AN ANALOGUE OF THE CARTAN DECOMPOSITION FOR p-ADIC REDUCTIVE SYMMETRIC SPACES

by

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Abstract. — Let k be a non Archimedean locally compact field of residue characteristic different from 2, let G be a connected reductive group defined over k, let σ be an involutive k-automorphism of G and H an open k-subgroup of the fixed points group of σ. We denote by G_k (resp. H_k) the group of k-points of G (resp. H). In this paper, we obtain an analogue of the Cartan decomposition for the reductive symmetric space $H_k\backslash G_k$. More precisely, we obtain a decomposition of G_k as a union of H_k-cosets which is related to the H_k-conjugacy classes of maximal σ-anti-invariant k-split tori in G. When G is k-split, we get a more precise result, involving the stabilizer of a special point of the Bruhat-Tits building of G over k.

Résumé. — Soit k un corps localement compact non archimédien de caractéristique résiduelle impaire, soit G un groupe réductif connexe défini sur k, soient σ un k-automorphisme involutif de G et H un k-sous-groupe ouvert du groupe des points de G fixes par σ. On note G_k (resp. H_k) le groupe des k-points de G (resp. H). Dans cet article, nous obtenons un analogue de la décomposition de Cartan pour l'espace symétrique réductif $H_k\backslash G_k$. Plus précisément, nous obtenons une décomposition de G_k sous la forme d'une réunion de classes modulo H_k reliée aux classes de H_k-conjugaison de tores k-déployés σ-anti-invariants maximaux de G. Lorsque G est déployé sur k, nous obtenons un résultat plus précis impliquant le stabilisateur d'un point spécial de l'immeuble de Bruhat-tits de G sur k.

Introduction

Let k be a non Archimedean locally compact field of residue characteristic different from 2. Let G be a connected reductive group defined over k, let σ be an involutive k-automorphism of G and H an open k-subgroup of the fixed points group of σ. We denote by G_k (resp. H_k) the group of k-points of G (resp. H). Harmonic analysis on the reductive symmetric space $\mathrm{H}_k\backslash\mathrm{G}_k$ is the study of the action of G_k on the space of complex square integrable functions on $\mathrm{H}_k\backslash\mathrm{G}_k$. This study is related to the classification of H_k-distinguished representations of G_k, that is representations having a nonzero space of H_k-invariant linear forms. The question of distinguishedness has been studied intensively for GL_n and related groups. See for instance [**1, 2, 12, 13, 14, 23, 24**] for a (non exhaustive) list of works on this question. Some other aspects of that problem, including the Plancherel formula, have been studied by Offen [**22**] for spherical representations, in three particular cases related to GL_n. Blanc and Delorme [**5**] have studied parabolically induced representations for a general reductive symmetric space $\mathrm{H}_k\backslash\mathrm{G}_k$. In this paper, we investigate the geometry of the space $\mathrm{H}_k\backslash\mathrm{G}_k$.

Connected reductive groups over k can be considered as reductive symmetric spaces. Indeed, if G' is such a group, the map $\sigma : (x,y) \mapsto (y,x)$ defines a k-involution of the connected reductive group $\mathrm{G} = \mathrm{G}' \times \mathrm{G}'$ whose fixed points group H is the diagonal image of G' in G. Hence the reductive symmetric space $\mathrm{H}_k\backslash\mathrm{G}_k$ naturally identifies with the group G'_k. Moreover, if K' is a subgroup of G'_k and if we set $\mathrm{K} = \mathrm{K}' \times \mathrm{K}'$, then the $(\mathrm{H}_k, \mathrm{K})$-double cosets of G_k correspond to the K'-double cosets of G'_k. In particular, if K' is the stabilizer in G'_k of a special point in the (reduced) Bruhat-Tits building of G' over k, the decomposition of $\mathrm{H}_k\backslash\mathrm{G}_k$ into K-orbits corresponds to the Cartan decomposition of G'_k relative to K' (see [**8**, Proposition 4.4.3]).

In this paper, we obtain an analogue of the Cartan decomposition for a general reductive symmetric space $\mathrm{H}_k\backslash\mathrm{G}_k$. In [**15, 16, 17**] A. and G. Helminck and Wang studied two kinds of objects which are related to our problem:

(i) H_k-conjugacy classes of maximal σ-anti-invariant k-split tori of G (called maximal (σ, k)-*split* tori in [**15**], see also Definition 3.5);

(ii) H_k-conjugacy classes of the parabolic k-subgroups P of G which are opposite to $\sigma(\mathrm{P})$ (called σ-*split* parabolic k-subgroups in [**17**] and σ-*parabolic* k-subgroups in this paper, see Definition 3.7).

Let $\{A^j \mid j \in J\}$ be a set of representatives of the H_k-conjugacy classes of maximal (σ, k)-split tori in G. For each j, we denote by $W_{G_k}(A^j)$ (resp. $W_{H_k}(A^j)$) the quotient of the normalizer of A^j in G_k (resp. in H_k) by its centralizer. According to Helminck and Wang [**17**], the set J is finite and, for $j \in J$, the group $W_{G_k}(A^j)$ is the Weyl group of a root system. Moreover, let A be a maximal (σ, k)-split torus of G, let S be a σ-stable maximal k-split torus of G containing A and P a minimal σ-parabolic k-subgroup of G containing S. Then, according to [**16**, Theorem 3.6], the finite union:

$$\bigcup_{j \in J} W_{H_k}(A^j) \backslash W_{G_k}(A^j) \tag{0.1}$$

classifies the open (H_k, P_k)-double cosets of G_k. For each $j \in J$, we choose:

(1) a set $\mathscr{N}_j \subset N_{G_k}(A^j)$ of representatives of $W_{H_k}(A^j) \backslash W_{G_k}(A^j)$;
(2) an element $y_j \in G_k$ such that $y_j A y_j^{-1} = A^j$;

and we denote by $\mathscr{N}$ the set of all ny_j for $j \in J$ and $n \in \mathscr{N}_j$. Note that $\mathscr{N}$ is a set of representatives of (0.1). Let ϖ be a uniformizer of k, let Λ be the lattice formed by the images of ϖ by the various algebraic one-parameter subgroups of A and let Λ^- denote the subset of anti-dominant elements of Λ relative to P. Then we can state our first main result (see Theorem 3.10):

Theorem 0.1. — *There exists a compact subset Ω of G_k such that:*

$$G_k = \bigcup_{n \in \mathscr{N}} H_k n \Lambda^- \Omega.$$

In order to prove this result, we make a large use of the Bruhat-Tits theory [**8, 9**]. Let $\mathscr{B}$ be the (reduced) Bruhat-Tits building of G over k. It is endowed with an action of σ. Then the proof of Theorem 0.1 is based on the following result (see Proposition 2.4):

Proposition 0.2. — *$\mathscr{B}$ is the union of its σ-stable apartments.*

This result can be rephrased as follows. Let S be a σ-stable maximal k-split torus of G, let N its normalizer in G and let $\mathscr{O}$ be the set of all $g \in G_k$ such that $g^{-1}\sigma(g) \in \mathrm{N}_k$. Then we have $\mathrm{G}_k = \mathscr{O}\mathrm{K}$, where K is the stabilizer in G_k of any point of the apartment corresponding to S (see Proposition 3.4).

Let us mention that the question of the disjointness of the various components appearing in the decomposition of G_k given by Theorem 0.1 has been investigated by Lagier [**19**].

When the group G is k-split, we obtain a refinement of Theorem 0.1, which is based on the following refinement of Proposition 0.2 (see Proposition 4.4):

Proposition 0.3. — *Let x be a special point of $\mathscr{B}$. There is a σ-stable maximal k-split torus* S *of* G *such that the apartment corresponding to* S *contains x, and such that the maximal σ-anti-invariant subtorus of* S *is a maximal (σ, k)-split torus of* G.

We thus obtain our second main result (see Theorem 4.8):

Theorem 0.4. — *Let* K *be the stabilizer in* G_k *of a special point in $\mathscr{B}$. Then:*

$$\mathrm{G}_k = \bigcup_{j \in \mathrm{J}} \mathrm{H}_k y_j \mathrm{S}_k \mathrm{K}.$$

Note that Proposition 0.3 is no longer true for non-split groups, as proven in §5.3.

The paper is organized as follows. In §1 we recall the main properties of the Bruhat-Tits building attached to a connected reductive group defined over k. In §2 we study the set of all apartments containing a given σ-stable subset of the building, and we prove Proposition 0.2. In §3 we prove our first main result (Theorem 0.1). In §4 we are devoted to the case where G is k-split. We prove Proposition 0.3 and Theorem 0.4. Finally, in §5 we study in more details the two following examples:

(1) $\mathrm{G}_k = \mathrm{GL}_n(k)$ and $\sigma(g) =$ transpose of g^{-1}.

(2) $\mathrm{G}_k = \mathrm{GL}_n(k')$ with k' quadratic over k and $\mathrm{id} \neq \sigma \in \mathrm{Gal}(k'/k)$.

When $n = 2$ and k' is totally ramified over k, Example (2) provides an example of a non-split group for which Proposition 0.3 is not satisfied.

After this work was finished, we learnt that Y. Benoist and H. Oh [**4**] *proved a result equivalent to Theorem 0.1, with a weaker assumption on k (they only* *assume that its characteristic is not* 2*). They also use the Bruhat-Tits building, but in a different way.*

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1. The Bruhat-Tits building

Let k be a non Archimedean non discrete locally compact field, and let ω be its normalized valuation. In this section, we recall the main properties of the Bruhat-Tits building attached to a connected reductive group defined over k. The reader may refer to the original construction of Bruhat-Tits [**8, 9**] or to more concise presentations [**20, 26, 29**].

If G is a linear algebraic group defined over k, the group of its k-points will be denoted by G_k or $\mathrm{G}(k)$, and its neutral component will be denoted by G°. If H is a subset of G, then $\mathrm{N_G(H)}$ (resp. $\mathrm{Z_G(H)}$) denotes the normalizer (resp. the centralizer) of H in G.

If X is a subset of G, then ${}^g\mathrm{X}$ denotes the left conjugate of X by $g \in \mathrm{G}$.

1.1. Let G be a connected reductive group defined over k, and let S be a maximal k-split torus of G. We denote by $\mathrm{X^*(S) = Hom(S, GL_1)}$ (resp. by $\mathrm{X_*(S) = Hom(GL_1, S)}$) the group of algebraic characters (resp. cocharacters) of S. We define a map:

$$\mathrm{X_*(S) \times X^*(S)} \to \mathbf{Z} \tag{1.1}$$

as follows. If $\lambda \in \mathrm{X_*(S)}$ and $\chi \in \mathrm{X^*(S)}$, then $\chi \circ \lambda$ is an endomorphism of the multiplicative group $\mathrm{GL_1}$, which corresponds to an endomorphism of the ring $\mathbf{Z}[t, t^{-1}]$. It is of the form $t \mapsto t^n$ for some $n \in \mathbf{Z}$. This integer n is denoted by $\langle \lambda, \chi \rangle$. The map (1.1) defines a perfect duality (see [**6**, §8.6]).

