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On intersections of orbital varieties and components of Springer fiber

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Abstract

We consider Springer fibers and orbital varieties for GL_n. We show that the irreducible components of an intersection of components of Springer fiber are in bijection with the irreducible components of intersection of orbital varieties; moreover, the corresponding irreducible components in this correspondence have the same codimension. Finally we give a sufficient condition to have an intersection in codimension one.

Keywords: Flag manifold; Springer fibers; Orbital varieties; Robinson–Schensted correspondence; Schubert cell

1. Introduction

1.1. Let G be a semisimple (connected) complex algebraic group with Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$ on which G acts by the adjoint action. For $g \in G$ and $u \in \mathfrak{g}$ we denote this action by $g.u := gug^{-1}$.

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Fix a Cartan subalgebra $\mathfrak{h}$. Let $\mathcal{W}$ denote the associated Weyl group. We have the Chevalley–Cartan decomposition of $\mathfrak{g}$:

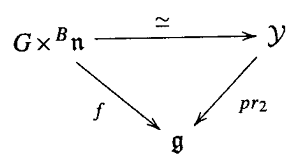
$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{R}} \mathfrak{g}_\alpha,$$

where $\mathcal{R}$ is the root system of $\mathfrak{g}$ relatively to $\mathfrak{h}$. Let Π be a set of simple roots of $\mathcal{R}$. Denote $\mathcal{R}^+$ (respectively $\mathcal{R}^-$) the positive roots (respectively negative roots) (w.r.t. Π). We sometimes prefer the notation $\alpha > 0$ (respectively $\alpha < 0$) to designate a positive (respectively negative) root. Let $\mathfrak{b} := \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{R}^+} \mathfrak{g}_\alpha$ be the standard Borel subalgebra (w.r.t. Π) and $\mathfrak{n} := \sum_{\alpha \in \mathcal{R}^+} \mathfrak{g}_\alpha$ its nilpotent radical. Let B be the Borel subgroup of G with $\mathrm{Lie}(B) = \mathfrak{b}$.

Let $G \times^B \mathfrak{n}$ be the space obtained as the quotient of $G \times \mathfrak{n}$ by the right action of B given by $(g, x).b := (gb, b^{-1}.x)$ with $g \in G$, $x \in \mathfrak{n}$ and $b \in B$. By the Killing form we get the following identification $G \times^B \mathfrak{n} \simeq T^*(G/B)$. Let $g * x$ denote the class of (g, x) and $\mathcal{F} := G/B$ the flag manifold. The map $G \times^B \mathfrak{n} \to \mathcal{F} \times \mathfrak{g}$, $g * x \mapsto (gB, g.x)$ is an embedding which identify $G \times^B \mathfrak{n}$ with the following closed subvariety of $\mathcal{F} \times \mathfrak{g}$ (see [16, p. 19]):

$$\mathcal{Y} := \{(gB, x) \mid x \in g.\mathfrak{n}\}.$$

The map $f : G \times^B \mathfrak{n} \to \mathfrak{g}$, $g * x \mapsto g.x$ is called the *Springer resolution* and we have the following commutative diagram:



where $pr_2 : \mathcal{F} \times \mathfrak{g} \to \mathfrak{g}$, $(gB, x) \mapsto x$. The map f is proper (because G/B is complete) and its image is exactly $G.\mathfrak{n} = \mathcal{N}$, the *nilpotent variety* of $\mathfrak{g}$ [21].

Let x be a nilpotent element in $\mathfrak{n}$. By the diagram above we have:

$$\mathcal{F}_x := f^{-1}(x) = \{gB \in \mathcal{F} \mid x \in g.\mathfrak{n}\} = \{gB \in \mathcal{F} \mid g^{-1}.x \in \mathfrak{n}\}. \tag{1.1}$$

The variety $\mathcal{F}_x$ is called the *Springer fiber* above x and has been studied by many authors. It was one of the most stimulating subjects during the last three decades, appearing in many areas, for example, in representation theory and singularity theory. But it remains a very mysterious object, and the major difficulty is its geometric description which is known in a few cases. For x in the regular orbit in $\mathfrak{g}$ it is reduced to one point. For x in the subregular orbit in $\mathfrak{g}$ it is a finite union of projective lines which intersect themselves transversally and is usually called the *Dynkin curve*, it was obtained by J. Tits (see e.g. [24, Theorem 2, p. 153]). For x in the minimal orbit its irreducible components are some Schubert varieties [2].

