2)$ such that $\mathrm{ad}(X)^2 Y = -2X$. Since $\mathrm{ad}(e)^2 \colon \overline{\mathfrak{g}}_x(-2) \to \overline{\mathfrak{g}}_x(2)$ is injective and $\mathrm{ad}(e)^2 f = -2e$, Y is necessarily a lift of f. Let $H = [X, Y] \in \mathfrak{g}_{x,0}(0)$. H is a lift of h. We also have $[H, X] = 2X$. We need to check that $[H, Y] = -2Y$.

It follows from Hypothesis 4.2.3 and Morosov's theorem (see [17, Lemma 7, p. 98] or [10, the proof of Proposition 5.3.1; in particular, pp. 140–141]) that there exists $Y' \in \mathfrak{g}$ such that (Y', H, X) is an $\mathfrak{sl}_2(k)$-triple. Since $H \in \mathfrak{g}(0)$, $X \in \mathfrak{g}(2)$, and $[\mathfrak{g}(i), \mathfrak{g}(j)] \subset \mathfrak{g}(i+j)$, we can assume that $Y' \in \mathfrak{g}(-2)$. However, since $\mathrm{ad}(X)^2 \colon \mathfrak{g}_{x,s-r}(-2) \to \mathfrak{g}_{x,s+r}(2)$ is injective for all $s \in \mathbb{R}$, we must have $Y' = Y$. □

4.4. *Some fixed-point results for one-parameter subgroups.* Fix $\lambda \in \mathbf{X}_*^k(\mathbf{G})$. Let $\mathbf{M}$ denote the k-Levi subgroup of $\mathbf{G}$ whose group of k-rational points is the Levi subgroup $M = C_{G^\circ}(\lambda)$. Note that for all $z \in \mathcal{B}(M)$ we have $d\lambda(R^\times) \subset \mathfrak{g}_{z,0} \smallsetminus \mathfrak{g}_{z,0^+}$.

The following lemma and its subsequent applications in Corollaries 4.4.2 and 4.4.3 arose from discussions with Gopal Prasad. The results of this subsection may be thought of as natural generalizations of the material in [24, §3.6].

LEMMA 4.4.1. *Suppose F is a codimension one 0-facet in $\mathcal{B}(\mathbf{M}, K)$. Either every 0-alcove of $\mathcal{B}(\mathbf{G}, K)$ which contains F in its closure belongs to $\mathcal{B}(\mathbf{M}, K)$ or exactly two of the 0-alcoves which contain F in their closure lie in $\mathcal{B}(\mathbf{M}, K)$.*

Proof. Since $F \subset \mathcal{B}(\mathbf{M}, K)$, there exists a maximal K-split torus $\mathbf{T}$ such that $\mathbf{T} \subset \mathbf{M}$ and $F \subset \mathcal{A}(\mathbf{T}, K)$. Since $\lambda \in \mathbf{X}_*(\mathbf{T})$, we can consider the image $\bar{\lambda}$ of λ in $\mathbf{X}_*(\mathsf{G}_F)$. Let C_1 and C_2 denote the 0-alcoves in $\mathcal{A}(\mathbf{T}, K)$ which contain F in their closure. Let B denote the Borel subgroup of G_F corresponding to C_1.

First suppose that $\bar{\lambda}$ lies in the center of G_F. Since every Borel subgroup of G_F is conjugate to B by an element of G_F, we conclude that every 0-alcove in $\mathcal{B}(\mathbf{G}, K)$ which contains F in its closure is conjugate to C_1 by an element of $\mathbf{G}(K)_F \cap \mathbf{M}(K)$.

Now suppose that $\bar{\lambda}$ does not lie in the center of G_F. Since the derived group of G_F is either $\mathbf{SL}_2$ or $\mathbf{PGL}_2$, it follows that there are exactly two Borel subgroups in G_F containing $\bar{\lambda}$. These Borel subgroups correspond to C_1 and C_2. □

COROLLARY 4.4.2. *Suppose that $\mathfrak{f}$ has more than three elements. We have*

$$\mathcal{B}(M) = \mathcal{B}(G)^{\lambda(R^\times)}.$$

Proof. "$\subset$": Since any natural embedding of $\mathcal{B}(M)$ into $\mathcal{B}(G)$ is M-equivariant, this follows from the fact that $\lambda(R^\times)$ fixes $\mathcal{B}(M)$ point-wise.

"$\supset$": Suppose $\mathcal{B}(M) \subsetneq \mathcal{B}(G)^{\lambda(R^\times)}$. We will obtain a contradiction.

Since the group $\lambda(R^\times)$ fixes $\mathcal{B}(\mathbf{M}, K)$, there exists a 0-alcove C in $\mathcal{B}(\mathbf{G}, K)$ such that $\bar{C} \cap \mathcal{B}(\mathbf{M}, K)$ has codimension one and $\lambda(R^\times)$ fixes $\bar{C}$.

Choose an apartment $\mathcal{A}$ in $\mathcal{B}(\mathbf{M}, K)$ such that $F = \bar{C} \cap \mathcal{B}(\mathbf{M}, K) \subset \mathcal{A}$. Let $\mathbf{T}$ be the maximal K-split torus in $\mathbf{G}$ corresponding to $\mathcal{A}$. Note that $\lambda \in \mathbf{X}_*(\mathbf{T})$. Since C does not lie in $\mathcal{B}(\mathbf{M}, K)$, from the proof of Lemma 4.4.1 we conclude that the image of λ in G_F does not lie in the center of G_F. Thus, since the derived group of G_F is either $\mathbf{SL}_2$ or $\mathbf{PGL}_2$, we conclude that if the cardinality of $\mathfrak{f}$ is greater than three, then the image of $\lambda(R^\times)$ in G_F lies only in those Borel subgroups of G_F corresponding to 0-alcoves in $\mathcal{B}(\mathbf{M},K)$. That is, with our restrictions on $\mathfrak{f}$, C cannot be fixed by $\lambda(R^\times)$. ☐

COROLLARY 4.4.3. *Suppose the characteristic of* $\mathfrak{f}$ *is not two. Let* $H = d\lambda(1)$. *If* $y \in \mathcal{B}(G)$, *then* $H \in \mathfrak{g}_{y,0}$ *if and only if* $y \in \mathcal{B}(M)$.

Proof. "$\Leftarrow$": This is immediate.

"$\Rightarrow$": Let $\mathcal{C}$ denote the set of $z \in \mathcal{B}(\mathbf{G},K)$ for which $H \in \mathfrak{g}(K)_{z,0}$. The set $\mathcal{C}$ is convex and contains $\mathcal{B}(\mathbf{M},K)$.

Suppose that $\mathcal{C} \neq \mathcal{B}(\mathbf{M},K)$. We will derive a contradiction. Since $\mathcal{C}$ is convex and contains $\mathcal{B}(\mathbf{M},K)$, there exists a 0-alcove C in $\mathcal{B}(\mathbf{G},K)$ such that $\bar{C} \cap \mathcal{B}(\mathbf{M},K)$ has codimension one and $H \in \mathfrak{g}(K)_{c,0}$ for all $c \in C$.

Choose an apartment $\mathcal{A}$ in $\mathcal{B}(\mathbf{M},K)$ such that $\bar{C} \cap \mathcal{B}(\mathbf{M},K) \subset \mathcal{A}$. Let $\mathbf{T}$ be the maximal K-split torus in $\mathbf{G}$ corresponding to $\mathcal{A}$. Note that $\lambda \in \mathbf{X}_*(\mathbf{T})$. Since C does not lie in $\mathcal{B}(\mathbf{M},K)$, from the proof of Lemma 4.4.1 we conclude that the image of λ in G_F does not lie in the center of G_F. Thus, since the derived group of G_F is either $\mathbf{SL}_2$ or $\mathbf{PGL}_2$ and since the characteristic of $\mathfrak{f}$ is not two, the image of H in $\mathrm{Lie}(\mathsf{G}_F)$ lies only in those Borel subalgebras of $\mathrm{Lie}(\mathsf{G}_F)$ corresponding to 0-alcoves in $\mathcal{B}(\mathbf{M},K)$. That is, with our restrictions on $\mathfrak{f}$, we cannot have $H \in \mathfrak{g}(K)_{c,o}$ for any $c \in C$. ☐

4.5. *From Jacobson-Morosov triples over* k *to Jacobson-Morosov triples over* $\mathfrak{f}$. Different versions of Lemma 4.5.1 were proved independently (and nearly simultaneously) by Eugene Kushnirsky and myself. The proof here is due to Eugene Kushnirsky; I thank him for allowing me to publish it here. My proof will appear elsewhere.