1.2. Let N (resp. Z) denote the normalizer (resp. the centralizer) of S in G. If we extend (1.1) by **R**-linearity, there exists a unique group homomorphism:

$$\nu : \mathrm{Z}_k \to \mathrm{X}_*(\mathrm{S}) \otimes_{\mathbf{Z}} \mathbf{R} \tag{1.2}$$

such that the condition:

$$\langle \nu(z), \chi \rangle = -\omega(\chi(z)) \tag{1.3}$$

holds for any $z \in \mathrm{Z}_k$ and any k-rational character $\chi \in \mathrm{X}^*(\mathrm{Z})_k$ (see [**29**, §1.2]). According to [**20**, Proposition 1.2], the kernel of (1.2) is the maximal compact subgroup of Z_k. It will be denoted by Z^1_k.

Remark 1.1. — Note that the intersection $\mathrm{S}_k \cap \mathrm{Z}^1_k$ is equal to the maximal compact subgroup of S_k, which we denote by S^1_k. Indeed S^1_k contains the compact subgroup $\mathrm{S}_k \cap \mathrm{Z}^1_k$ of S_k and is contained in the maximal compact subgroup Z^1_k of Z_k. According to [**29**, §1.2], the quotient $\Lambda_{\mathrm{Z}} = \mathrm{Z}_k/\mathrm{Z}^1_k$ is a free abelian group of rank $\dim \mathrm{S}$, and the image of S_k in Λ_{Z} has finite index.

1.3. Let C denote the connected centre of G and let $\mathrm{X}_*(\mathrm{C})$ be the group of its algebraic cocharacters. It is a subgroup of the free abelian group $\mathrm{X}_*(\mathrm{S})$. We denote by $\mathscr{A}$ the space:

$$\mathrm{V} = (\mathrm{X}_*(\mathrm{S}) \otimes_{\mathbf{Z}} \mathbf{R})/(\mathrm{X}_*(\mathrm{C}) \otimes_{\mathbf{Z}} \mathbf{R})$$

considered as an affine space on itself and by $\mathrm{Aff}(\mathscr{A})$ the group of its affine automorphisms. By making V act on $\mathscr{A}$ by translations, we can think to V as a subgroup of $\mathrm{Aff}(\mathscr{A})$. It is the kernel of the natural group homomorphism $\mathrm{Aff}(\mathscr{A}) \to \mathrm{GL}(\mathrm{V})$ which associates to any affine automorphism its linear part.

1.4. The map (1.2) induces a homomorphism:

$$\mathrm{Z}_k \to \mathrm{Aff}(\mathscr{A}) \tag{1.4}$$

which is still denoted by ν. Its image is contained in V. An important property of this homomorphism is that it extends to a homomorphism $\mathrm{N}_k \to \mathrm{Aff}(\mathscr{A})$ (see [**29**, §1.2]). It does not extend in a unique way, but two homomorphisms extending (1.4) to N_k are conjugated by a *unique* element of $\mathrm{Aff}(\mathscr{A})$ (see [**20**, Proposition 1.8]).

1.5. The affine space $\mathscr{A}$ endowed with an action of N_k defined by a group homomorphism $\nu : \mathrm{N}_k \to \mathrm{Aff}(\mathscr{A})$ extending the homomorphism (1.4) is called the (reduced) *apartment* attached to S. It satisfies the conditions:

A1 $\mathscr{A}$ is an affine space on V;

A2 ν is a group homomorphism $\mathrm{N}_k \to \mathrm{Aff}(\mathscr{A})$ extending the canonical homomorphism $\mathrm{Z}_k \to \mathrm{V}$.

It has the following unicity property. If $(\mathscr{A}', \nu')$ satisfy **A1** and **A2**, then there is a unique affine and N_k-equivariant isomorphism from $\mathscr{A}'$ to $\mathscr{A}$.

Remark 1.2. — We obtain the *non reduced* apartment $\mathscr{A}_{\mathrm{nr}}$ by replacing V by $\mathrm{X}_*(\mathrm{S}) \otimes_{\mathbf{Z}} \mathbf{R}$. This is the point of view of Tits [**29**]. The non reduced apartment is not as canonical as the reduced one: two homomorphisms extending the map $\nu_{\mathrm{nr}} : \mathrm{Z}_k \to \mathrm{Aff}(\mathscr{A}_{\mathrm{nr}})$ to N_k are conjugated by an element of $\mathrm{Aff}(\mathscr{A}_{\mathrm{nr}})$ which is not necessarily unique (see [**20**, §1] and also [**29**, §1.2]).

1.6. Let $\Phi = \Phi(\mathrm{G}, \mathrm{S})$ denote the set of roots of G relative to S. It is a subset of $\mathrm{X}^*(\mathrm{S})$. Therefore, any root $a \in \Phi$ can be seen as a linear form on $\mathrm{X}_*(\mathrm{S}) \otimes \mathbf{R}$ which is trivial on the subspace $\mathrm{X}_*(\mathrm{C}) \otimes \mathbf{R}$, hence as a linear form on V (see [**20**, §1]).

For $a \in \Phi$, we denote by U_a the root subgroup associated to a, which is a unipotent subgroup of G normalized by Z (see [**6**, Proposition 21.9]), and by s_a the reflection corresponding to a, considered as an element of $\mathrm{GL}(\mathrm{V})$ — or, more precisely, of the quotient of $\nu(\mathrm{N}_k)$ by $\nu(\mathrm{Z}_k)$.

1.7. Let $a \in \Phi$ and $u \in \mathrm{U}_a(k) - \{1\}$. The intersection:

$$\mathrm{U}_{-a}(k) u \mathrm{U}_{-a}(k) \cap \mathrm{N}_k \tag{1.5}$$

consists of a single element, called $m(u)$, whose image by ν is an affine reflection whose linear part is s_a (see [**7**, §5]). The set $\mathscr{H}_{a,u}$ of fixed points of $\nu(m(u))$ is an affine hyperplane of $\mathscr{A}$, which is called a *wall* of $\mathscr{A}$.

A *chamber* of $\mathscr{A}$ is a connected component of the complementary in $\mathscr{A}$ of the union of its walls. Note that a chamber is open in $\mathscr{A}$.

A point $x \in \mathscr{A}$ is said to be *special* if, for all root $a \in \Phi$, there is a root $b \in \Phi \cap \mathbf{R}_{+}a$ and an element $u \in \mathrm{U}_b(k) - \{1\}$ such that $x \in \mathscr{H}_{b,u}$ (see [**21**, §1.2.3] and also [**29**, §1.9]).

1.8. Let $\theta(a,u)$ denote the affine function $\mathscr{A} \to \mathbf{R}$ whose linear part is a and whose vanishing hyperplane is the wall $\mathscr{H}_{a,u}$ of fixed points of $\nu(m(u))$. We fix a base point in $\mathscr{A}$, so that $\mathscr{A}$ can be identified with the vector space V. For $r \in \mathbf{R}$, we set:

$$\mathrm{U}_a(k)_r = \{u \in \mathrm{U}_a(k) - \{1\} \mid \theta(a,u)(x) \geqslant a(x) + r \text{ for all } x \in \mathscr{A}\} \cup \{1\}.$$

Thus we obtain a filtration of $\mathrm{U}_a(k)$ by subgroups. If we change the base point in $\mathscr{A}$, this filtration is only modified by a translation of the indexation.

1.9. Let Ω be a nonempty subset of $\mathscr{A}$. We set:

$$\mathrm{N}_\Omega = \{n \in \mathrm{N}_k \mid \nu(n)(x) = x \text{ for all } x \in \Omega\},$$

and we denote by U_Ω the subgroup of G_k generated by all the $\mathrm{U}_a(k)_r$ such that the affine function $x \mapsto a(x) + r$ is non negative on Ω. According to [**20**, §12], this subgroup is compact in G_k, and we have $n\mathrm{U}_\Omega n^{-1} = \mathrm{U}_{\nu(n)(\Omega)}$ for any $n \in \mathrm{N}_k$. In particular N_Ω normalizes U_Ω.

The subgroup $\mathrm{P}_\Omega = \mathrm{N}_\Omega \mathrm{U}_\Omega$ is open in G_k (see *loc.cit.*, Corollary 12.12).

1.10. Let $\Phi = \Phi^- \cup \Phi^+$ be a decomposition of Φ into positive and negative roots. We denote by U^+ (resp. U^-) the subgroup of G_k generated by the U_a for $a \in \Phi^+$ (resp. $a \in \Phi^-$), and we set $\mathrm{U}_\Omega^+ = \mathrm{U}_\Omega \cap \mathrm{U}^+$ (resp. $\mathrm{U}_\Omega^- = \mathrm{U}_\Omega \cap \mathrm{U}^-$). Then the group P_Ω has the following Iwahori decomposition:

$$\mathrm{P}_\Omega = \mathrm{U}_\Omega^- \mathrm{U}_\Omega^+ \mathrm{N}_\Omega \tag{1.6}$$

(see [**20**, Corollary 12.6] and also [**8**, §7.1.4]).

1.11. In [**8, 9**], Bruhat and Tits associate to the apartment $(\mathscr{A}, \nu)$ a G_k-set $\mathscr{B} = \mathscr{B}(\mathrm{G}, k)$ containing $\mathscr{A}$, called the (reduced) *building* of G over k and satisfying the following conditions:

B1 The set $\mathscr{B}$ is the union of the $g \cdot \mathscr{A}$ with $g \in \mathrm{G}_k$.

B2 The subgroup N_k is the stabilizer of $\mathscr{A}$ in G_k, and $n \cdot x = \nu(n)(x)$ for all $x \in \mathscr{A}$ and $n \in \mathrm{N}_k$.

B3 For all $a \in \Phi$ and $r \in \mathbf{R}$, the subgroup $\mathrm{U}_a(k)_r$ defined in §1.8 fixes the subset $\{x \in \mathscr{A} \mid a(x) + r \geqslant 0\}$ pointwise.

The building has the following unicity property. If $\mathscr{B}'$ is a G_k-set containing $\mathscr{A}$ and satisfying **B1**, **B2** and **B3**, then there is a unique G_k-equivariant bijection from $\mathscr{B}'$ to $\mathscr{B}$ (see [**29**, §2.1] and also [**25**, §1.9]).

1.12. The subsets of $\mathscr{B}$ of the form $g \cdot \mathscr{A}$ with $g \in \mathrm{G}_k$ are called *apartments*. According to **B1** the building is the union of its apartments.

For $g \in \mathrm{G}_k$, the apartment $g \cdot \mathscr{A}$ can be naturally endowed with a structure of affine space and an action of ${}^g\mathrm{N}_k$ by affine isomorphisms. Upto unique isomorphism, it is the apartment attached to the maximal k-split torus ${}^g\mathrm{S}$ (see §1.5). This defines a unique G_k-equivariant map:

$$\mathrm{G} \supset \mathrm{S}' \mapsto \mathscr{A}(\mathrm{S}') \subset \mathscr{B} \tag{1.7}$$

between maximal k-split tori of G and apartments of $\mathscr{B}$, such that S maps to $\mathscr{A}$.