The Springer fibers arise in many contexts. They arise as fibers of Springer's resolution of singularities of the nilpotent variety in [16,17,21]. In the course of these investigations, Springer defined $\mathcal{W}$-module structures on the rational homology groups $H_*(\mathcal{F}_x, \mathbb{Q})$ on which also the finite group $A(x) = Z_G(x)/Z_G^o(x)$ (where $Z_G(x)$ is a stabilizer of x and $Z_G^o(x)$ is its neutral component) acts compatibly. Set $d = \dim(\mathcal{F}_x)$, the $A(x)$-fixed subspace $H_{2d}(\mathcal{F}_x, \mathbb{Q})^{A(x)}$ of the top homology is known to be irreducible [22].

In [8], D. Kazhdan and G. Lusztig tried to understand Springer's work connecting nilpotent classes and representations of Weyl groups. Among problems they have posed, the conjecture 6.3 in [8] has stimulated much research into the relation between the Kazhdan–Lusztig basis and the Springer fibers.

1.2. More known for $G = \mathrm{GL}_n$. For $x \in \mathfrak{n}$ its only characteristic value is 0, so that its Jordan form is completely defined by $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition of n where λ_i is the length of ith Jordan block. Arrange the numbers in a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ in the decreasing order (that is $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 1$) and write $J(x) = \lambda$. In turn an ordered partition can be presented as a Young diagram D_λ—an array with k rows of boxes starting on the left with the ith row containing λ_i boxes. In such a way there is a bijection between Springer fibers and Young diagrams.

Fill the boxes of Young diagram D_λ with n distinct positive integers. If the entries increase in rows from left to right and in columns from top to bottom we call such an array a Young tableau or simply a tableau of shape λ. Let $\mathbf{Tab}_\lambda$ be the set of all Young tableaux of shape λ.

Given $x \in \mathfrak{n}$ such that $J(x) = \lambda$ by Spaltenstein [18] and Steinberg [26] there is a bijection between components of $\mathcal{F}_x$ and $\mathbf{Tab}_\lambda$ (cf. 2.5). For $T \in \mathbf{Tab}_\lambda$ set $\mathcal{F}_T$ to be the corresponding component of $\mathcal{F}_x$.

For GL_n the conjecture of Kazhdan and Lusztig mentioned in 1.1 is equivalent to the irreducibility of certain characteristic varieties [1, Conjecture 4]. It was shown to be reducible in general by Kashiwara and Saito [7]. Nevertheless, the description of pairwise intersections of the irreducible components of the Springer fibers is still open. In particular the determination in terms of Young tableaux of a pair of irreducible components with the intersections in codimension 1 is unknown in general. The search of these intersections is the main motivation of our paper. The general answer seems to be beyond our means but we can address these questions in some special cases.

Let us first describe the answers in the special cases which are already known.

1.3. The description of the Springer fiber was completely done for the hook and two-row Young diagrams in [4,27]. P. Lorist studied the Springer fiber of dimension 2, [10]. He showed in that case that all the irreducible components of the Springer fiber are either the product of two projective lines or are ruled surfaces over a projective line with $e = 2$ and he also gave the complete description of the intersection between them; his method is very basic but very cumbersome, it consists of calculations of the different intersections of the Springer fiber with every Schubert cell and then pasting them together.

For one of us this work was motivated by Lorist's work, by the desire to find a more efficient way of computation of the Springer fiber (cf. [14, p. 108]). The idea is to find the unique Schubert cell which intersects generically with a given irreducible component. Obviously the determination of such Schubert cell depends on the choice of the point above which we are looking at the Springer fiber, another point will generate another Schubert cell. In this work we will determine all these possibilities, in fact it will be realized just by interpreting in a geometric way the notion of Young cell (see Theorem 2.13). Actually this interpretation helps to understand a work of Tits [21] who showed that any two points of $\mathcal{F}_x$ can be connected by a finite union of projective lines. An immediate application of this interpretation is the sufficient condition for the intersection of two irreducible components of the Springer fiber to be in codimension one (see Remark 3.4).

1.4. Let us return to a semisimple algebraic group G. Let $x \in \mathfrak{n}$ be some nilpotent element and let $\mathcal{O}_x = G.x$ be its orbit. Consider $\mathcal{O}_x \cap \mathfrak{n}$. Its irreducible components are called orbital varieties associated to $\mathcal{O}_x$. By Spaltenstein's construction [19] there is a tight connection between $\mathcal{F}_x$ and $\mathcal{O}_x \cap \mathfrak{n}$. We explain it in 2.1.