LEMMA 4.5.1. *For any* $x, y \in \mathcal{B}(G)$, *we have* $\mathrm{stab}_G(x) \cap G_y = G_x \cap \mathrm{stab}_G(y) = G_{\{x,y\}}$.

Here $G_{\{x,y\}}$ is the $\mathrm{Gal}(K/k)$-fixed points of the group of R_K-rational points of the identity component of the group scheme associated to the set $\{x,y\}$ (see [24, §3.4] and [9, §1.2.12]).

Proof (*Eugene Kushnirsky*). Without loss of generality, we work over K. Let $\mathcal{A}$ be an apartment in $\mathcal{B}(\mathbf{G},K)$ containing x and y. Let $\mathbf{T}$ denote the maximal K-split torus of $\mathbf{G}$ corresponding to $\mathcal{A}$ and let $\mathbf{Z}$ denote the centralizer in $\mathbf{G}^\circ$ of $\mathbf{T}$.

For $\alpha \in \Phi(\mathbf{T},K)$, let $U_\alpha \subset \mathbf{G}^\circ(K)$ denote the corresponding root subgroup. For a fixed ordering on $\Phi(\mathbf{T},K)$ we define U^+ (resp., U^-) to be the group generated by $\{U_\alpha\}_{\alpha>0}$ (resp., $\{U_\alpha\}_{\alpha<0}$). For $z \in \mathcal{A}$ we define $U_z^\pm = \mathrm{stab}_G(z) \cap U^\pm$. Choose an ordering on $\Phi(\mathbf{T},K)$ so that $U_y^+ \subset U_x^+$; this implies that $U^+_{\{x,y\}} = U^+_y$. According to [9, *Corollaire* 4.6.7], $G_y = U^+_y U^-_y U^+_y \mathcal{I}^0(R_K)$. Here $\mathcal{I}$ is the smooth R_K-model for $\mathbf{Z}$ constructed in [9, §4.4]. Another application of [9, *Corollaire* 4.6.7] produces

$$\begin{aligned} \mathrm{stab}_G(x) \cap G_y &= (\mathrm{stab}_G(x) \cap U^+_y U^-_y U^+_y)\mathcal{I}^0(R_K) \\ &= U^+_{\{x,y\}}(\mathrm{stab}_G(x) \cap U^-_y) U^+_{\{x,y\}} \mathcal{I}^0(R_K) \\ &= U^+_{\{x,y\}} U^-_{\{x,y\}} U^+_{\{x,y\}} \mathcal{I}^0(R_K) \\ &= G_{\{x,y\}}. \end{aligned}$$ □

Remark 4.5.2. Recall the definition of unipotent in Section 2.1. Since every unipotent element belongs to some parahoric subgroup, Lemma 4.5.1 implies that if $u \in G$ is unipotent and $x \in \mathcal{B}(G)$ such that $ux = x$, then $u \in G_x$.

For the remainder of this subsection we fix a nontrivial $X \in \mathcal{N}$, and we suppose that Hypothesis 4.2.5 holds. Let (Y, H, X) be an $\mathfrak{sl}_2(k)$-triple completing X. Suppose that φ is a homomorphism for (Y, H, X) as described in Hypothesis 4.2.5. We have $H = d\varphi(\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right))$ and $Y = d\varphi(\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right))$. We wish to find a point $y \in \mathcal{B}(G)$ such that $Y \in \mathfrak{g}_{y,-r}$, $H \in \mathfrak{g}_{y,0}$, and $X \in \mathfrak{g}_{y,r}$.

I thank Gopal Prasad for explaining to me the proof of the following lemma; this lemma occurs without proof in [6, Corollary 3.7 (1)].

LEMMA 4.5.3 (Dan Barbasch and Allen Moy). *Suppose Hypothesis* 4.2.5 *holds. There exists* $x \in \mathcal{B}(G)$ *such that* $Y, H, X \in \mathfrak{g}_{x,0}$.

Proof (*Gopal Prasad*). Let $J = \varphi\Big(\mathbf{SL}_2(R_K)\Big) \subset \mathbf{G}^\circ(K)$. The group $J \rtimes \mathrm{Gal}(K/k)$ acts on $\mathcal{B}(\mathbf{G}, K)$. Moreover, since $J \rtimes \mathrm{Gal}(K/k)$ is bounded, its action has a fixed point [24, §2.3.1]. Let $x \in \mathcal{B}(\mathbf{G}, K)$ be such a fixed-point.

Let $\mathcal{G}$ denote the R-group scheme associated to $\mathrm{stab}_{\mathbf{G}^\circ(K)}(x)$ (see [9]). The generic fiber $\mathcal{G} \otimes_R k$ is $\mathbf{G}^\circ$ and the group of R_K-rational points of $\mathcal{G}$ is $\mathrm{stab}_{\mathbf{G}^\circ(K)}(x)$. Let $L(\mathcal{G})$ denote the Lie algebra of $\mathcal{G}$. $L(\mathcal{G})$ is a lattice in $\mathfrak{g}$ and $\mathfrak{g}(K)_{x,0} = L(\mathcal{G}) \otimes_R R_K$. Let $\mathcal{J}$ denote the R-group scheme associated to $\mathbf{SL_2}(R_K)$. From [9, Proposition 1.7.6] the map φ induces an R_K-scheme homomorphism of $\mathcal{J}$ into $\mathcal{G}$. Consequently, $d\varphi\Big(\mathfrak{sl}_2(R_K)\Big) \subset \mathfrak{g}(K)_{x,0}$.

Since x is fixed by $\mathrm{Gal}(K/k)$, we have $x \in \mathcal{B}(G)$ and $Y, H, X \in \mathfrak{g}_{x,0}$. □

Remark 4.5.4. The images of Y, H, and X in $V_{x,0}$ form an $\mathfrak{sl}_2(\mathfrak{f})$-triple.

COROLLARY 4.5.5. *Suppose Hypotheses* 4.2.3 *and* 4.2.5 *hold. If* $x \in \mathcal{B}(G) = \mathcal{B}(\mathbf{G}, K)^{\mathrm{Gal}(K/k)}$, *then*

$$x \in \mathcal{B}(\mathbf{G}, K)^{\varphi(\mathbf{SL}_2(R_K))} \text{ if and only if } d\varphi\Big(\mathfrak{sl}_2(R)\Big) \subset \mathfrak{g}_{x,0}.$$

Proof. "⇒": This follows from the proof of Lemma 4.5.3.

"$\Leftarrow$": Suppose F is a $\mathrm{Gal}(K/k)$-invariant 0-facet in $\mathcal{B}(\mathbf{G},K)$ such that $x \in F^{\mathrm{Gal}(K/k)}$. Since $d\varphi\Big(\mathfrak{sl}_2(R)\Big) \subset \mathfrak{g}_{x,0}$, it follows from Hypotheses 4.2.3 and 4.2.5 that for all $t \in R_K$ both $\varphi(\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right))$ and $\varphi(\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right))$ lie in $N_{\mathbf{G}(K)}(\mathfrak{g}(K)_{x,0}) \cap N_{\mathbf{G}(K)}(\mathfrak{g}(K)_{x,0^+})$. Since these elements generate $\varphi\Big(\mathbf{SL}_2(R_K)\Big)$, it follows from Lemma 3.2.8 that $\varphi\Big(\mathbf{SL}_2(R_K)\Big) \subset \mathrm{stab}_{\mathbf{G}^\circ(K)}(F)$. From Remark 4.5.2 it follows that $\varphi\Big(\mathbf{SL}_2(R_K)\Big) \subset \mathrm{stab}_{\mathbf{G}^\circ(K)}(F)$ if and only if $\varphi\Big(\mathbf{SL}_2(R_K)\Big) \subset \mathbf{G}(K)_F$. Thus, $\varphi\Big(\mathbf{SL}_2(R_K)\Big) \subset \mathbf{G}(K)_F = \mathbf{G}(K)_x \subset \mathrm{stab}_{\mathbf{G}(K)}(x)$. ☐

Let $\lambda \in \mathbf{X}_*^k(\mathbf{G})$ be the one-parameter subgroup derived from φ. That is, $\lambda(t) = \varphi(\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right))$ for all $t \in k^\times$.