Note that $\mathscr{B}$ does not depend on the maximal k-split torus S. Indeed, let S' be a maximal k-split torus of G, let $(\mathscr{A}', \nu')$ be the apartment attached to S' and $\mathscr{B}'$ the building of G over k relative to this apartment (see §1.11). If we identify $\mathscr{A}'$ with the unique apartment of $\mathscr{B}$ corresponding to S' *via* (1.7), then $\mathscr{B}' = \mathscr{B}$.

1.13. The building has the following important properties (see [**8**, §7.4] and also [**20**, §13]):

(1) Let Ω be a nonempty subset of $\mathscr{A}$. Then P_Ω is the subgroup of G_k made of all elements fixing Ω pointwise.

(2) Let $g \in \mathrm{G}_k$. There exists $n \in \mathrm{N}_k$ such that $g \cdot x = n \cdot x$ for any element $x \in \mathscr{A} \cap g^{-1} \cdot \mathscr{A}$.

In particular, Property (1) together with **B2** imply that $\mathrm{N}_\Omega = \mathrm{N}_k \cap \mathrm{P}_\Omega$.

1.14. Let σ be a k-automorphism of G. There is a unique bijective map from $\mathscr{B}$ to itself, which we still denote by σ, such that:

(i) the condition:

$$\sigma(g \cdot x) = \sigma(g) \cdot \sigma(x)$$

holds for any $g \in \mathrm{G}_k$ and $x \in \mathscr{B}$;

(ii) the map σ permutes the apartments and, for any apartment $\mathscr{A}$, the restriction of σ to $\mathscr{A}$ is an affine isomorphism from $\mathscr{A}$ onto its image.

This gives us an action of the group $\mathrm{Aut}_k(\mathrm{G})$ of k-automorphisms of G on the building (see [**9**, §4.2.12]).

1.15. Let V^1 denote the dual space of $\mathrm{X}^*(\mathrm{G}) \otimes \mathbf{R}$. The *extended building* of G over k is the product $\mathscr{B}^1 = \mathscr{B} \times \mathrm{V}^1$, where G_k acts on V^1 by:

$$g \cdot \chi = -\omega(\chi(g)),$$

for any $g \in \mathrm{G}_k$ and any k-rational character $\chi \in \mathrm{X}^*(\mathrm{G})_k$. The G_k-stabilizer of the reduced building $\mathscr{B} \times \{0\}$, considered as a subset of the extended building $\mathscr{B}^1$, is denoted by G_k^1. It is the subgroup of all $g \in \mathrm{G}_k$ such that $\omega(\chi(g)) = 0$ for any $\chi \in \mathrm{X}^*(\mathrm{G})_k$.

Remark 1.3. — Let D denote the maximal k-split torus of the connected centre C of G. Then the quotient $\Lambda_\mathrm{G} = \mathrm{G}_k/\mathrm{G}_k^1$ is a free abelian group of rank $\dim \mathrm{D}$, and the image of D_k in Λ_G has finite index.

The action of $\mathrm{Aut}_k(\mathrm{G})$ on G induces an action of $\mathrm{Aut}_k(\mathrm{G})$ on V^1, hence on the extended building $\mathscr{B}^1$.

Remark 1.4. — Let Γ be a finite subgroup of $\mathrm{Aut}_k(\mathrm{G})$ whose order is prime to the residue characteristic of k, and let H be the neutral component of the fixed points subgroup G^Γ. Prasad and Yu [**25**] proved the existence of a H_k-equivariant map $\iota : \mathscr{B}^1(H, k) \to \mathscr{B}^1(G, k)$ whose image is the set of Γ-invariant points. Moreover, such a map is *toral* in the sense of [**20**], which means that for any maximal k-split torus T of H, there is a maximal k-split torus S of G containing T such that ι maps $\mathscr{A}_{nr}(H, T)$ to $\mathscr{A}_{nr}(G, S)$ by an affine transformation (see [**25**, Theorem 1.9], and Remark 1.2 for the definition of the non reduced apartment $\mathscr{A}_{nr}$).

2. Existence of σ-stable apartments

From now on, k will be a non-Archimedean locally compact field of residue characteristic different from 2. Let G be connected reductive group defined over k and let σ be a k-involution on G.

According to §1.14, the building $\mathscr{B}$ of G over k is endowed with an action of σ. In this section, we prove that, for any $x \in \mathscr{B}$, there exists a σ-stable apartment containing x. We keep using notations of Section 1.

2.1. Let Ω be a nonempty σ-stable subset of $\mathscr{B}$ contained in some apartment, and let $\mathrm{App}(\Omega)$ be the set of all apartments of $\mathscr{B}$ containing Ω. It is a nonempty set on which the group P_Ω acts transitively (see [**20**, Corollary 13.7]). Because Ω is σ-stable, both P_Ω and $\mathrm{App}(\Omega)$ are σ-stable. Note that the σ-stable apartments containing Ω are exactly the σ-invariant points in $\mathrm{App}(\Omega)$.

2.2. Let us fix an apartment $\mathscr{A} \in \mathrm{App}(\Omega)$ and an element $u \in P_\Omega$ such that $\sigma(\mathscr{A}) = u \cdot \mathscr{A}$. Let N denote the normalizer in G of the maximal k-split torus of G corresponding to $\mathscr{A}$. As σ is involutive, we have:

$$\sigma(u)u \in P_\Omega \cap N_k = N_\Omega. \tag{2.1}$$

The map $\rho : g \mapsto g \cdot \mathscr{A}$ induces a P_Ω-equivariant bijection between the homogeneous spaces P_Ω/N_Ω and $\mathrm{App}(\Omega)$. The automorphism:

$$\theta : x \mapsto u^{-1}\sigma(x)u \tag{2.2}$$

of the group G_k stabilises P_Ω and N_Ω. Indeed $\sigma(N_k) = uN_ku^{-1}$, and:

$$\theta(N_\Omega) = u^{-1}\sigma(P_\Omega \cap N_k)u = P_\Omega \cap u^{-1}\sigma(N_k)u = N_\Omega.$$

Note that the condition (2.1) means that $\theta \circ \theta$ is conjugation by some element of N_Ω. As N_Ω is θ-stable, the map:

$$(2.3) \qquad (\sigma, g\mathrm{N}_\Omega) \mapsto u\theta(g\mathrm{N}_\Omega), \quad g \in \mathrm{P}_\Omega,$$

defines an action of σ on $\mathrm{P}_\Omega/\mathrm{N}_\Omega$, making ρ into a σ-equivariant bijection. Note that this action differs from the natural action of σ on $\mathrm{P}_\Omega/\mathrm{N}_\Omega$ (which obviously has fixed points).

2.3. Let Ω be a nonempty σ-stable subset of $\mathscr{B}$ contained in some apartment.

Proposition 2.1. — *Assume that Ω contains a point of a chamber of $\mathscr{B}$. Then Ω is contained in some σ-stable apartment.*

Proof. — First we describe $\mathrm{P}_\Omega/\mathrm{N}_\Omega$ as a projective limit of finite σ-sets. According to [**11**, §1.2], Example (f), the group G_k is locally compact and totally disconnected. Therefore we can choose a decreasing filtration $(\mathrm{Q}^i)_{i\geqslant 0}$ of the open subgroup P_Ω of G_k satisfying the following properties:

(A) The intersection of the Q^i is reduced to $\{1\}$.

(B) For any $i \geqslant 0$, the subgroup Q^i is compact open and normal in P_Ω.

For $i \geqslant 0$, let P^i_Ω denote the intersection of the subgroups $\mathrm{N}_\Omega\mathrm{Q}^i$ and $\theta(\mathrm{N}_\Omega\mathrm{Q}^i)$. The P^i_Ω form a decreasing filtration of P_Ω, and we claim that such a filtration satisfies the following properties:

(1) The intersection of the P^i_Ω is reduced to N_Ω.

(2) For any $i \geqslant 0$, the subgroup P^i_Ω is θ-stable and of finite index in P_Ω.

As N_Ω is θ-stable, it is contained in the intersection of the P^i_Ω. Let g be in this intersection. For any $i \geqslant 0$, there exist $n_i \in \mathrm{N}_\Omega$ and $q_i \in \mathrm{Q}^i$ such that $g = n_iq_i$. Because of Property (A) above, q_i converges to 1. Therefore n_i converges to a limit contained in the closed subgroup N_Ω, and this limit is g. This proves Property (1).

Now recall that $\theta \circ \theta$ is conjugation by some element of N_Ω. This implies that P^i_Ω is θ-stable. As P^i_Ω is open in P_Ω and contains N_Ω, the quotient $\mathrm{P}_\Omega/\mathrm{P}^i_\Omega$ can be identified with the quotient of U_Ω, which is compact, by some open subgroup. This gives the expected result.

Because of Property (2), the map:

$$(\sigma, g\mathrm{P}_\Omega^i) \mapsto u\theta(g\mathrm{P}_\Omega^i), \quad g \in \mathrm{P}_\Omega,$$

defines an action of σ on the finite quotient $\mathrm{P}_\Omega/\mathrm{P}_\Omega^i$. We get a projective system $(\mathrm{P}_\Omega/\mathrm{P}_\Omega^i)_{i\geqslant 0}$ of finite σ-sets. Because P_Ω is complete, and thanks to Property (1), the natural σ-equivariant map from $\mathrm{P}_\Omega/\mathrm{N}_\Omega$ to the projective limit of the $\mathrm{P}_\Omega/\mathrm{P}_\Omega^i$ is bijective.

Lemma 2.2. — *Let $(\mathscr{X}^i)_{i\geqslant 0}$ be a projective system of finite σ-sets, and let $\mathscr{X}$ be its projective limit. Assume that, for each $i \geqslant 0$, the cardinal of $\mathscr{X}^i$ is odd. Then $\mathscr{X}$ has a σ-invariant point.*

Proof. — Because each of the $\mathscr{X}^i$ has an odd cardinal, each of them contains a σ-invariant element. Suppose that we have constructed for some $i \geqslant 1$ a σ-invariant element $x_i \in \mathscr{X}^i$. The fiber of x_i in $\mathscr{X}^{i+1}$ is σ-stable and its cardinal is the quotient of the cardinal of $\mathscr{X}^{i+1}$ by the one of $\mathscr{X}^i$. Therefore it is odd. We deduce from this that there exists a σ-invariant element $x_{i+1} \in \mathscr{X}^{i+1}$ whose image in $\mathscr{X}^i$ is x_i. By induction, we get a σ-invariant element $x \in \mathscr{X}$. □

Let p denote the residue characteristic of k. Recall that p is assumed to be odd.

Lemma 2.3. — *Let* K *be a normal subgroup of finite index in* P_Ω *containing* N_Ω. *Then the index of* K *in* P_Ω *is a power of* p.