In particular, for $G = \mathrm{GL}_n$ the Spaltenstein's construction provides the bijection between the orbital varieties associated to $\mathcal{O}_x$ and components of $\mathcal{F}_x$. That is let $J(x) = \lambda$ then there is a natural bijection ϕ between $\{\mathcal{F}_T\}_{T\in\mathbf{Tab}_\lambda}$ and the set of orbital varieties associated to $\mathcal{O}_x$. Let us denote the set of orbital varieties by $\{\mathcal{V}_T\}_{T\in\mathbf{Tab}_\lambda}$ where $\mathcal{V}_T = \phi(\mathcal{F}_T)$. As a straightforward corollary of this construction we get in Proposition 2.2 that the number of irreducible components and their codimensions of $\mathcal{F}_T \cap \mathcal{F}_{T'}$ are equal to the number of irreducible components and their codimensions of $\mathcal{V}_T \cap \mathcal{V}_{T'}$. Thus from our point of view orbital varieties are equivalent to the components of Springer fibre.

1.5. The body of the paper consists of three sections. In Section 2 we explain Spaltenstein's and Steinberg's constructions and show that on the level of intersections the components of Springer fibre and orbital varieties are the same objects. Finally in Section 3 we give an sufficient condition to have an intersection in codimension one.

2. The Spaltenstein's and Steinberg's constructions

2.1. We start with the Spaltenstein's construction [19]. Recall notation from 1.1 and from 1.4. Given $x \in \mathfrak{n}$ put $G_x = \{g \in G\colon\ g^{-1}xg \in \mathfrak{n}\}$. Set $f_1 : G_x \to \mathcal{O}_x \cap \mathfrak{n}$ by $f_1(g) = g^{-1}xg$. Note that f_1 is a surjection. Let $\{\mathcal{V}_i\}_{i=1}^k$ be the set of orbital varieties associated to $\mathcal{O}_x$ and $Y_i = f_1^{-1}(\mathcal{V}_i)$ its preimage in G_x. One has Y_i is closed in G_x and $G_x = \bigcup_{i=1}^k Y_i$.

Set $f_2 : G_x \to \mathcal{F}_x$ by $f_2(g) = gB$. Again, f_2 is a surjection. Let $\{\mathcal{F}_\sigma\}_{\sigma\in S}$ be the set of components of $\mathcal{F}_x$ and $Y_\sigma = f_2^{-1}(\mathcal{F}_\sigma)$ its preimage in G_x. Again, Y_σ is closed in G_x and $G_x = \bigcup_{\sigma\in S} Y_\sigma$.

Let $Z_G(x) := \{g \in G\colon\ g^{-1}xg = g\}$ be the stabilizer of x and $Z_G^o(x)$ be its neutral component. Let $A(x) := Z_G(x)/Z_G^o(x)$ be the component group. Note that since $\mathcal{V}_i$ is B stable one has $Z_G(x)Y_iB = Y_i$. On one hand, if $\theta : G \to G/B$ is the natural projection we have $Y_\sigma = \theta^{-1}(\mathcal{F}_\sigma)$, since θ is a locally trivial fibration with fiber isomorphic to B we deduce that Y_σ is irreducible and $\dim(Y_\sigma) = \dim(\mathcal{F}_\sigma) + \dim(B)$; on the other hand, the obvious identity $Z_g^o(x)Y_\sigma B = Y_\sigma$ allows us to define a natural action $A(x) \times \{Y_\sigma\}_{\sigma\in S} \to \{Y_\sigma\}_{\sigma\in S}$, $(a, Y_\sigma) \mapsto Y_{a(\sigma)} := gZ_G^o(x)Y_\sigma B$, with $a = gZ_G^o(x)$. As it is shown in [19] for any i there exists σ such that

$$Y_i = \bigcup_{a \in A(x)} Y_{a(\sigma)}, \tag{2.1}$$

in particular Y_i is equidimensional, $\dim(Y_i) = \dim(Y_\sigma)$ and one has

Theorem *(Spaltenstein). $\mathcal{F}_x$ and $\mathcal{O}_x \cap \mathfrak{n}$ are equidimensional and*

$$\begin{aligned} \dim(\mathcal{O}_x \cap \mathfrak{n}) + \dim(Z_G(x)) &= \dim(\mathcal{F}_x) + \dim(B), \\ \dim(\mathcal{O}_x \cap \mathfrak{n}) + \dim(\mathcal{F}_x) &= \dim(\mathfrak{n}), \\ \dim(\mathcal{O}_x \cap \mathfrak{n}) &= \frac{1}{2}\dim(\mathcal{O}_x). \end{aligned}$$