Definition 4.5.6. The one-parameter subgroup λ constructed in the preceding paragraph is said to be *adapted* to the $\mathfrak{sl}_2(k)$-triple (Y,H,X).

Define the Levi subgroup $M = C_{G^\circ}(\lambda)$. We now present two corollaries of the results in subsection 4.4.

Corollary 4.5.7. *Suppose Hypotheses* 4.2.3 *and* 4.2.5 *hold. There exists* $y \in \mathcal{B}(G)$ *such that* $Y \in \mathfrak{g}_{y,-r}$, $H \in \mathfrak{g}_{y,0}$, *and* $X \in \mathfrak{g}_{y,r}$.

Proof. The hypotheses imply that $\mathfrak{f}$ has more than three elements.

Choose $x \in \mathcal{B}(G)$ as in Lemma 4.5.3. Since x is fixed by the group $\lambda(R^\times)$, Corollary 4.4.2 tells us $x \in \mathcal{B}(M)$. Fix an apartment $\mathcal{A}$ in $\mathcal{B}(M)$ which contains x. Since the group $\lambda(k^\times)$ lies in the center of M, λ acts on every apartment in $\mathcal{B}(M)$ by translation. It therefore makes sense to define $y = x + (r/2)\cdot\lambda \in \mathcal{A}$. This y satisfies the requirements of the lemma. ☐

Remark 4.5.8. The image of (Y,H,X) in $V_{y,-r} \times V_{y,0} \times V_{y,r}$ forms an $\mathfrak{sl}_2(\mathfrak{f})$-triple under the inherited Lie algebra operation.

The following result may be interpreted as a sharpening of Corollary 4.5.7.

Corollary 4.5.9. *Suppose Hypotheses* 4.2.3 *and* 4.2.5 *hold. If* $y \in \mathcal{B}(G)$ *such that* $Y \in \mathfrak{g}_{y,-r}$, $H \in \mathfrak{g}_{y,0}$, *and* $X \in \mathfrak{g}_{y,r}$, *then* y *must lie in* $\mathcal{B}(M)$.

Proof. The hypotheses imply that the characteristic of $\mathfrak{f}$ is not two. Therefore, the result follows from Corollary 4.4.3. ☐

5. The parametrization

Fix $r \in \mathbb{R}$. In this section we combine the material of the previous two sections and produce a parametrization of the nilpotent orbits in $\mathfrak{g}$.

5.1. *A "building set" related to an* $\mathfrak{sl}_2(k)$*-triple.* Suppose Hypotheses 4.2.3 and 4.2.5 hold. Given an $\mathfrak{sl}_2(k)$-triple in $\mathfrak{g}$, we want to produce a nice subset of $\mathcal{B}(G)$. The idea for the definitions in this subsection originated in [21].

Fix $Z \in \mathcal{N}$ and $s \in \mathbb{R}$.

Definition 5.1.1.

$$\mathcal{B}(Z, s) := \{z \in \mathcal{B}(G) \mid Z \in \mathfrak{g}_{z,s}\}.$$

The set $\mathcal{B}(Z, s)$ is a nonempty and convex subset of $\mathcal{B}(G)$. Moreover, it is the union of generalized s-facets.

LEMMA 5.1.2. *$\mathcal{B}(Z, s)$ is closed.*

Proof. Suppose that $y \in \overline{\mathcal{B}(Z, s)}$. Let F^* be the generalized s-facet containing y. From Lemma 3.2.20, the union of all generalized s-facets that contain y in their closure is an open neighborhood of y. Consequently, there exists a generalized s-facet H^* such that $H^* \subset \mathcal{B}(Z, s)$ and $F^* \subset \overline{H^*}$. From Corollary 3.2.19, we have $Z \in \mathfrak{g}_{H^*,s} \subset \mathfrak{g}_{F^*,s} = \mathfrak{g}_{y,s}$. Thus $y \in \mathcal{B}(Z, s)$. $\square$

Suppose (Y, H, X) is an (possibly trivial) $\mathfrak{sl}_2(k)$-triple in $\mathfrak{g}$.

Definition 5.1.3.

$$\mathcal{B}(Y, H, X) := \mathcal{B}(X, r) \cap \mathcal{B}(Y, -r).$$

The set $\mathcal{B}(Y, H, X)$ is a nonempty (Corollary 4.5.7), closed (Lemma 5.1.2), and convex subset of $\mathcal{B}(G)$. Moreover, it is also the union of generalized r-facets.

These properties imply the following result (see also [6, Lemma 3.6]).

LEMMA 5.1.4. *Suppose $F_1^*, F_2^* \in \mathcal{F}(r)$ and $F_i^* \subset \mathcal{B}(Y, H, X)$. If F_1^* and F_2^* are maximal generalized r-facets in $\mathcal{B}(Y, H, X)$, then F_1^* and F_2^* are strongly r-associated.*

Proof. Choose $x_i \in F_i^*$. Let $\mathcal{A}$ be an apartment in $\mathcal{B}(G)$ containing x_1 and x_2. Since F_i^* is maximal in $\mathcal{B}(Y, H, X)$ and since $\mathcal{B}(Y, H, X)$ is convex, we have $\emptyset \neq F_j^* \cap \mathcal{A} \subset A(\mathcal{A}, F_i^*)$ for $i, j \in \{1, 2\}$. $\square$

Remark 5.1.5. In the language of Section 4.5, if X is not trivial, then from Corollaries 4.4.2, 4.5.5 and 4.5.9 we have

$$\mathcal{B}(Y,H,X) = \mathcal{B}(\mathbf{G},K)^{\varphi(\mathbf{SL}_2(R_K))\rtimes \mathrm{Gal}(K/k)} + (r/2)\cdot\lambda.$$

The sum on the right-hand side occurs in $\mathcal{B}(G)^{\lambda(R^\times)} = \mathcal{B}\Big(C_{G^\circ}(\lambda)\Big)$.

5.2. *An extension of some work of J.-L. Waldspurger.* We assume that Hypotheses 4.2.3, 4.2.5, and 4.2.7 are in effect.

Fix $X \in \mathcal{N} \smallsetminus \{0\}$. Suppose (Y,H,X) is an $\mathfrak{sl}_2(k)$-triple in $\mathfrak{g}$. Suppose $\lambda \in \mathbf{X}_*^k(\mathbf{G})$ is adapted to (Y,H,X). Fix $x \in \mathcal{B}(Y,H,X)$.

We will explore the relationship between the coset $X + \mathfrak{g}_{x,r^+}$ and the nilpotent orbit ${}^GX$. For example, from [6] we expect that ${}^GX$ is the unique nilpotent orbit of minimal dimension which intersects $X + \mathfrak{g}_{x,r^+}$ nontrivially. This result requires some work; we follow J.-L. Waldspurger's presentation [26, §IX.4].