Proof. — Let S be the maximal k-split torus associated to $\mathscr{A}$, let Φ denote the set of roots of G relative to S and let $\Phi = \Phi^- \cup \Phi^+$ be a decomposition of Φ into positive and negative roots. According to §1.10, the group P_Ω has the following Iwahori decomposition:

$$\mathrm{P}_\Omega = \mathrm{U}_\Omega^- \mathrm{U}_\Omega^+ \mathrm{N}_\Omega. \tag{2.4}$$

The fact that Ω contains a point of a chamber of $\mathscr{B}$ implies that the group N_Ω is reduced to $\mathrm{Ker}(\nu)$, hence normalizes the groups U_Ω^+ and U_Ω^-. The index of

K in P_Ω can be decomposed as follows:

$$(\mathrm{P}_\Omega : \mathrm{K}) = (\mathrm{P}_\Omega : \mathrm{U}_\Omega^+\mathrm{K}) \cdot (\mathrm{U}_\Omega^+\mathrm{K} : \mathrm{K}). \tag{2.5}$$

In a first hand, the index $(\mathrm{U}_\Omega^+\mathrm{K} : \mathrm{K}) = (\mathrm{U}_\Omega^+ : \mathrm{U}_\Omega^+ \cap \mathrm{K})$ is a power of p, because U_Ω^+ is a pro-p-group (*i.e.* a projective limit of finite discrete p-groups). In the other hand, the index $(\mathrm{P}_\Omega : \mathrm{U}_\Omega^+\mathrm{K})$ is equal to $(\mathrm{U}_\Omega^- : \mathrm{U}_\Omega^- \cap \mathrm{U}_\Omega^+\mathrm{K})$, which is a power of p because U_Ω^- is a pro-p-group. The result follows. □

According to Lemma 2.3, the cardinal of each set $\mathrm{P}_\Omega/\mathrm{P}_\Omega^i$ with $i \geqslant 0$ is odd. Proposition 2.1 now follows from Lemma 2.2. □

2.4. We now prove the main result of this section.

Proposition 2.4. — *For any $x \in \mathscr{B}$, there exists a σ-stable apartment containing x.*

Proof. — Let x be a point in $\mathscr{B}$, and let y be a point of a chamber of $\mathscr{B}$ whose adherence contains x. The set $\Omega = \{y, \sigma(y)\}$ is a σ-stable subset of $\mathscr{B}$ satisfying the conditions of Proposition 2.1. Hence we get a σ-stable apartment of $\mathscr{B}$ containing y. Such an apartment contains the adherence of the chamber of y. In particular, it contains x. □

3. Decomposition of $\mathrm{H}_k\backslash\mathrm{G}_k$

Let k be a non-Archimedean locally compact field of residue characteristic different from 2. Let G be a connected reductive group defined over k, let σ be an involutive k-automorphism of G and let H be an open k-subgroup of the fixed points group G^σ. Equivalently, H is a k-subgroup of G^σ containing the neutral component $(\mathrm{G}^\sigma)^\circ$ (see [**3**]).

3.1. Let S be a maximal k-split torus of G, and let $\mathscr{A}$ denote the corresponding apartment.

Lemma 3.1. — *$\mathscr{A}$ is σ-stable if, and only if* S *is σ-stable.*

Proof. — This comes from the fact that the apartment corresponding to $\sigma(\mathrm{S})$ is the image of $\mathscr{A}$ by σ. □

3.2. We now assume that S is σ-stable. Let N (resp. Z) denote the normalizer (resp. the centralizer) of S in G. Let $\mathscr{O} = \mathscr{O}_{\mathrm{S}}$ denote the set of all $g \in \mathrm{G}_k$ such that $g^{-1}\sigma(g) \in \mathrm{N}_k$.

Proposition 3.2. — *$\mathscr{O}$ is a finite union of $(\mathrm{H}_k, \mathrm{Z}_k)$-double cosets.*

Proof. — Let us fix a minimal parabolic k-subgroup P of G containing S. According to [**17**, Proposition 6.8], the map $g \mapsto \mathrm{H}_k g \mathrm{P}_k$ induces a bijection between the $(\mathrm{H}_k, \mathrm{Z}_k)$-double cosets in $\mathscr{O}$ and the $(\mathrm{H}_k, \mathrm{P}_k)$-double cosets in G_k. The result then follows from [**17**, Corollary 6.16].

We now give a direct proof of this result. We have an exact sequence:

$$\mathrm{G}_k^\sigma = \mathrm{H}^0(\mathrm{G}_k) \to \mathrm{H}^0(\mathrm{G}_k/\mathrm{N}_k) \xrightarrow{\delta} \mathrm{H}^1(\mathrm{N}_k) \to \mathrm{H}^1(\mathrm{G}_k),$$

where H^0 and H^1 denote respectively the set of σ-fixed points and the first set of nonabelian cohomology of σ (see [**28**, Chapter I, §5]). The transition map δ induces an injective map from $\mathrm{G}_k^\sigma \backslash \mathrm{H}^0(\mathrm{G}_k/\mathrm{N}_k)$, which is the set of $(\mathrm{G}_k^\sigma, \mathrm{N}_k)$-double cosets of $\mathscr{O}$, into $\mathrm{H}^1(\mathrm{N}_k)$. Because Z_k (resp. H_k) is of finite index in N_k (resp. in G_k^σ), the finiteness of the number of $(\mathrm{G}_k^\sigma, \mathrm{N}_k)$-double cosets of $\mathscr{O}$ is equivalent to the finiteness of the number of $(\mathrm{H}_k, \mathrm{Z}_k)$-double cosets of $\mathscr{O}$. Therefore, it will be enough to prove that $\mathrm{H}^1(\mathrm{N}_k)$ is finite.

Let M be a group with an action of $\sigma \in \mathrm{Aut}(\mathrm{M})$, and let M' be a σ-stable normal subgroup of M. We can form the following exact sequence:

$$\mathrm{H}^1(\mathrm{M}') \to \mathrm{H}^1(\mathrm{M}) \to \mathrm{H}^1(\mathrm{M}/\mathrm{M}'),$$

which proves that the finiteness of $\mathrm{H}^1(\mathrm{M}')$ and $\mathrm{H}^1(\mathrm{M}/\mathrm{M}')$ implies the finiteness of $\mathrm{H}^1(\mathrm{M})$. Therefore we are reduced to proving that:

$$\mathrm{H}^1(\mathrm{N}_k/\mathrm{Z}_k), \quad \mathrm{H}^1(\mathrm{Z}_k/\mathrm{Z}_k^1), \quad \mathrm{H}^1(\mathrm{Z}_k^1)$$

are finite sets. Recall (see §1.2) that Z_k^1 denotes the maximal compact subgroup of Z_k. Because $\mathrm{N}_k/\mathrm{Z}_k$ is finite, the first case is immediate. Next, the quotient $\Lambda = \mathrm{Z}_k/\mathrm{Z}_k^1$ is a finitely generated free abelian group. We have an exact sequence:

$$\mathrm{H}^1(2\Lambda)\xrightarrow{a}\mathrm{H}^1(\Lambda) \to \mathrm{H}^1(\Lambda/2\Lambda).$$

Let $2m \in 2\Lambda$ be a cocycle, that is $2m + \sigma(2m) = 0$, and consider it as a cocycle in Λ. The identity $2m = m - \sigma(m)$ implies that the class $a(2m)$ is trivial in $\mathrm{H}^1(\Lambda)$, hence that the map a is null. Therefore $\mathrm{H}^1(\Lambda)$ is embedded in $\mathrm{H}^1(\Lambda/2\Lambda)$, which is finite because $\Lambda/2\Lambda$ is finite.

Now we treat the case of the compact subgroup Z_k^1. Let M be an open pro-p-subgroup of Z_k^1. (Its existence is a topological property of G_k asserted in [**11**, §1.2], Example (f).) The normalizer of M in Z_k^1 is open, hence of finite index, in Z_k^1. We can therefore assume that M is normal (if not, we replace it by the intersection of the finitely many ${}^g\mathrm{M}$ with $g \in \mathrm{Z}_k^1$). Moreover, we assume that M is stable by σ (if not, we replace it by $\mathrm{M} \cap \sigma(\mathrm{M})$). Then $\mathrm{H}^1(\mathrm{M})$ is trivial because M is a pro-p-group and p is odd, and $\mathrm{H}^1(\mathrm{Z}_k^1/\mathrm{M})$ is finite because M is of finite index in Z_k^1. The finiteness of $\mathrm{H}^1(\mathrm{Z}_k^1)$ follows. This ends our alternative proof of Lemma 3.2. □

3.3. Let $\mathscr{A}$ denote the σ-stable apartment corresponding to S.

Lemma 3.3. — *We have $g \in \mathscr{O}$ if and only if $g \cdot \mathscr{A}$ is σ-stable.*

Proof. — As $\mathscr{A}$ is σ-stable, the apartment $g \cdot \mathscr{A}$ is σ-stable if and only if $\sigma(g) \cdot \mathscr{A} = g \cdot \mathscr{A}$. This amounts to saying that $g^{-1}\sigma(g) \in \mathrm{N}_k$. □

For $x \in \mathscr{A}$, let P_x denote the subgroup P_Ω (see §1.10) with $\Omega = \{x\}$.

Proposition 3.4. — *Let x be in $\mathscr{A}$. Then we have $\mathrm{G}_k = \mathscr{O}\mathrm{P}_x$.*

Proof. — For $g \in \mathrm{G}_k$, we set $x' = g \cdot x$. According to Proposition 2.4, there is a σ-stable apartment $\mathscr{A}'$ containing x'. Let $g' \in \mathscr{O}$ be such that $\mathscr{A}' = g' \cdot \mathscr{A}$. According to Property (2) of §1.13, there exists $n \in \mathrm{N}_k$ such that we have $g'^{-1}g \cdot x = n \cdot x$. Hence we get $g \in \mathscr{O}\mathrm{N}_k\mathrm{P}_x$. As $\mathscr{O}\mathrm{N}_k = \mathscr{O}$, we obtain the expected result. □

3.4. If T is a σ-stable torus in G, we denote by T^+ (resp. T^-) the neutral component of $\mathrm{T} \cap \mathrm{H}$ (resp. of the subgroup $\{t \in \mathrm{T} \mid \sigma(t) = t^{-1}\}$). Note that, as T^+ is open in the fixed points subgroup T^σ, we have $\mathrm{T}^+ = (\mathrm{T}^\sigma)^\circ$. The torus T is the almost direct product (see [**6**, xi]) of T^+ and T^-, which means that T is equal to the product $\mathrm{T}^+\mathrm{T}^-$ and the intersection $\mathrm{T}^+ \cap \mathrm{T}^-$ is finite.

Definition 3.5 **(Helminck-Wang [17], §4.4)**. — A σ-stable torus T of G is said to be *(σ, k)-split* if it is k-split and if $\mathrm{T} = \mathrm{T}^-$.

Let us recall (see [**17**, Proposition 10.3]) that two arbitrary maximal (σ, k)-split tori of G are G_k-conjugated.

3.5. Let T be a k-split torus of G, and let T_k^1 denote its maximal compact subgroup. Let ϖ be a uniformizer of k. The images of ϖ by the various algebraic cocharacters of T form a σ-stable lattice in T_k, which will be denoted by $\Lambda(\mathrm{T}_k)$.