2.2. In particular, if $G = \mathrm{GL}_n$ then $Z_G(x)$ is connected and $A(x)$ is trivial so that there exists a bijection $\pi : \{\mathcal{F}_i\}_{i=1}^k \to \{\mathcal{V}_i\}_{i=1}^k$ where $\pi(\mathcal{V}_i) := f_1(f_2^{-1}(\mathcal{F}_i)) = \mathcal{V}_i$.

As a straightforward corollary of Spaltenstein's construction for the case GL_n we get

Proposition. *Let $x \in \mathfrak{n}$ and let $\mathcal{F}_1, \mathcal{F}_2$ be two irreducible components of $\mathcal{F}_x$ and $\{\mathcal{E}_l\}_{l=1}^t$ the set of irreducible components of $\mathcal{F}_1 \cap \mathcal{F}_2$. Then $\{\pi(\mathcal{E}_l)\}_{l=1}^t$ is exactly the set of irreducible components of $\mathcal{V}_1 \cap \mathcal{V}_2$ and $\mathrm{codim}_{\mathcal{F}_1}(\mathcal{E}_l) = \mathrm{codim}_{\mathcal{V}_1}(\pi(\mathcal{E}_l))$.*

Proof. Denote $\{\mathcal{W}_l\}_{l=1}^s$ the set of irreducible components of $\mathcal{V}_1 \cap \mathcal{V}_2$. Put $Y_1 \cap Y_2 := f_2^{-1}(\mathcal{V}_1 \cap \mathcal{V}_2)$. By (2.1) we have $Y_1 \cap Y_2 = \bigcup_{l=1}^s f_1^{-1}(\mathcal{W}_l) = f_1^{-1}(\mathcal{V}_1) \cap f_1^{-1}(\mathcal{V}_2) = \bigcup_{a,a' \in A(x)} f_2^{-1}(\mathcal{F}_{a(1)}) \cap f_2^{-1}(\mathcal{F}_{a'(2)})$, since $A(x)$ is trivial we have $Y_1 \cap Y_2 = f_2^{-1}(\mathcal{F}_{(1)}) \cap f_2^{-1}(\mathcal{F}_{(2)}) = \bigcup_{l=1}^t \{f_2^{-1}(\mathcal{E}_l)\}$, where $\{\mathcal{E}_l\}_{l=1}^t$ is the set of irreducible components of $\mathcal{F}_1 \cap \mathcal{F}_2$. In the same spirit as before each subset $f_2^{-1}(\mathcal{E}_l) = \theta^{-1}(\mathcal{E}_l)$ is irreducible and we have

$$\dim(f_2^{-1}(\mathcal{E}_l)) = \dim(\mathcal{E}_l) + \dim(B) \tag{2.2}$$

and for $i = 1, 2$

$$\dim(f_2^{-1}(\mathcal{F}_i)) = \dim(\mathcal{F}_i) + \dim(B). \tag{2.3}$$

If $f_2^{-1}(\mathcal{E}_l) \subset C$, where C is an irreducible component of $Y_1 \cap Y_2$, then $\theta(C)$ is irreducible and we necessary have $\theta(f_2^{-1}(\mathcal{E}_l)) = \theta(\theta^{-1}(\mathcal{E}_l)) = \mathcal{E}_l \subset \theta(C)$, therefore we have $\mathcal{E}_1 = \theta(C)$, $C = f_2^{-1}(\mathcal{E}_l)$ and $\{f_2^{-1}(\mathcal{E}_l)\}_{l=1}^t$ is exactly the set of distinct irreducible components of $Y_1 \cap Y_2$. We can suppose that $x \in \mathcal{V}_1 \cap \mathcal{V}_2$, then if we notice that f_1 is the restriction of the orbit map $\varphi : G \to \mathcal{O}_x$, $g \mapsto g^{-1}xg$ which is open, we deduce that $f_1(f_2^{-1}(\mathcal{E}_l))$ is closed and irreducible in $\mathcal{V}_1 \cap \mathcal{V}_2$. We can also easily deduce that $\{f_1(f_2^{-1}(\mathcal{E}_l))\}_{l=1}^t$ is the set (maybe redundant) of irreducible components of $\mathcal{V}_1 \cap \mathcal{V}_2$, therefore $t \leqslant s$.