Since $H \in \mathfrak{g}_{x,0}$, it follows from Corollary 4.5.9 that there exists a maximal k-split torus $\mathbf{S}$ in $\mathbf{G}$ such that $x \in \mathcal{A}(\mathbf{S},k)$ and $\lambda \in \mathbf{X}_*(\mathbf{S})$. For $i \in \mathbb{Z}$ and $s \in \mathbb{R}$, define (for λ) the objects $\mathfrak{g}(i)$ and $\mathfrak{g}_{x,s}(i)$ as in Section 4.3. As before we have

$$\mathfrak{g}_{x,s} = \bigoplus_i \mathfrak{g}_{x,s}(i). \tag{2}$$

Lemma 5.2.1. *Assume that Hypotheses* 4.2.3, 4.2.5, *and* 4.2.7 *hold.*

$${}^{G_x^+}\Big(X + C_{\mathfrak{g}_{x,r^+}}(Y)\Big) = X + \mathfrak{g}_{x,r^+}.$$

Proof (*A generalization of an argument of J.-L. Waldspurger*).

"$\subset$": There is nothing to prove here.

"$\supset$": From Hypothesis 4.2.3 and [10, Proposition 5.4.8] we can write $\mathfrak{g}$ as a direct sum of irreducible (Y,H,X)-modules of highest weight at most $(p-3)$. Write

$$\mathfrak{g} = \bigoplus_{\rho\in\mathbb{Z}} \mathfrak{g}_\rho$$

where $\mathfrak{g}_\rho$ denotes the isotypic component consisting of irreducible (Y,H,X)-modules of highest weight ρ. For $i,\rho \in \mathbb{Z}$ we define $\mathfrak{g}(\rho,i) := \mathfrak{g}_\rho \cap \mathfrak{g}(i)$. We have $\mathfrak{g}(i) = \oplus_\rho \mathfrak{g}(\rho,i)$ and so $\mathfrak{g} = \oplus_{\rho,i}\mathfrak{g}(\rho,i)$. For $i,\rho\in\mathbb{Z}$ and $s \in \mathbb{R}$ we define

$$\mathfrak{g}_{x,s}(\rho,i) := \mathfrak{g}_{x,s} \cap \mathfrak{g}(\rho,i).$$

We first want to show

$$\mathfrak{g}_{x,s} = \bigoplus_{\rho,i} \mathfrak{g}_{x,s}(\rho,i). \tag{3}$$

A calculation shows that if $\mathfrak{g}(\rho,i)$ is nontrivial, then

$$\mathfrak{g}(\rho,i)=\{Z\in\mathfrak{g}(i)\mid\Big(\mathrm{ad}(X)\circ\mathrm{ad}(Y)\Big)(Z)=j(\rho,i)\cdot Z\},$$

where $j(\rho,i):=\Big((\rho+1)^2-(i-1)^2\Big)/4$. Note that if $\mathfrak{g}(\rho,i)$ and $\mathfrak{g}(\rho',i)$ are nontrivial and $\rho\neq\rho'$, then $(j(\rho,i)-j(\rho',i),p)=1$.

Fix $\rho,i\in\mathbb{Z}$ such that $\mathfrak{g}(\rho,i)$ is nontrivial. Define the nonzero integer

$$C(\rho):=\prod_{\rho'\neq\rho;\ \mathfrak{g}(\rho',i)\neq\{0\}}\Big(j(\rho,i)-j(\rho',i)\Big).$$

From the previous paragraph $(C(\rho),p)=1$ and so $C(\rho)\in R^{\times}$. Write $Z\in\mathfrak{g}(i)$ as $Z=\sum Z_{\rho'}$ where $Z_{\rho'}\in\mathfrak{g}(\rho',i)$. Since the operator

$$C(\rho)^{-1}\cdot\prod_{\rho'\neq\rho;\ \mathfrak{g}(\rho',i)\neq\{0\}}\Big(\mathrm{ad}(X)\circ\mathrm{ad}(Y)-j(\rho',i)\Big)$$

maps Z to Z_ρ and preserves depth, we have

$$\mathfrak{g}_{x,s}(i)=\bigoplus_{\rho}\mathfrak{g}_{x,s}(\rho,i). \tag{4}$$

Equations (4) and (2) imply that equation (3) is valid.

Note that $C_{\mathfrak{g}}(X)=\oplus_i\mathfrak{g}(i,i)$ and $C_{\mathfrak{g}}(Y)=\oplus_i\mathfrak{g}(-i,i)$. Equation (3) tells us that

$$C_{\mathfrak{g}_{x,s}}(Y)=\bigoplus_{i}\mathfrak{g}_{x,s}(-i,i). \tag{5}$$

From equation (4) and its proof we have

$$\mathfrak{g}_{x,s}(i)=\mathfrak{g}_{x,s}(-i,i)+\mathrm{ad}(X)\Big(\mathfrak{g}_{x,(s-r)}(i-2)\Big).$$

Combining this, equation (2), and equation (5) yields

$$\mathfrak{g}_{x,s}=C_{\mathfrak{g}_{x,s}}(Y)+\mathrm{ad}(X)(\mathfrak{g}_{x,(s-r)}). \tag{6}$$

Suppose $Z\in\mathfrak{g}_{x,r^+}$. We wish to produce an $h\in G_x^+$ and $C\in C_{\mathfrak{g}_{x,r^+}}(Y)$ such that ${}^h(X+C)=X+Z$. Let $h_0=1$ and $C_0=0$.

Fix $s_1>r$ such that $\mathfrak{g}_{x,r^+}=\mathfrak{g}_{x,s_1}\neq\mathfrak{g}_{x,s_1^+}$. From equation (6), we can write $Z=C_1'+\mathrm{ad}(X)P_1$ with $C_1'\in C_{\mathfrak{g}_{x,s_1}}(Y)$ and $P_1\in\mathfrak{g}_{x,(s_1-r)}$. From Hypothesis 4.2.7, there exists $h_1'=\phi_x(-P_1)\in G_{x,(s_1-r)}\subset G_x^+$ such that

$$\begin{aligned}{}^{h_1'h_0}(X+C_0+C_1') &= X+C_1'+\mathrm{ad}(X)P_1 \mod \mathfrak{g}_{x,s_1^+}\\ &= X+Z-Z_1\end{aligned}$$

with $Z_1\in\mathfrak{g}_{x,s_1^+}$. Let $h_1=h_1'\cdot h_0$ and $C_1=C_0+C_1'$

Fix $s_2 > s_1$ such that $\mathfrak{g}_{x,s_1^+} = \mathfrak{g}_{x,s_2} \neq \mathfrak{g}_{x,s_2^+}$. From equation (6), we can write $Z_1 = C_2' + \mathrm{ad}(X)P_2$ with $C_2' \in C_{\mathfrak{g}_{x,s_2}}(Y)$ and $P_2 \in \mathfrak{g}_{x,(s_2-r)}$. From Hypothesis 4.2.7, there exists $h_2' = \phi_x(-P_2) \in G_{x,(s_2-r)} \subset G_{x,(s_1-r)}$ such that

$$\begin{aligned} {}^{h_2'h_1}(X + C_1 + C_2') &= {}^{h_2'}(X + Z - Z_1 + C_2') \mod \mathfrak{g}_{x,s_2^+} \\ &= X + Z - Z_1 + C_2' + \mathrm{ad}(X)P_2 \mod \mathfrak{g}_{x,s_2^+} \\ &= X + Z - Z_2 \end{aligned}$$

with $Z_2 \in \mathfrak{g}_{x,s_2^+}$. Let $h_2 = h_2' \cdot h_1$ and $C_2 = C_1 + C_2'$.