Lemma 3.6. — (i) *T_k is the direct product of $\Lambda(\mathrm{T}_k)$ and T_k^1.*

(ii) *For any $g \in \mathrm{G}_k$, we have $\Lambda({}^g\mathrm{T}_k) = {}^g\Lambda(\mathrm{T}_k)$.*

(iii) *The subgroup generated by $\Lambda(\mathrm{T}_k^+)$ and $\Lambda(\mathrm{T}_k^-)$ has finite index in $\Lambda(\mathrm{T}_k)$.*

Proof. — Only (iii) is not immediate. First note that, as k is a non Archimedean locally compact field of characteristic different from 2, the subgroup of squares of $k^\times$ is of finite index in $k^\times$. This implies that $\mathrm{T}_k^2 = \{t^2 \mid t \in \mathrm{T}_k\}$ is of finite index in T_k.

For any $t \in \mathrm{T}_k$, the element t^2 can be decomposed as the product of $t\sigma(t) \in \mathrm{T}_k^+$ and $t\sigma(t)^{-1} \in \mathrm{T}_k^-$. Indeed the image of T by the map $t \mapsto t\sigma(t)$ is connected and contained in T^σ, thus in T^+. By a similar argument, the image of T by $t \mapsto t\sigma(t)^{-1}$ is contained in T^-.

Therefore T_k^2 is contained in $\mathrm{T}_k^+\mathrm{T}_k^-$, thus there is some finite subset $\mathscr{F}$ of T_k such that $\mathrm{T}_k = \mathrm{T}_k^+\mathrm{T}_k^-\mathscr{F}$. According to (i), this gives:

$$\begin{aligned} \Lambda(\mathrm{T}_k)\mathrm{T}_k^1 &= \Lambda(\mathrm{T}_k^+)(\mathrm{T}_k^+)^1\Lambda(\mathrm{T}_k^-)(\mathrm{T}_k^-)^1\mathscr{F} \\ &= \Lambda(\mathrm{T}_k^+)\Lambda(\mathrm{T}_k^-)\mathrm{T}_k^1\mathscr{F}. \end{aligned}$$

We obtain the expected result by computing the quotient of this equality by the subgroup $\Lambda(\mathrm{T}_k^+)\Lambda(\mathrm{T}_k^-)\mathrm{T}_k^1$. □

3.6. Let $\{\mathrm{A}^j \mid j \in \mathrm{J}\}$ be a set of representatives of the H_k-conjugacy classes of maximal (σ, k)-split tori in G. We denote by $\mathrm{W}_{\mathrm{G}_k}(\mathrm{A}^j)$ (resp. $\mathrm{W}_{\mathrm{H}_k}(\mathrm{A}^j)$) the quotient of the normalizer of A^j in G_k (resp. in H_k) by its centralizer. According to [**17**, Proposition 5.9], the group $\mathrm{W}_{\mathrm{G}_k}(\mathrm{A}^j)$ is the Weyl group of a root system. In particular, it is a finite group. (If σ is trivial on the isotropic factor of G over k, then this group is trivial.)

Definition 3.7. — A parabolic subgroup P of G is said to be *σ-parabolic* if it is opposite to $\sigma(\mathrm{P})$, that is if $\mathrm{P} \cap \sigma(\mathrm{P})$ is a Levi subgroup of P and $\sigma(\mathrm{P})$.

Remark 3.8. — This differs from the terminology used in [**17**], where such parabolic subgroups are said to be *σ-split*.

3.7. Let A be a maximal (σ, k)-split torus of G.

Lemma 3.9. — *There is a σ-stable maximal k-split torus of* G *containing* A.

Proof. — Let G′ denote the neutral component of the centralizer of A in G. It is a connected reductive k-group. Let S be a σ-stable maximal k-split torus of G′, whose existence is asserted by Proposition 2.4 and Lemma 3.1 together. Such a torus S is a σ-stable maximal k-split torus of G containing A. □

Let S be a σ-stable maximal k-split torus of G containing A and P a minimal σ-parabolic k-subgroup of G containing S (see [**17**, §4]). Let ϖ be a uniformizer of k, set $\Lambda = \Lambda(\mathrm{A}_k)$ and let Λ^- denote the subset of anti-dominant elements of Λ relative to P.

Theorem 3.10. — *For $j \in \mathrm{J}$, let $\mathscr{N}_j \subset \mathrm{N}_{\mathrm{G}_k}(\mathrm{A}^j)$ be a set of representatives of $\mathrm{W}_{\mathrm{H}_k}(\mathrm{A}^j)\backslash\mathrm{W}_{\mathrm{G}_k}(\mathrm{A}^j)$ and $y_j \in \mathrm{G}_k$ such that ${}^{y_j}\mathrm{A} = \mathrm{A}^j$. There exists a compact subset Ω of G_k such that:*

$$\mathrm{G}_k = \bigcup_{j\in\mathrm{J}} \bigcup_{n\in\mathscr{N}_j} \mathrm{H}_k n y_j \Lambda^-\Omega.$$

Proof. — First let $\{u_i \mid i \in \mathrm{I}\}$ be a set of representatives of $(\mathrm{H}_k, \mathrm{Z}_k)$-double cosets in $\mathscr{O}$. According to Lemma 3.2, such a set is finite. Let $\mathscr{A}$ denote the apartment corresponding to S, and let K be the stabilizer of x in G_k. Then Proposition 3.4 can be rephrased as follows:

$$\mathrm{G}_k = \bigcup_{i\in\mathrm{I}} \mathrm{H}_k u_i \mathrm{Z}_k \mathrm{K}. \tag{3.1}$$

Let K^1 denote the intersection $\mathrm{K} \cap \mathrm{G}_k^1$ (see §1.15). It is the maximal compact subgroup of K.

Lemma 3.11. — *We have* $\mathrm{Z}_k\mathrm{K} = \Lambda(\mathrm{S}_k)\mathscr{F}\mathrm{K}^1$ *for some finite subset* $\mathscr{F} \subset \mathrm{G}_k$.

Proof. — First note that $\mathrm{Z}_k \cap \mathrm{K}^1 = \mathrm{Z}_k^1$. Indeed, any element of the group Z_k^1, which is the kernel of (1.2), acts trivially on x. Therefore Z_k^1 is contained in K, hence in its maximal compact subgroup K^1. Inversely, the compact group $\mathrm{Z}_k \cap \mathrm{K}^1$ is contained in Z_k, hence in its maximal compact subgroup Z_k^1. According to Remark 1.1, the group $\mathrm{S}_k\mathrm{Z}_k^1$ has finite index in Z_k. Thus there exists a finite subset $\mathscr{F}_1 \subset \mathrm{Z}_k$ such that $\mathrm{Z}_k = \mathscr{F}_1\mathrm{S}_k(\mathrm{Z}_k \cap \mathrm{K}^1)$.

Let D denote the maximal k-split torus of the connected centre C of G. According to Remark 1.3, the image of D_k in $\mathrm{G}_k/\mathrm{G}_k^1$ has finite index, thus its image in K/K^1 too. This implies that $\mathrm{D}_k\mathrm{K}^1 = \Lambda(\mathrm{D}_k)\mathrm{K}^1$ has finite index in K, thus that there exists a finite subset $\mathscr{F}_2 \subset \mathrm{K}$ such that $\mathrm{K} = \mathscr{F}_2\Lambda(\mathrm{D}_k)\mathrm{K}^1$.

Finally, we have:

$$\begin{aligned} \mathrm{Z}_k\mathrm{K} &= \mathscr{F}_1\mathrm{S}_k\mathrm{K} \\ &= \mathscr{F}_1\Lambda(\mathrm{S}_k)\mathrm{K} \\ &= \mathscr{F}_1\Lambda(\mathrm{S}_k)\mathscr{F}_2\Lambda(\mathrm{D}_k)\mathrm{K}^1 \end{aligned}$$

which gives the expected result with $\mathscr{F} = \mathscr{F}_1\mathscr{F}_2$. □

For $i \in \mathrm{I}$, we set $\mathrm{S}^i = {}^{u_i}\mathrm{S}$. According to Lemmas 3.11 and 3.6(iii), there are finite subsets $\mathscr{F} \subset \mathrm{G}_k$ and $\mathscr{V}_i \subset \Lambda(\mathrm{S}_k^i)$, for $i \in \mathrm{I}$, such that:

$$\mathrm{H}_k u_i \mathrm{Z}_k \mathrm{K} = \mathrm{H}_k \Lambda(\mathrm{S}_k^{i-})\mathscr{V}_i u_i \mathscr{F}\mathrm{K}^1. \tag{3.2}$$

According to [**16**, Lemma 2.2], the (σ,k)-split torus S^{i-} is H_k-conjugated to a subtorus of A^j for some $j\in\mathrm{J}$. We can therefore assume that, for a suitable choice of the representative u_i, the (σ,k)-split torus S^{i-} is contained in A^j for some $j\in\mathrm{J}$. For $j\in\mathrm{J}$, let $\mathscr{U}_j$ be the union of the $\mathscr{V}_i u_i\mathscr{F}$ such that A^j contains S^{i-}. Together with (3.1) and (3.2), this gives:

$$\mathrm{G}_k=\bigcup_{j\in\mathrm{J}}\mathrm{H}_k\Lambda(\mathrm{A}_k^j)\mathscr{U}_j\mathrm{K}^1. \tag{3.3}$$

For $j\in\mathrm{J}$, we fix a set $\mathscr{N}_{\mathrm{H}_k,j}$ of representatives of $\mathrm{W}_{\mathrm{H}_k}(\mathrm{A}^j)$ and we denote by $\tilde{\mathscr{N}_j}$ the set $\{hn \mid h\in\mathscr{N}_{\mathrm{H}_k,j}, n\in\mathscr{N}_j\}$. It is a set of representatives of $\mathrm{W}_{\mathrm{G}_k}(\mathrm{A}^j)$. From (3.3) we have:

$$\mathrm{G}_k=\bigcup_{j\in\mathrm{J}}\bigcup_{n\in\tilde{\mathscr{N}_j}}\mathrm{H}_k n\Lambda(\mathrm{A}_k^j)^- n^{-1}\mathscr{U}_j\mathrm{K}^1, \tag{3.4}$$

where $\Lambda(\mathrm{A}_k^j)^-$ denotes the subset of anti-dominant elements of $\Lambda(\mathrm{A}_k^j)$ relative to the parabolic subgroup ${}^{y_j}\mathrm{P}$. If we remark that $\Lambda(\mathrm{A}_k^j)^- = {}^{y_j}\Lambda^-$, and if we denote by Ω the union of the $y_j^{-1}n^{-1}\mathscr{U}_j\mathrm{K}^1$ for $j\in\mathrm{J}$ and $n\in\tilde{\mathscr{N}_j}$, then (3.4) becomes:

$$\mathrm{G}_k=\bigcup_{j\in\mathrm{J}}\bigcup_{n\in\mathscr{N}_j}\mathrm{H}_k n y_j\Lambda^-\Omega. \tag{3.5}$$

This gives us the expected result. □

4. The split case

In this section, we keep using notations of Section 3. Moreover, we assume that the reductive group G is split over k. Therefore, for any root a of G relative to some maximal k-split torus of G, the root subgroup U_a is k-isomorphic to the additive group.