Continuing in this way we produce a sequence $r < s_1 < s_2 < \cdots < s_n \cdots$ with $s_n \to \infty$, elements $h_n = h_n' h_{(n-1)} \in G_x^+$ with $h_n' \in G_{x,(s_n-r)}$, and elements $C_n = C_{(n-1)} + C_n' \in C_{\mathfrak{g}_{x,r^+}}(Y)$ with $C_n' \in C_{\mathfrak{g}_{x,s_n}}(Y)$ such that

$${}^{h_n}(X + C_n) = X + Z \mod \mathfrak{g}_{x,s_n^+}.$$

Let $h = \lim_{n\to\infty} h_n$ and $C = \lim_{n\to\infty} C_n$. Then $h \in G_x^+$, $C \in C_{\mathfrak{g}_{x,r^+}}(Y)$ and ${}^h(X + C) = X + Z$. □

LEMMA 5.2.2. *Suppose that Hypothesis* 4.2.5 *is valid.*

$$\Big(X + C_{\mathfrak{g}}(Y)\Big) \cap {}^G X = \{X\}.$$

Proof. See, for example, [26, V.7 (9)]. □

COROLLARY 5.2.3. *Assume that Hypotheses* 4.2.3, 4.2.5, *and* 4.2.7 *hold.*

$$(X + \mathfrak{g}_{x,r^+}) \cap {}^G X = {}^{G_x^+} X.$$

COROLLARY 5.2.4. *Assume that Hypotheses* 4.2.3, 4.2.5, *and* 4.2.7 *hold. If* $\mathcal{O} \in \mathcal{O}(0)$ *such that*

$$(X + \mathfrak{g}_{x,r^+}) \cap \mathcal{O} \neq \emptyset,$$

then ${}^G X \subset \overline{\mathcal{O}}$.

Here the closure is taken in the p-adic topology on $\mathfrak{g}$.

Proof (J.-L. Waldspurger). There exist $h \in G_x^+$ and $C \in C_{\mathfrak{g}_{x,r^+}}(Y)$ such that ${}^h(X + C) \in \mathcal{O}$. Thus, $X + C \in \mathcal{O}$. Since $X + C$ is nilpotent, there exists (from Hypothesis 4.2.5) a $\mu \in \mathbf{X}_*^k(\mathbf{G})$ such that ${}^{\mu(t)}(X + C) = t^2 \cdot (X + C)$ for all $t \in k^\times$. Since $C \in \oplus_{i\leq 0}\mathfrak{g}(i)$, we have

$$\begin{aligned} \lim_{t\to 0} {}^{\lambda(t)^{-1}\mu(t)}(X + C) &= X + \lim_{t\to 0} {}^{\lambda(t)^{-1}}(t^2 \cdot C) \\ &= X. \end{aligned}$$

□

We close with a corollary to the above corollary.

Corollary 5.2.5. *Assume that Hypotheses* 4.2.3, 4.2.5, *and* 4.2.7 *hold. Choose* $F^* \in \mathcal{F}(r)$ *such that* $F^* \subset \mathcal{B}(Y, H, X)$. *If* $\mathcal{O} \in \mathcal{O}(0)$ *such that*

$$(X + \mathfrak{g}^+_{F^*}) \cap \mathcal{O} \neq \emptyset,$$

then ${}^G X \subset \overline{\mathcal{O}}$.

Proof. Note that in this entire subsection the only assumption on x was that $x \in \mathcal{B}(Y, H, X)$. The result follows from Corollary 5.2.4. □

5.3. *A map for degenerate cosets.* We assume that all of the hypotheses stated in Section 4.2 hold.

The set I_r is too large. We first restrict to degenerate cosets of depth r.

Definition 5.3.1.

$$I^n_r := \{(F^*, v) \in I_r \mid v \text{ is a degenerate element of } V_{F^*}\}.$$

Remark 5.3.2. Suppose $(F^*_i, v_i) \in I_r$ for $i = 1, 2$ and $(F^*_1, v_1) \sim (F^*_2, v_2)$. We have $(F^*_1, v_1) \in I^n_r$ if and only if $(F^*_2, v_2) \in I^n_r$.

Suppose that $(F^*, e) \in I^n_r$. We wish to attach to (F^*, e) a nilpotent orbit $\mathcal{O}(F^*, e) \in \mathcal{O}(0)$.

Suppose $x \in F^*$. We adopt the following conventions. If e is trivial, then we declare that the $\mathfrak{sl}_2(\mathfrak{f})$-triple $(f, h, e) \in V_{x,-r} \times V_{x,0} \times V_{x,r}$ completing e is the trivial triple. Moreover, given a trivial $\mathfrak{sl}_2(\mathfrak{f})$-triple (f, h, e) as above, we declare that the $\mathfrak{sl}_2(k)$-triple lifting (f, h, e) is the trivial $\mathfrak{sl}_2(k)$-triple.

Lemma 5.3.3. *Suppose all the Hypotheses of Section* 4.2 *hold. Suppose* $(F^*, e) \in I^n_r$.

1. *Fix* $x \in F^*$. *There exists an* $\mathfrak{sl}_2(\mathfrak{f})$*-triple* $(f, h, e) \in V_{x,-r} \times V_{x,0} \times V_{x,r}$ *completing* e *and an* $\mathfrak{sl}_2(k)$*-triple* (Y, H, X) *which lifts* (f, h, e).
2. *For any* $x \in F^*$, *for any* $\mathfrak{sl}_2(\mathfrak{f})$*-triple* $(f, h, e) \in V_{x,-r} \times V_{x,0} \times V_{x,r}$ *completing* e, *and for any* $\mathfrak{sl}_2(k)$*-triple* (Y, H, X) *which lifts* (f, h, e) *we have* $F^* \subset \mathcal{B}(Y, H, X)$ *and* ${}^G X$ *is the unique nilpotent orbit of minimal dimension which intersects the coset* e *nontrivially.*

Proof. We first prove (1). Fix $x \in F^*$. If e is trivial, there is nothing to do. Suppose e is nontrivial. Hypothesis 4.2.1 says that there exist $f \in V_{x,-r}$ and $h \in V_{x,0}$ such that (f, h, e) is an $\mathfrak{sl}_2(\mathfrak{f})$-triple (under the inherited Lie algebra operation). From Corollary 4.3.2 we know that a lift (Y, H, X) of (f, h, e) exists.

Now we prove (2). Suppose $x \in F^*$, the $\mathfrak{sl}_2(\mathfrak{f})$-triple $(f, h, e) \in V_{x,-r} \times V_{x,0} \times V_{x,r}$ completes e, and (Y, H, X) is an $\mathfrak{sl}_2(k)$-triple which lifts (f, h, e). We have $F^* \subset \mathcal{B}(Y, H, X)$. It follows from Corollary 5.2.5 that ${}^G X$ is the unique nilpotent orbit of minimal dimension which intersects the coset e nontrivially. □

The following definition now makes sense.

Definition 5.3.4. Suppose all the Hypotheses of Section 4.2 hold. For $(F^*, e) \in I_r^n$ let $\mathcal{O}(F^*, e)$ denote the unique nilpotent orbit of minimal dimension which intersects the coset e nontrivially.

Remark 5.3.5. If $g \in G$ and $(F^*, e) \in I_r^n$, then $\mathcal{O}(gF^*, {}^g e) = \mathcal{O}(F^*, e)$.

5.4. *The map is well defined.* Recall the equivalence relation on I_r defined in Section 3.6.

LEMMA 5.4.1. *We assume that all of the hypotheses of Section* 4.2 *hold. The map from I_r^n to $\mathcal{O}(0)$ which sends (F^*, e) to $\mathcal{O}(F^*, e)$ induces a well-defined map from $I_r^n/\sim$ to $\mathcal{O}(0)$.*

Proof. Suppose $(F_i^*, e_i) \in I_r^n$ for $i = 1, 2$. We need to show that if $(F_1^*, e_1) \sim (F_2^*, e_2)$, then $\mathcal{O}(F_1^*, e_1) = \mathcal{O}(F_2^*, e_2)$. We may assume that $e_i \in V_{F_i^*}$ is not trivial.

Choose $x_i \in C(F_i^*)$. Since $(F_1^*, e_1) \sim (F_2^*, e_2)$, there exist $g \in G$ and an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ such that

$$\emptyset \neq A(\mathcal{A}, F_1^*) = A(\mathcal{A}, gF_2^*)$$

and

$$e_1 \stackrel{i}{=} {}^g e_2 \quad \text{in } V_{F_1^*} \stackrel{i}{=} V_{gF_2^*}.$$

From Remark 5.3.5 we can assume that $g = 1$.