The main results of this section are Proposition 4.4 and Theorem 4.8.

4.1. Let S be a σ-stable maximal k-split torus of G, let $\mathscr{A}$ be the apartment corresponding to S and Φ the set of roots of G relative to S.

Let $x \in \mathscr{A}$ be a special point, and let U_x denote the subgroup U_Ω (see §1.10) with $\Omega = \{x\}$. Let $a \in \Phi$ be a σ-invariant root, which means that $a \circ \sigma = a$.

Lemma 4.1. — *Assume that* $\mathrm{U}_{-a}(k)$ *is contained in* $\{g \in \mathrm{G}_k \mid \sigma(g) = g^{-1}\}$. *Then there are* $n \in \mathrm{N}_k$ *and* $c \in \mathrm{U}_x$ *such that* $n = c^{-1}\sigma(c)$ *and* $\nu(n)$ *is the affine reflection of* $\mathscr{A}$ *which let* x *invariant and whose linear part is* s_a.

Proof. — We fix a base point in the apartment $\mathscr{A}$, so that it can be identified with the vector space V. For any $b \in \Phi$, this defines a filtration of the group $\mathrm{U}_b(k)$ (see §1.8). For $u \in \mathrm{U}_b(k) - \{1\}$, we denote by $\varphi_b(u)$ the greatest real number $r \in \mathbf{R}$ such that $u \in \mathrm{U}_b(k)_r$.

Let us choose $w \in \mathrm{U}_{-a}(k) - \{1\}$ such that x is contained in the wall $\mathscr{H}_{-a,w}$. Thus $\nu(m(w))$ is the affine reflection of $\mathscr{A}$ which fixes x and whose linear part is s_a, and we can set:

$$n = m(w) \in \mathrm{N}_k.$$

Moreover $\theta(-a, w)$, which is the unique affine function from $\mathscr{A}$ to $\mathbf{R}$ whose linear part is $-a$ and whose vanishing hyperplane is $\mathscr{H}_{-a,w}$, vanishes on x. Therefore it is equal to the map:

$$y \mapsto -a(y) + a(x),$$

which implies that $\varphi_{-a}(w) = a(x)$. According to **B3** (see §1.11), it follows that w fixes x.

The subgroup $\mathrm{U}_{-a}(k)$ is isomorphic to the additive group of k. Thus, for any $r \in \mathbf{R}$, the subgroup $\mathrm{U}_{-a}(k)_r$ corresponds through this isomorphism to a nontrivial $\mathfrak{o}$-submodule of k, where $\mathfrak{o}$ denotes the ring of integers of k (see [**20**, Proposition 7.7]). Therefore there is a unique element $v \in \mathrm{U}_{-a}(k)$ such that $w = v^2$ and $\varphi_{-a}(v) = \varphi_{-a}(w)$. Thus $v \in \mathrm{U}_x$.

The map $\mathrm{U}_a(k) \times \mathrm{U}_a(k) \to \mathrm{G}_k$ defined by $(u, u') \mapsto uwu'$ is injective and the intersection given by (1.5) consists of a single element, namely n. If we choose $u, u' \in \mathrm{U}_a(k)$ such that $uwu' = n$, then the element:

$$\sigma(u')^{-1} w \sigma(u)^{-1} = \sigma(n)^{-1} \tag{4.1}$$

is contained in the intersection (1.5). Hence $\sigma(n)^{-1}$ is equal to n, and the unicity property implies that $u' = \sigma(u)^{-1}$. Moreover, according to [**20**, Lemma 7.4(ii)], the real numbers $\varphi_a(u)$ and $\varphi_a(\sigma(u))$ are both equal to $-\varphi_{-a}(w)$. This implies that u and $\sigma(u)$ are contained in U_x. Since v is σ-anti-invariant and $w = v^2$, we get the expected result with $c = (uv)^{-1}$. □

Remark 4.2. — Note that $\sigma(c) \in \mathrm{U}_x$. Indeed we have $\sigma(v) = v^{-1} \in \mathrm{U}_x$ and $\sigma(u) \in \mathrm{U}_x$. Hence $n = c^{-1}\sigma(c) \in \mathrm{N}_k \cap \mathrm{U}_\Omega$, which is contained in N_Ω with $\Omega = \{x, \sigma(x)\}$.

4.2. Let $\mathscr{D}\mathrm{G}$ denote the derived subgroup of G, and recall that C denotes the connected centre of G. This latter subgroup is a k-split torus of G.

Lemma 4.3. — *Let* T *be a k-split torus of* G.

(i) *There is a k-subtorus* T' *of* C *such that the groups* $\mathrm{T} \cdot \mathscr{D}\mathrm{G}$ *and* $\mathrm{T}' \cdot \mathscr{D}\mathrm{G}$ *are equal.*

(ii) *If* T *is (σ, k)-split, then any* T' *satisfying (i) is (σ, k)-split.*

(iii) *Assume that* $\mathscr{D}\mathrm{G}$ *is contained in* H *and* T *is (σ, k)-split. Then any* T' *satisfying (i) is (σ, k)-split and has the same dimension as* T.

Proof. — We set $\tilde{\mathrm{G}} = \mathrm{G}/\mathscr{D}\mathrm{G}$ and, for any k-subgroup K of G, we denote by $\tilde{\mathrm{K}}$ the image of K in $\tilde{\mathrm{G}}$. According to [**6**, Proposition 14.2], the group G is the almost direct product of C and $\mathscr{D}\mathrm{G}$, which means that G is equal to the product $\mathrm{C} \cdot \mathscr{D}\mathrm{G}$ and that the intersection $\mathrm{C} \cap \mathscr{D}\mathrm{G}$ is finite. This implies that $\tilde{\mathrm{C}} = \tilde{\mathrm{G}}$. Let f denote the k-rational map $\mathrm{C} \to \tilde{\mathrm{C}}$. It is surjective with finite kernel. Hence $\tilde{\mathrm{G}}$ is a k-split torus, and we denote by $\tilde{\sigma}$ the involutive k-automorphisme of $\tilde{\mathrm{G}}$ induced by σ. We now prove the lemma in three steps.

(i) According to [**6**, Proposition 8.2(c)], the neutral component of the inverse image $f^{-1}(\tilde{\mathrm{T}})$ is a k-split subtorus of C which we denote by T'. It has finite index in $f^{-1}(\tilde{\mathrm{T}})$. The image $f(\mathrm{T}')$ is then a subtorus of finite index in the connected group $\tilde{\mathrm{T}}$, so that $\tilde{\mathrm{T}}' = \tilde{\mathrm{T}}$.

(ii) Now assume that T is (σ, k)-split, and let T' satisfy (i). Let us consider the map $t \mapsto t\sigma(t)$ from T' to itself. As $\tilde{\mathrm{T}}' = \tilde{\mathrm{T}}$ is a $(\tilde{\sigma}, k)$-split torus, the image of this map is a connected k-subgroup contained in the kernel of f, which is finite.

(iii) Assume that $\mathscr{D}$G is contained in H and T is (σ,k)-split. Then the map $\mathrm{T}\to\tilde{\mathrm{T}}$ has finite kernel, which implies that T and $\tilde{\mathrm{T}}$ have the same dimension. Now let T′ satisfy (i). According to (ii), such a torus is (σ,k)-split, and it has the same dimension as $\tilde{\mathrm{T}}'=\tilde{\mathrm{T}}$.

This ends the proof of Lemma 4.3. □

4.3. Let $\mathscr{B}$ denote the building of G over k.

Proposition 4.4. — *Let x be a special point of $\mathscr{B}$. There exists a σ-stable maximal k-split torus* S *of* G *such that the apartment corresponding to* S *contains x and such that* S^- *is a maximal (σ,k)-split torus of* G.

Remark 4.5. — In §5.3 we give an example of a *non split* k-group G such that Proposition 4.4 does not hold.

Proof. — Let $\mathscr{A}$ be a σ-stable apartment containing x (see Proposition 2.4) and let S be the corresponding maximal k-split torus of G. Assume that $\mathscr{A}$ has been chosen such that the dimension of the (σ,k)-split torus S^- is maximal. If it is a maximal (σ,k)-split torus of G, then Proposition 4.4 is proved. Assume that this is not the case, and let A be a maximal (σ,k)-split torus of G containing S^-. The dimension of A is greater than $\dim \mathrm{S}^-$ (if not, the containment $\mathrm{S}^-\subset\mathrm{A}$ would imply that $\mathrm{S}^-=\mathrm{A}$). If we get a contradiction, the proposition will be proved.

Let G′ be the neutral component of the centralizer of S^- in G. It is a k-split connected reductive subgroup of G containing S and A, which is naturally endowed with a nontrivial action of σ. Let C′ denote the connected center of G′.

Lemma 4.6. — *There is $a\in\Phi(\mathrm{G}',\mathrm{S})$ such that the corresponding root subgroup U'_a is not contained in* H, *and such a root is σ-invariant.*

Proof. — Assume that $\mathrm{U}'_a\subset\mathrm{H}$ for each root $a\in\Phi(\mathrm{G}',\mathrm{S})$. Thus the derived subgroup $\mathscr{D}\mathrm{G}'$, which is generated by the U'_a for $a\in\Phi(\mathrm{G}',\mathrm{S})$, is contained in H (see [**18**, Theorem 27.5(e)]). According to Lemma 4.3(iii), there exists a (σ, k)-subtorus A′ of C′ such that $\mathrm{A} \cdot \mathscr{D}\mathrm{G}' = \mathrm{A}' \cdot \mathscr{D}\mathrm{G}'$ and $\dim(\mathrm{A}) = \dim(\mathrm{A}')$.

The subgroup generated by C′ and S is a k-torus of G′. As G′ is k-split, S is a maximal torus of G′, hence it contains C′. Therefore S^- contains A′ which has the same dimension as A, and this dimension is greater than $\dim \mathrm{S}^-$. This gives us a contradiction.

Now let a be a root in $\Phi(\mathrm{G}', \mathrm{S})$ such that U'_a is not contained in H. The root a and its conjugate $a \circ \sigma$ coincide on S^+ and are both trivial on S^-. As S is the almost direct product of S^+ and S^- (see §3.4), they are equal. Therefore a is σ-invariant. This ends the proof of Lemma 4.6. □

Let $a \in \Phi(\mathrm{G}', \mathrm{S})$ as in Lemma 4.6. If we think to a as a root in $\Phi(\mathrm{G}, \mathrm{S})$, the root subgroup U_a is σ-stable and is not contained in H. Moreover, we have the following result.