Let $\mathbf{S}$ denote the maximal k-split torus in $\mathbf{G}$ corresponding to $\mathcal{A}$. Let S denote the maximal $\mathfrak{f}$-split torus in G_{x_1} corresponding to $\mathbf{S}$.

Complete e_1 to an $\mathfrak{sl}_2(\mathfrak{f})$-triple $(f_1, h_1, e_1) \in V_{x_1,-r} \times V_{x_1,0} \times V_{x_1,r}$ and suppose $\bar{\lambda} \in \mathbf{X}_*^{\mathfrak{f}}(\mathsf{G}_{x_1})$ is adapted to this triple. There exists $h \in G_{x_1}$ such that ${}^{\bar{h}}\bar{\lambda} \in \mathbf{X}_*(\mathsf{S})$. (Here $\bar{h}$ denotes the image of h in $\mathsf{G}_{x_1}(\mathfrak{f})$.) Since F_1^* and F_2^* are strongly r-associated and $e_1 \stackrel{i}{=} e_2$ in $V_{F_1^*} \stackrel{i}{=} V_{F_2^*}$, it follows from Lemma 3.5.4 that there exists $h' \in G_{x_1} \cap G_{x_2}$ such that

$${}^h e_1 \stackrel{i}{=} {}^{h'} e_1 \stackrel{i}{=} {}^{h'} e_2 \ \text{ in } V_{F_1^*} \stackrel{i}{=} V_{F_1^*} \stackrel{i}{=} V_{F_2^*}.$$

Let $\lambda \in \mathbf{X}_*(\mathbf{S})$ be the lift of ${}^{\bar{h}}\bar{\lambda}$.

Let $\mathfrak{g} = \oplus_j \mathfrak{g}(j)$ be the decomposition of $\mathfrak{g}$ arising from λ. We have $\mathfrak{g}_{F_i^*} = \oplus_j \mathfrak{g}_{F_i^*}(j)$. There exists an $X \in \mathfrak{g}_{F_1^*}(2) \cap \mathfrak{g}_{F_2^*}(2)$ such that the image of X in $V_{F_i^*}$ is ${}^{h'}e_i$.

It follows from Corollary 4.3.2 and Lemma 5.3.3 that

$$\mathcal{O}(F_i^*, e_i) = \mathcal{O}(F_i^*, {}^{h'}e_i) = {}^G X. \qquad \square$$

5.5. *Distinguished cosets.* We assume that all of the hypotheses of Section 4.2 hold.

The set $I_r^n/\sim$ is too large. We now restrict our attention to *distinguished* cosets of depth r.

Definition 5.5.1. We define $I_r^d \subset I_r^n$ to be those pairs $(F^*, e) \in I_r^n$ such that for any $x \in F^*$, for any $\mathfrak{sl}_2(\mathfrak{f})$-triple $(f, h, e) \in V_{x,-r} \times V_{x,0} \times V_{x,r}$ completing e, and for any $\mathfrak{sl}_2(k)$-triple (Y, H, X) which lifts (f, h, e), we have that F^* is a maximal generalized r-facet in $\mathcal{B}(Y, H, X)$.

Remark 5.5.2. If $r = 0$, then it can be shown that this definition of distinguished is equivalent to the usual one. That is, if $(F^*, e) \in I_0^d$, then e does not lie in any proper Levi subalgebra of the $\mathfrak{f}$-Lie algebra $V_{F^*,0}$.

Lemma 5.5.3. *Suppose all of the hypotheses of Section* 4.2 *hold. If* $(F^*, e) \in I_r^n$ *and* e *is nontrivial, then* $(F^*, e) \in I_r^d$ *if and only if there exist an* $x \in F^*$, *an* $\mathfrak{sl}_2(\mathfrak{f})$-*triple* $(f, h, e) \in V_{x,-r} \times V_{x,0} \times V_{x,r}$ *completing* e, *and an* $\mathfrak{sl}_2(k)$-*triple* (Y, H, X) *in* $\mathfrak{g}$ *lifting* (f, h, e) *such that* F^* *is a maximal generalized* r-*facet in* $\mathcal{B}(Y, H, X)$.

Proof. "$\Rightarrow$": This follows from the definitions.

"$\Leftarrow$": Suppose we have an $x \in F^*$, an $\mathfrak{sl}_2(\mathfrak{f})$-triple $(f, h, e) \in V_{x,-r} \times V_{x,0} \times V_{x,r}$ completing e, and an $\mathfrak{sl}_2(k)$-triple (Y, H, X) in $\mathfrak{g}$ lifting (f, h, e) such that F^* is a maximal generalized r-facet in $\mathcal{B}(Y, H, X)$.

Suppose we also have data $x' \in F^*$, an $\mathfrak{sl}_2(\mathfrak{f})$-triple $(f', h', e) \in V_{x',-r} \times V_{x',0} \times V_{x',r}$ completing e, and an $\mathfrak{sl}_2(k)$-triple (Y', H', X') in $\mathfrak{g}$ which lifts (f', h', e) such that F^* is not a maximal generalized r-facet in $\mathcal{B}(Y', H', X')$. We will derive a contradiction.

Since $\mathfrak{g}_{x',-r} = \mathfrak{g}_{x,-r}$ (from Lemma 3.2.5) and $\mathfrak{g}_{x',r} = \mathfrak{g}_{x,r}$, we have

$$[\mathfrak{g}_{x',-r}, \mathfrak{g}_{x',r}] \subset \mathfrak{g}_{x,0}.$$

Thus, we can assume that $x = x'$.

From Lemma 5.3.3 we have

$${}^G X = {}^G X' = \mathcal{O}(F^*, e).$$

From Corollary 5.2.3 we can assume (after replacing (Y', H', X') with a G_x^+-conjugate) that $X = X'$. From Hypothesis 4.2.5, there exists a $g \in C_G(X)$ such that $Y' = {}^gY$ and $H' = {}^gH$. Consequently, $\mathcal{B}(Y', H', X) = g\mathcal{B}(Y, H, X)$.

By assumption, $g^{-1}F^* \subset g^{-1}\mathcal{B}(Y', H', X) = \mathcal{B}(Y, H, X)$ is not a maximal generalized r-facet in $\mathcal{B}(Y, H, X)$. Since $\dim g^{-1}F^* = \dim F^*$, this is a contradiction. □

Remark 5.5.4. Suppose $(F_i^*, e_i) \in I_r^n$ for $i = 1, 2$ and $(F_1^*, e_1) \sim (F_2^*, e_2)$. From Lemma 5.4.1 we have $\mathcal{O}(F_1^*, e_1) = \mathcal{O}(F_2^*, e_2)$. From the proof above, we conclude that $(F_1^*, e_1) \in I_r^d$ if and only if $(F_2^*, e_2) \in I_r^d$.

5.6. *A bijective correspondence.* We assume that all of the hypotheses of Section 4.2 hold. In this subsection we establish the following theorem.

Theorem 5.6.1. *Assume that all of the hypotheses of Section* 4.2 *hold. There is a bijective correspondence between* $I_r^d/\sim$ *and* $\mathcal{O}(0)$ *given by the map which sends* (F^*, e) *to* $\mathcal{O}(F^*, e)$.

Proof. We have already seen that the map is well defined.

We first show that the map is injective. We need to show that if (F_1^*, e_1), $(F_2^*, e_2) \in I_r^d$ and $\mathcal{O}(F_1^*, e_1) = \mathcal{O}(F_2^*, e_2)$, then $(F_1^*, e_1) \sim (F_2^*, e_2)$.

If $\mathcal{O}(F_i^*, e_i) = \{0\}$, then F_i^* is open in $\mathcal{B}(G)$ and the result follows. Thus, we may assume that e_i is not trivial.

Fix $x_i \in C(F_i^*)$. Complete e_i to an $\mathfrak{sl}_2(\mathfrak{f})$-triple $(f_i, h_i, e_i) \in V_{x_i,-r} \times V_{x_i,0} \times V_{x_i,r}$. From Corollary 4.3.2 we may lift (f_i, h_i, e_i) to an $\mathfrak{sl}_2(k)$-triple (Y_i, H_i, X_i) in $\mathfrak{g}$. From Lemma 5.3.3 we have $\mathcal{O}(F_i^*, e_i) = {}^GX_i$.