Lemma 4.7. — *$\mathrm{U}_a(k)$ is contained in $\{g \in \mathrm{G}_k \mid \sigma(g) = g^{-1}\}$.*

Proof. — As G is k-split, U_a is k-isomorphic to the additive group. Thus the action of σ on $\mathrm{U}_a(k)$ corresponds to an involutive automorphism of the k-algebra $k[t]$. It has the form $t \mapsto \lambda t$ for some $\lambda \in k^\times$ with $\lambda^2 = 1$. As U_a is not contained in H, we have $\lambda = -1$. This gives the expected result. □

According to Lemma 4.1, there are $n \in \mathrm{N}_k$ and $c \in \mathrm{U}_x$ such that $n = c^{-1}\sigma(c)$ and $\nu(n)$ is the affine reflection of $\mathscr{A}$ which let x invariant and whose linear part is s_a. For any $t \in \mathrm{S}$, note that we have:

$$\begin{aligned} \sigma(ctc^{-1}) &= cn\sigma(t)n^{-1}c^{-1} \\ &= cs_a(\sigma(t))c^{-1}. \end{aligned}$$

Let $\mathscr{A}'$ denote the apartment $c \cdot \mathscr{A}$ and let $\mathrm{S}' = {}^c\mathrm{S}$ be the corresponding maximal k-split torus of G. Then $\mathscr{A}'$ contains x and is σ-stable. Moreover, as the root a is trivial on S^- and s_a fixes the kernel of a pointwise, the conjugate ${}^c(\mathrm{S}^-)$ is a (σ, k)-split subtorus of S′. Thus S'^- has dimension not smaller than $\dim \mathrm{S}^-$.

Now let S_a denote the maximal k-split torus in the set of all $t \in \mathrm{S}$ such that $s_a(t) = t^{-1}$. As a is σ-invariant, such a torus is σ-stable. Moreover, it is one dimensional and its intersection with $\mathrm{Ker}(a)$ is finite. Therefore the conjugate ${}^c\mathrm{S}_a$ is a nontrivial (σ, k)-split subtorus of S' which is not contained in ${}^c(\mathrm{S}^-)$. Thus the dimension of S'^-, which contains ${}^c(\mathrm{S}_a\mathrm{S}^-)$, is greater than $\dim \mathrm{S}^-$, which contradicts the maximality property of $\mathscr{A}$. This ends the proof of Proposition 4.4. □

4.4. Let A be a maximal (σ, k)-split torus of G and S a σ-stable maximal k-split torus of G containing A. Let $\{\mathrm{A}^j \mid j \in \mathrm{J}\}$ be a set of representatives of the H_k-conjugacy classes of maximal (σ, k)-split tori in G. Let x be a special point of the building and let K be its stabilizer in G_k.

Theorem 4.8. — *For $j \in \mathrm{J}$, let $y_j \in \mathrm{G}_k$ such that ${}^{y_j}\mathrm{A} = \mathrm{A}^j$. We have:*

$$\mathrm{G}_k = \bigcup_{j \in \mathrm{J}} \mathrm{H}_k y_j \mathrm{S}_k \mathrm{K}.$$

Proof. — We fix $g \in \mathrm{G}_k$. According to Proposition 4.4, there is a σ-stable maximal k-split torus S' of G such that the apartment corresponding to it contains $g \cdot x$ and such that S'^- is a maximal (σ, k)-split torus of G. Let $j \in \mathrm{J}$ be such that S'^- is H_k-conjugate to A^j. According to [**16**, Lemma 2.2], there is $h \in \mathrm{H}_k$ such that $\mathrm{S}' = {}^{hy_j}\mathrm{S}$. Hence $g \cdot x$ is contained in $hy_j \cdot \mathscr{A}$. According to Property (2) of §1.13, there exists $n \in \mathrm{N}_k$ such that $g \cdot x = hy_j n \cdot x$.

Therefore G_k is the union of the $\mathrm{H}_k y_j \mathrm{N}_k \mathrm{K}$ for $j \in \mathrm{J}$. As x is special, we have $\mathrm{N}_k\mathrm{K} = \mathrm{S}_k\mathrm{K}$ and we get the expected result. □

5. Examples

Let k be a non-Archimedean locally compact field of residue characteristic different from 2. Let $\mathfrak{o}$ be its ring of integers and $\mathfrak{p}$ its maximal ideal.

5.1. Here we consider the connected reductive k-group $\mathrm{G} = \mathrm{GL}_n$, endowed with the k-involution $\sigma : g \mapsto {}^t g^{-1}$, where ${}^t g$ denotes the transpose of $g \in \mathrm{G}$.

We set $\mathrm{K} = \mathrm{GL}_n(\mathfrak{o})$ and $\mathrm{H} = \mathrm{G}^\sigma$, which is an orthogonal group, and we denote by S the diagonal torus of G.

We start with the following lemma.

Lemma 5.1. — *Let* V *be a finite dimensional k-vector space and* B *a symmetric bilinear form on* V*. Then any free $\mathfrak{o}$-submodule of finite rank of* V *has a basis which is orthogonal relative to* B*.*

Proof. — Let Λ be a free $\mathfrak{o}$-submodule of finite rank of V. The proof goes by induction on the rank of Λ. If B is null, then the result is trivial. If not, we denote by B_Λ the restriction of B to $\Lambda \times \Lambda$. Its image is of the form $\mathfrak{p}^m$ for some integer $m \in \mathbf{Z}$. If ϖ is a uniformizer of k, then the form $\mathrm{B}_\Lambda^0 = \varpi^{-m}\mathrm{B}_\Lambda$ has image $\mathfrak{o}$ on $\Lambda \times \Lambda$. Therefore, it defines a non trivial bilinear form:

$$\bar{\mathrm{B}}_\Lambda^0 : \Lambda/\mathfrak{p}\Lambda \times \Lambda/\mathfrak{p}\Lambda \to \mathfrak{o}/\mathfrak{p}.$$

Let $e \in \Lambda$ be a vector whose reduction mod. $\mathfrak{p}$ is not isotropic relative to $\bar{\mathrm{B}}_\Lambda^0$, which means that $\mathrm{B}_\Lambda^0(e,e)$ is a unit of $\mathfrak{o}$. Then Λ is the direct sum of $\mathfrak{o}e$ and $\Lambda \cap ke^\perp$, where $ke^\perp$ denotes the orthogonal of ke in V. Indeed, it follows from the decomposition:

$$x = \frac{\mathrm{B}(e,x)}{\mathrm{B}(e,e)}e + \Big(x - \frac{\mathrm{B}(e,x)}{\mathrm{B}(e,e)}e\Big)$$

for any $x \in \Lambda$. As $\Lambda \cap ke^\perp$ is a free $\mathfrak{o}$-submodule of finite rank of V whose rank is smaller than the rank of Λ, we conclude by induction. □

We introduce the set $\mathscr{E}$ of all $g \in \mathrm{G}_k$ such that ${}^tgg \in \mathrm{S}_k$ (compare §3.2). We have the following decomposition of G_k, which is more precise than the one given by Proposition 3.4.

Proposition 5.2. — *We have* $\mathrm{G}_k = \mathscr{E}\mathrm{K}$*.*

Proof. — We make G_k act on the quotient G_k/K, which can be identified to the set of all $\mathfrak{o}$-lattices (that is, cocompact free $\mathfrak{o}$-submodules) of the k-vector space $\mathrm{V} = k^n$. Let B denote the symmetric bilinear form on V making the canonical basis of V into an orthonormal basis. According to Lemma 5.1, for any $g \in \mathrm{G}_k$, the $\mathfrak{o}$-lattice Λ corresponding to the class $g\mathrm{K}$ has a basis which is orthogonal relative to B. This means that there exists $u \in \mathrm{K}$ such that the element $g' = gu^{-1} \in g\mathrm{K}$ maps the canonical basis of V to an orthogonal basis of Λ. Therefore we have $g' \in \mathscr{E}$, thus $g \in \mathscr{E}\mathrm{K}$. □

We now investigate the maximal (σ, k)-split tori of G. Note that S is a maximal (σ, k)-split torus of G.

Proposition 5.3. — *The map $g \mapsto {}^g\mathrm{S}$ induces a bijection between $(\mathrm{H}_k, \mathrm{N}_k)$-double cosets of $\mathscr{E}$ and H_k-conjugacy classes of maximal (σ, k)-split tori of* G.

Proof. — One immediately checks that this map is well defined and injective. For $g \in \mathrm{G}_k$, the conjugate ${}^g\mathrm{S}$ is a maximal (σ, k)-split torus of G if anf only if $g^{-1}\sigma(g) \in \mathrm{S}_k$, which amounts to saying that $g \in \mathscr{E}$ and proves surjectivity. □

Let $\mathscr{Q}$ denote the set of all equivalence classes of non degenerate quadratic forms on k^n. For $a = \mathrm{diag}(a_1, \ldots, a_n) \in \mathrm{S}_k$ we denote by Q_a the diagonal quadratic form $a_1\mathrm{X}_1^2 + \ldots + a_n\mathrm{X}_n^2$. Note that the map $a \mapsto \mathrm{Q}_a$ induces a surjective map from S_k to $\mathscr{Q}$.

Proposition 5.4. — (i) *The map $g \mapsto {}^tgg$ induces an injection ι from the set of $(\mathrm{H}_k, \mathrm{N}_k)$-double cosets of $\mathscr{E}$ to $\mathrm{H}^1(\mathrm{N}_k)$.*

(ii) *For $a \in \mathrm{S}_k$, the class of a in $\mathrm{H}^1(\mathrm{N}_k)$ is in the image of ι if and only if $\mathrm{Q}_a \sim \mathrm{X}_1^2 + \ldots + \mathrm{X}_n^2$.*

Proof. — We have an exact sequence:

$$\mathrm{H}_k \to \mathrm{H}^0(\mathrm{G}_k/\mathrm{N}_k) \to \mathrm{H}^1(\mathrm{N}_k) \to \mathrm{H}^1(\mathrm{G}_k),$$

where the map from $\mathrm{H}^0(\mathrm{G}_k/\mathrm{N}_k)$ to $\mathrm{H}^1(\mathrm{N}_k)$ is induced by $g \mapsto {}^tgg$. As the set of $(\mathrm{H}_k, \mathrm{N}_k)$-double cosets of $\mathscr{E}$ is a subset of $\mathrm{H}_k\backslash\mathrm{H}^0(\mathrm{G}_k/\mathrm{N}_k)$, we get (i). To get (ii), it is enough to remark that $\mathrm{H}^1(\mathrm{G}_k)$ canonically identifies with $\mathscr{Q}$. □

Remark 5.5. — Recall (see [**27**, IV §2.3]) that for $a, b \in \mathrm{S}_k$, the nondegenerate quadratic forms $\mathrm{Q}_a, \mathrm{Q}_b$ are equivalent if, and only if they have the same discriminant and the same Hasse invariant.