Since $\mathcal{O}(F_1^*, e_1) = \mathcal{O}(F_2^*, e_2)$, there exists a $g \in G$ such that $({}^gY_2, {}^gH_2, {}^gX_2) = (Y_1, H_1, X_1)$. Consequently, since $(F_i^*, e_i) \in I_r^d$, from Lemma 5.1.4 we have that F_1^* and gF_2^* are strongly r-associate. Thus, there exists an apartment $\mathcal{A}$ in $\mathcal{B}(G)$ such that

$$\emptyset \neq A(\mathcal{A}, F_1^*) = A(\mathcal{A}, gF_2^*).$$

Moreover, since X_1 has image e_1 in $V_{F_1^*}$ and X_1 has image ${}^ge_2$ in $V_{gF_2^*}$, we have $X_1 \in \mathfrak{g}_{F_1^*} \cap \mathfrak{g}_{gF_2^*}$ and

$$e_1 \overset{i}{=} {}^ge_2 \ \text{ in } \ V_{F_1^*} \overset{i}{=} V_{gF_2^*}.$$

Thus, the map is injective.

We now show that the map is surjective. Suppose $\mathcal{O} \in \mathcal{O}(0)$.

If $\mathcal{O}$ is trivial, then let F^* be an open generalized r-facet and let e be trivial in V_{F^*}. We have $(F^*, e) \in I_r^d$ and $\mathcal{O}(F^*, e) = \{0\}$.

Suppose $\mathcal{O}$ is not trivial. Fix $X \in \mathcal{O}$. Complete X to an $\mathfrak{sl}_2(k)$-triple (Y, H, X) in $\mathfrak{g}$. Let F^* be a maximal generalized r-facet in $\mathcal{B}(Y, H, X)$ and let e denote the image of X in V_{F^*}. We will be done if we can show that $\mathcal{O}(F^*, e) = {}^G X$. This, however, follows from Lemma 5.3.3 (2). □

For future reference, we record the following corollary of the proof of Theorem 5.6.1.

Corollary 5.6.2. *Assume that all of the hypotheses of Section* 4.2 *hold. Suppose* $(F_1^*, e_1), (F_2^*, e_2) \in I_r^d$ *and* $(F_1^*, e_1) \sim (F_2^*, e_2)$. *There exists* $g \in G$ *and an* $\mathfrak{sl}_2(k)$*-triple* (Y, H, X) *such that*

1. $X \in \mathfrak{g}_{F_1^*} \cap \mathfrak{g}_{gF_2^*}$,
2. X *has image* e_1 *in* $V_{F_1^*}$ *and image* ${}^g e_2 \in V_{gF_2^*}$, *and*
3. F_1^* *and* gF_2^* *are maximal generalized* r*-facets in* $\mathcal{B}(Y, H, X)$.

Appendix A. Some comments on Hypothesis 4.2.1

A.1. *Introduction.* We will show that, subject to some conditions on k and $\mathbf{G}$, Hypothesis 4.2.1 is valid. This result is related to material in [16] and [19, §2] . No attempt has been made to produce an optimal set of conditions.

Fix x, r, and X as in the statement of Hypothesis 4.2.1. Without loss of generality, we assume throughout this appendix that $\mathbf{G}$ is connected.

A.2. *An* $\mathfrak{sl}_2(\mathfrak{f})$*-triple.* In this subsection, we establish the existence of Y and H in $\mathfrak{g}$ satisfying the requirements of Hypothesis 4.2.1.

We let $\mathfrak{g}'$ (resp., $\mathfrak{z}$) denote the Lie algebra of the group of k-rational points of the the derived group of $\mathbf{G}$ (resp., the connected component of the center of $\mathbf{G}$). From [3, Proposition 3.1] if p is larger than some constant which may be determined by examining the absolute root datum of $\mathbf{G}$, then we may assume that $\mathfrak{g} = \mathfrak{z} + \mathfrak{g}'$. Thus, without loss of generality, we may assume that $\mathbf{G}$ is semisimple.

We claim that, under suitable conditions on k and $\mathbf{G}$, the Killing form κ identifies $\mathfrak{g}_{x,s}$ with $\mathfrak{g}_{x,s}^*$ for all $s \in \mathbb{R}$; in particular, for all $Z \in \mathfrak{g}_{x,s} \setminus \mathfrak{g}_{x,s^+}$ there exists a $W \in \mathfrak{g}_{x,-s} \setminus \mathfrak{g}_{x,(-s)^+}$ such that $\kappa(Z, W) \in R^\times$. Indeed, since $\mathfrak{g}_{x,s}$ [1, Proposition 1.4.1] and κ behave well with respect to Galois descent, we may reduce to the case when $\mathbf{G}$ is k-split. If $\mathbf{G}$ is k-split, we can fix a Chevalley basis for $\mathfrak{g}$. In this situation the statement follows if p is greater than some constant which may be derived from the absolute root datum of $\mathbf{G}$.

Recall the definition of the finite-dimensional $\mathfrak{f}$-Lie algebra $\overline{\mathfrak{g}}_x$ from Section 4.2. From the previous paragraph we see that, under suitable conditions on k and $\mathbf{G}$, the representation ad of $\overline{\mathfrak{g}}_x$ has a nondegenerate trace-form. Let $e \in \overline{\mathfrak{g}}_x$ denote the image of X in $V_{x,r}$. From Hypothesis 4.2.3 we have that $\mathrm{ad}(e)^m = 0$ for some $m \leq (p-2)$. Thus from [10, Proposition 5.3.1] there exists an $\mathfrak{sl}_2(\mathfrak{f})$-triple (f, h, e) in $\overline{\mathfrak{g}}_x$ completing e.

We claim that we may assume that $f \in V_{x,-r}$ and $h \in V_{x,0}$. (We already know that $e \in V_{x,r}$.) Indeed, let f_{-r} denote the image of f under the projection of $\overline{\mathfrak{g}}_x$ onto $V_{x,-r} \subset \overline{\mathfrak{g}}_x$. Since $e \in V_{x,r} \subset \overline{\mathfrak{g}}_x$, it follows that $\mathrm{ad}(e)^2 f_{-r} = \mathrm{ad}(e)^2 f = -2e$. Let h_0 denote the image of h under the projection of $\overline{\mathfrak{g}}_x$ onto $V_{x,0}$. We have $\mathrm{ad}(e)f_{-r} = h_0$ and $\mathrm{ad}(e)h_0 = -2e$. From Morosov's theorem (see [17, Lemma 7, p. 98] or [10, the proof of Proposition 5.3.1; in particular, pp. 140–141]) there exists $f' \in \overline{\mathfrak{g}}_x$ such that (f', h_0, e) is an $\mathfrak{sl}_2(\mathfrak{f})$-triple. Since $h_0 \in V_{x,0}$, $e \in V_{x,r}$, and $[V_{x,s}, V_{x,t}] \subset V_{x,(s+t)}$ for all $s, t \in \mathbb{R}$, we can assume that $f' \in V_{x,-r}$. Thus we may assume that $f \in V_{x,-r}$ and $h \in V_{x,0}$.

Consequently, we can choose lifts $Y \in \mathfrak{g}_{x,-r}$ of f and $H \in \mathfrak{g}_{x,0}$ of h which satisfy the initial requirements of Hypothesis 4.2.1. From Lemma 4.1.2 we may assume that Y is nilpotent. Thus from Hypothesis 4.2.3 we have that $\mathrm{ad}(f)^m = \mathrm{ad}(e)^m = 0$ for some $m \leq (p-2)$. Therefore, from [10, Proposition 5.4.8] we have that $\overline{\mathfrak{g}}_x$ is a direct sum of irreducible (f,h,e)-modules of highest weight at most $(p-3)$.

A.3. *A one-parameter subgroup in G_x.* We now wish to establish the existence of $\bar{\lambda} \in \mathbf{X}_*^{\mathfrak{f}}(\mathsf{G}_x)$ satisfying the requirements of Hypothesis 4.2.1. We continue to use the notation introduced above, but we now remove the assumption that $\mathbf{G}$ is semisimple.

Since h is semisimple, from [7, Proposition 11.8 and its proof] there exists a maximal $\mathfrak{f}$-torus T of G_x such that $h \in \mathrm{Lie}(\mathsf{T})$. Let $\mathbf{T}$ be a maximal K-split k-torus of $\mathbf{G}$ associated to T. (That is, we have $x \in \mathcal{A}(\mathbf{T}, K)$ and the image of $\mathbf{T}(K) \cap \mathbf{G}(K)_x$ in $\mathsf{G}_x(\mathfrak{F})$ is $\mathsf{T}(\mathfrak{F})$. That the torus $\mathbf{T}$ exists follows from the argument made in the final paragraph of the proof of [9, Proposition 5.1.10].)

Let

$$\overline{\mathfrak{g}}(\mathfrak{F})_x := \bigoplus_{s \in \mathbb{R}/\ell\cdot\mathbb{Z}} \mathfrak{g}(K)_{x,s}/\mathfrak{g}(K)_{x,s^+}.$$

As in Section 4.2, we give $\overline{\mathfrak{g}}(\mathfrak{F})_x$ a natural Lie algebra structure; in fact, $\overline{\mathfrak{g}}_x$ may be identified with the set of $\mathrm{Gal}(\mathfrak{F}/\mathfrak{f})$-fixed points of the $\mathfrak{f}$-Lie algebra $\overline{\mathfrak{g}}(\mathfrak{F})_x$. Since $\overline{\mathfrak{g}}(\mathfrak{F})_x$ decomposes into irreducible (f,h,e)-modules, it follows from [10, Lemmas 5.5.3 and 5.5.4] that there exists a one-parameter subgroup $\bar{\lambda}_2\colon \mathrm{GL}_1 \to \mathrm{GL}(\overline{\mathfrak{g}}(\mathfrak{F})_x)$ defined over $\mathfrak{f}$ such that for $v \in \overline{\mathfrak{g}}(\mathfrak{F})_x$

$$\text{if } \bar{\lambda}_2(t)\cdot v = t^i v, \text{ then } |i| \leq (p-3) \text{ and } [h,v] = iv.$$

For $\alpha \in \Phi(\mathbf{T}, K)$, we denote by $(\overline{\mathfrak{g}}(\mathfrak{F})_x)_\alpha$ the (nontrivial) subspace of $\overline{\mathfrak{g}}(\mathfrak{F})_x$ on which T acts by α. We define a linear map λ_2 from the K-root lattice in $\mathbf{X}^*(\mathbf{T})$ to $\mathbb{Z}$ via $\bar{\lambda}_2$. That is, for $\alpha \in \Phi(\mathbf{T}, K)$ we define $\langle \lambda_2, \alpha \rangle \in \mathbb{Z}$ by the equality

$$\bar{\lambda}_2(t)\cdot v = t^{\langle \lambda_2, \alpha\rangle} v$$

for all $v \in (\overline{\mathfrak{g}}(\mathfrak{F})_x)_\alpha$ and extend linearly. Note that λ_2 is $\mathrm{Gal}(K/k)$-invariant.

For $i \in \mathbb{Z}$, we define

$$\mathfrak{g}(K)(i) := \bigoplus_{\alpha \in \Phi(\mathbf{T},K)\cup\{0\};\langle\lambda_2,\alpha\rangle = i} \mathfrak{g}(K)_\alpha.$$

Write $\mathfrak{g}(i)$ for the $\mathrm{Gal}(K/k)$-fixed points of $\mathfrak{g}(K)(i)$. Let $X \in \mathfrak{g}(2) \cap \mathfrak{g}_{x,r}$ be any lift of e. Since $\bar{\mathfrak{g}}_x$ decomposes into irreducible (e,h,f)-modules of highest weight at most $(p-3)$, we conclude that $\mathrm{ad}(e)^2 \colon \bar{\mathfrak{g}}_x(-2) \to \bar{\mathfrak{g}}_x(2)$ is an isomorphism (here $\bar{\mathfrak{g}}_x(\pm 2) = \{v \in \bar{\mathfrak{g}}_x \mid \mathrm{ad}(h)v = \pm 2v\}$). Thus, since k is complete, for all $s \in \mathbb{R}$ the map $\mathrm{ad}(X)^2 \colon \mathfrak{g}(-2) \cap \mathfrak{g}_{x,s-r} \to \mathfrak{g}(2) \cap \mathfrak{g}_{x,s+r}$ is an isomorphism (see also the proof of Lemma 4.3.1). Hence, there exists $Y \in \mathfrak{g}(2) \cap \mathfrak{g}_{x,-r}$ such that $\mathrm{ad}(X)^2 Y = -2X$. Let $H = \mathrm{ad}(X)Y$. Note that H is a lift of h and Y is a lift of f. We claim that (Y,H,X) is an $\mathfrak{sl}_2(k)$-triple. From Hypothesis 4.2.3 and Morosov's theorem, there exists $Y' \in \mathfrak{g}$ such that (Y',H,X) is an $\mathfrak{sl}_2(k)$-triple. However, as usual, we conclude that we may assume that $Y' = Y$ (see also the proof of Corollary 4.3.2).

From Hypothesis 4.2.5 there exists a k-homomorphism $\varphi \colon \mathbf{SL}_2 \to \mathbf{G}$ such that $d\varphi\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X$, $d\varphi\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = Y$, and $d\varphi\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = H$. Let λ denote the one-parameter subgroup $t \mapsto \varphi\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. Statements (1) and (2) of Hypothesis 4.2.5 along with Hypothesis 4.2.3 imply that for all $v \in \mathfrak{g}$

$$(*) \qquad \text{if } {}^{\lambda(t)}v = t^i v, \text{ then } |i| \leq (p-3) \text{ and } \mathrm{ad}(H)v = iv.$$

From Corollary 4.5.9 we have $x \in \mathcal{B}(C_G(\lambda))$. Thus, there exists a maximal k-split torus $\mathbf{S}$ such that $x \in \mathcal{A}(\mathbf{S},k)$ and $\lambda \in \mathbf{X}_*(\mathbf{S})$. Let $\bar{\lambda}$ denote the image of λ in $\mathbf{X}_*^{\mathfrak{f}}(\mathsf{G}_x)$. The image of $d\bar{\lambda}$ in $\mathrm{Lie}(\mathsf{G}_x)$ coincides with the one-dimensional subspace spanned by h. From (*) and that fact that H is a lift of h we have that for all $v \in \bar{\mathfrak{g}}_x$

$$\text{if } {}^{\bar{\lambda}(t)}v = t^i v, \text{ then } |i| \leq (p-3) \text{ and } \mathrm{ad}(h)v = iv.$$

Finally, we consider the uniqueness statement. Fix $i \in \mathbb{Z}$ such that $-2 \leq i \leq 2$. Note that if $m \in \mathbb{N}$ and $v \in \bar{\mathfrak{g}}_x$ such that $\mathrm{ad}(h)v = iv$, then ${}^{\bar{\lambda}(t)}\Big(\mathrm{ad}(e)^m v\Big) = t^{i+2m}\Big(\mathrm{ad}(e)^m v\Big)$ and ${}^{\bar{\lambda}(t)}\Big(\mathrm{ad}(f)^m v\Big) = t^{i-2m}\Big(\mathrm{ad}(f)^m v\Big)$. Since $\bar{\mathfrak{g}}_x$ is spanned by the set of all vectors of the form $\mathrm{ad}(e)^m v$ or $\mathrm{ad}(f)^m v$ (m and v as above), we conclude that $\bar{\lambda}$ is uniquely determined up to an element of $\mathbf{X}_*(\mathsf{Z}_x)$ whose differential is zero.

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