Proposition 5.6. — *Let* $\{a^j \mid j \in \mathrm{J}\} \subset \mathrm{S}_k$ *form a set of representatives of* $\mathrm{Im}(\iota)$ *in* $\mathrm{H}^1(\mathrm{N}_k)$. *For* $j \in \mathrm{J}$, *we choose* $y_j \in \mathscr{E}$ *such that* ${}^t y_j y_j = a^j$. *Then:*

$$\mathrm{G}_k = \bigcup_{j \in \mathrm{J}} \mathrm{H}_k y_j \mathrm{S}_k \mathrm{K}. \tag{5.1}$$

Proof. — Propositions 5.2 and 5.3 imply that G_k is the union of the components $\mathrm{H}_k y_j \mathrm{N}_k \mathrm{K}$ for $j \in \mathrm{J}$. As $\mathrm{N}_k\mathrm{K} = \mathrm{S}_k\mathrm{K}$ we get the expected result. □

Example 5.7. — In the case where $n = 2$, we give an explicit description of $\mathrm{Im}(\iota)$. Let ϖ denote a uniformizer of $\mathfrak{o}$ and $\xi \in \mathfrak{o}^\times$ a non square unit of $\mathfrak{o}$, so that $\{1, \xi, \varpi, \xi\varpi\}$ is a set of representatives of $k^\times$ modulo $k^{\times 2}$. The set of elements of $k^\times$ which are represented by the quadratic form $\mathrm{Q}_1 = \mathrm{X}^2 + \mathrm{Y}^2$ depends on the image of p in $\mathbf{Z}/4\mathbf{Z}$. If $p \equiv 1$ mod. 4, all elements of $k^\times$ are represented by Q_1. If $p \equiv 3$ mod. 4, an element of $k^\times$ is represented by Q_1 if and only if its normalized valuation if even. We set:

$$\mathrm{J} = \begin{cases} \{1, \xi, \varpi, \xi\varpi\} & \text{if } p \equiv 1 \text{ mod. } 4, \\ \{1, \xi\} & \text{if } p \equiv 3 \text{ mod. } 4. \end{cases}$$

For each $j \in \mathrm{J}$, set $a^j = \mathrm{diag}(j, j)$. Then the elements a^j form a set of representatives of $\mathrm{Im}(\iota)$ in $\mathrm{H}^1(\mathrm{N}_k)$.

5.2. In this paragraph we consider the connected reductive k-group $\mathrm{G} = \mathrm{Res}_{k'/k}\mathrm{GL}_n$, where k' is a quadratic extension of k, endowed with the involutive k-automorphism σ of G induced by the nontrivial element of $\mathrm{Gal}(k'/k)$.

We set $\mathrm{H} = \mathrm{G}^\sigma$, so that we have $\mathrm{G}_k = \mathrm{GL}_n(k')$ and $\mathrm{H}_k = \mathrm{GL}_n(k)$. We denote by S the diagonal torus of G and by K the maximal compact subgroup $\mathrm{GL}_n(\mathfrak{o}')$ of G_k, where $\mathfrak{o}'$ denotes the ring of integers of k'. Note that S is σ-invariant, that is $\mathrm{S} = \mathrm{S}^+$.

As usual, N (resp. Z) denotes the normalizer (resp. the centralizer) of S in G. Let $\mathfrak{S}_n$ denote the group of permutation matrices in G_k, so that N_k is the semidirect product of $\mathfrak{S}_n$ by Z_k. Note that S_k (resp. Z_k) is the subgroup of all diagonal matrices of G_k with entries in k (resp. in k').

Lemma 5.8. — $\mathrm{H}^1(\mathrm{N}_k)$ *can be identified with the set of conjugacy classes of elements of* $\mathfrak{S}_n$ *of order* 1 *or* 2.

Proof. — According to Hilbert's Theorem 90, the group $\mathrm{H}^1(\mathrm{Z}_k)$ is trivial. Therefore we have an exact sequence:

$$1 \to \mathrm{H}^1(\mathrm{N}_k) \to \mathrm{H}^1(\mathrm{N}_k/\mathrm{Z}_k). \tag{5.2}$$

As σ acts trivially on $\mathrm{N}_k/\mathrm{Z}_k \simeq \mathfrak{S}_n$, the set $\mathrm{H}^1(\mathrm{N}_k/\mathrm{Z}_k)$ can be identified to the set of $\mathfrak{S}_n$-conjugacy classes of $\mathrm{Hom}(\mathbf{Z}/2\mathbf{Z}, \mathfrak{S}_n)$, that is, to the set of conjugacy classes of elements of $\mathfrak{S}_n$ of order $\leqslant 2$. This proves that $\mathrm{H}^1(\mathrm{N}_k)$ can be naturally embedded in the set of conjugacy classes of elements of $\mathfrak{S}_n$ of order $\leqslant 2$.

Now two elements $w, w' \in \mathfrak{S}_n$ define the same class in $\mathrm{H}^1(\mathrm{N}_k)$ if and only if they are conjugate in $\mathfrak{S}_n$, thus if and only if $w\mathrm{Z}_k$ and $w'\mathrm{Z}_k$ define the same class in $\mathrm{H}^1(\mathrm{N}_k/\mathrm{Z}_k)$. Therefore (5.2) is a bijection. □

Proposition 5.9. — (i) *The number of* H_k*-conjugacy classes of* σ*-stable maximal* k*-split tori in* G_k *is* $[n/2]+1$.

(ii) *There is a unique* H_k*-conjugacy class of maximal* (σ,k)*-split tori in* G_k.

Proof. — (i) Let $\mathscr{O}$ denote the set of all $g \in \mathrm{G}_k$ such that $g^{-1}\sigma(g) \in \mathrm{N}_k$. Then the map $g \mapsto {}^g\mathrm{S}$ defines an injective map from the set of $(\mathrm{H}_k, \mathrm{N}_k)$-double cosets of $\mathscr{O}$ to $\mathrm{H}^1(\mathrm{N}_k)$. Therefore we are reduced to proving that this map is surjective, and (i) will follow from Lemma 5.8

For $n = 2$, let τ denote the nontrivial element of $\mathfrak{S}_2$ and choose an element $a \in k'$ which is not in k. Then the element:

$$u = \begin{pmatrix} a & \sigma(a) \\ 1 & 1 \end{pmatrix} \in \mathrm{GL}_2(k') \tag{5.3}$$

satisfies the relation $u^{-1}\sigma(u) = \tau$. For an arbitrary integer $n \geqslant 2$, let $w \in \mathfrak{S}_n$ have order $\leqslant 2$. Then there is an integer $0 \leqslant i \leqslant [n/2]$ such that w is conjugate to the element:

$$\tau_i = \mathrm{diag}(\tau, \dots, \tau, 1, \dots, 1) \in \mathrm{GL}_n(k'),$$

where $\tau \in \mathrm{GL}_2(k')$ appears i times and $1 \in \mathrm{GL}_1(k')$ appears $n-2i$ times. Thus the matrice:

$$u_i = \mathrm{diag}(u, \dots, u, 1, \dots, 1) \in \mathrm{GL}_n(k') \tag{5.4}$$

satisfies the relation $u_i^{-1}\sigma(u_i) = \tau_i$. Therefore any cocycle in N_k is G_k-cohomologous to the neutral element $1 \in \mathrm{G}_k$, which proves (i).

(ii) For any $0 \leqslant i \leqslant [n/2]$, the dimension of the (σ, k)-split torus $({}^{u_i}\mathrm{S})^-$ is equal to i. According to (i), the map:

$$\mathrm{H}_k g \mathrm{N}_k \mapsto \text{class of } g^{-1}\sigma(g) \text{ in } \mathrm{H}^1(\mathrm{N}_k)$$

is a bijection from the set of $(\mathrm{H}_k, \mathrm{N}_k)$-double cosets of $\mathscr{O}$ to $\mathrm{H}^1(\mathrm{N}_k)$, and the elements of this latter set are the classes of the τ_i for $0 \leqslant i \leqslant [n/2]$. This gives us the expected result.

This ends the proof of Proposition 5.9. □

Proposition 5.10. — *For* $0 \leqslant i \leqslant [n/2]$, *let* u_i *denote the element defined by (5.3) and (5.4). Then we have:*

$$\mathrm{G}_k = \bigcup_{i=0}^{[n/2]} \mathrm{H}_k u_i \mathrm{Z}_k \mathrm{K}.$$

Proof. — According to the proof of Proposition 5.9, the set $\mathscr{O}$ is the union of the double cosets $\mathrm{H}_k u_i \mathrm{N}_k$ with $0 \leqslant i \leqslant [n/2]$. The result then follows from Proposition 3.4 and from the fact that $\mathrm{N}_k\mathrm{K} = \mathrm{Z}_k\mathrm{K}$. □

5.3. Here we give an example (due to Bertrand Lemaire) of a non-split k-group such that Proposition 4.4 does not hold. We set $\mathrm{G} = \mathrm{Res}_{k'/k}\mathrm{GL}_2$, where k' is now a *totally ramified* quadratic extension of k. The k-involution σ is still induced by the nontrivial element of $\mathrm{Gal}(k'/k)$ and we set $\mathrm{H} = \mathrm{GL}_2$. Let $\mathscr{B}'$ (resp. $\mathscr{B}$) denote the building of G (resp. H) over k.

In [**10**], Bruhat and Tits give a description of the faces of $\mathscr{B}$ in terms of hereditary $\mathfrak{o}$-orders of $\mathrm{M}_2(k)$. More precisely, there is a bijective correspondence:

$$\mathrm{F} \mapsto \mathscr{M}_{\mathrm{F}} \tag{5.5}$$

between the faces of $\mathscr{B}$ and the hereditary $\mathfrak{o}$-orders of $M_2(k)$, such that the stabilizer of F in $GL_2(k)$ in the normalizer of $\mathscr{M}_F$ in $GL_2(k)$. For $x \in \mathscr{B}$, we will denote by $\mathscr{M}_x$ the hereditary order corresponding to the face of $\mathscr{B}$ which contains x. Of course, we have a similar correspondence between faces of $\mathscr{B}'$ and hereditary $\mathfrak{o}'$-orders of $M_2(k')$. Moreover, as k' is tamely ramified over k, there is a bijective correspondence j from the set $\mathscr{B}'^{\sigma}$ of σ-invariant points of $\mathscr{B}'$ to $\mathscr{B}$ such that, for any $x \in \mathscr{B}'^{\sigma}$, we have:

$$\mathscr{M}_{j(x)} = \mathscr{M}_x \cap M_2(k).$$

Let q denote the cardinal of the residue field of k. As k' is totally ramified over k, any vertex of $\mathscr{B}$ (resp. $\mathscr{B}'$) has exactly $q+1$ neighbours in $\mathscr{B}$ (resp. in $\mathscr{B}'$). Let x be a σ-invariant point of $\mathscr{B}'$. Recall that, according to Proposition 2.4, it is contained in a σ-stable apartment.

(1) If $j(x)$ is in a chamber of $\mathscr{B}$, then x has $q+1$ neighbours in $\mathscr{B}'$ but only two σ-invariant ones. Thus x has non-σ-invariant neighbours.

(2) If $j(x)$ is a vertex of $\mathscr{B}$, then x has $q+1$ neighbours in $\mathscr{B}'$ as in $\mathscr{B}$. Thus any neighbour of x in $\mathscr{B}'$ is σ-invariant, which implies that any σ-stable apartment containing x is σ-invariant. For instance, this is the case of the vertex x corresponding to the $\mathfrak{o}'$-order $M_2(\mathfrak{o}')$, because its image $j(x)$ corresponds to the maximal $\mathfrak{o}$-order $M_2(\mathfrak{o}') \cap M_2(k) = M_2(\mathfrak{o})$. Such a special point does not satisfy Proposition 4.4.

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