On the other hand, the identity $Z_G^o(x) f_2^{-1}(\mathcal{E}_l)B = f_2^{-1}(\mathcal{E}_l)$ gives us a natural action of $\tilde{A}(x) := Z_{A(x)}(\mathcal{F}_1) \cap Z_{A(x)}(\mathcal{F}_2)$ on the set $\{f_2^{-1}(\mathcal{E}_l)\}_{l=1}^t$. Moreover, for any $g \in G_x$ we have $f_1^{-1}(f_1(g)) = \varphi^{-1}(\varphi(g)) = Z_G(x)g$, therefore we have $f_1^{-1}(f_1(f_2^{-1}(\mathcal{E}_l))) \cap Y_1 \cap$

$Y_2 = \bigcup_{a\in\tilde{A}(x)} f_2^{-1}(\mathcal{E}_{a(l)})$, since $A(x)$ is trivial, we deduce that $f_1^{-1}(f_1(f_2^{-1}(\mathcal{E}_l))) \cap Y_1 \cap Y_2 = f_2^{-1}(\mathcal{E}_l)$; by this observation we deduce that $\{f_1(f_2^{-1}(\mathcal{E}_l))\}_{l=1}^t$ is exactly the set of distinct irreducible components of $\mathcal{V}_1 \cap \mathcal{V}_2$, therefore $t = s$ and

$$\dim\bigl(f_2^{-1}(\mathcal{E}_l)\bigr) = \dim\bigl(f_1^{-1}\bigl(f_1\bigl(f_2^{-1}(\mathcal{E}_l)\bigr)\bigr)\bigr) = \dim\bigl(f_1\bigl(f_2^{-1}(\mathcal{E}_l)\bigr)\bigr) + \dim\bigl(Z_G(x)\bigr) \tag{2.4}$$

and for $i = 1, 2$

$$\dim\bigl(f_2^{-1}(\mathcal{F}_i)\bigr) = \dim\bigl(f_1^{-1}\bigl(f_1\bigl(f_2^{-1}(\mathcal{F}_i)\bigr)\bigr)\bigr) = \dim\bigl(f_1\bigl(f_2^{-1}(\mathcal{F}_i)\bigr)\bigr) + \dim\bigl(Z_G(x)\bigr). \tag{2.5}$$

By (2.2)–(2.5) we get

$$\operatorname{codim}_{\mathcal{V}_i}\bigl(f_1\bigl(f_2^{-1}(\mathcal{E}_l)\bigr)\bigr) = \operatorname{codim}_{Y_i}\bigl(f_2^{-1}(\mathcal{E}_l)\bigr) = \operatorname{codim}_{\mathcal{F}_i}(\mathcal{E}_l). \qquad \square \tag{2.6}$$

This simple proposition shows that in $G = \mathrm{GL}_n$ orbital varieties associated to $\mathcal{O}_x$ are equivalent to the components of $\mathcal{F}_x$.

2.3. In what follows we fix the standard triangular decomposition of $\mathfrak{gl}_n$, namely $\mathfrak{gl}_n = \mathfrak{n}_n^- \oplus \mathfrak{h}_n \oplus \mathfrak{n}_n$ where $\mathfrak{n}_n^-$ is the subalgebra of strictly lower triangular $n \times n$ matrices, $\mathfrak{h}_n$ is the subalgebra of diagonal $n \times n$ matrices and $\mathfrak{n}$ is the subalgebra of strictly upper triangular $n \times n$ matrices. (As well in what follows we omit index n in the cases where it is clear what is our n.) Accordingly we put B_n (or simply B) to be the subgroup of all upper-triangular invertible matrices in GL_n and $\mathfrak{b} := \mathrm{Lie}(B) = \mathfrak{n} \oplus \mathfrak{h}$.

Let $e_{i,j}$ be an $n \times n$ matrix having 1 in the ijth entry and 0 elsewhere. Then $\{e_{i,j}\}_{i,j=1, i\neq j}^n \cup \{e_{i,i} - e_{i+1,i+1}\}_{i=1}^{n-1}$ is a basis of $\mathfrak{sl}_n$.

Take $i < j$ and let $\alpha_{i,j}$ be the root which is the weight of $e_{i,j}$. Set $\alpha_{j,i} = -\alpha_{i,j}$. We write $\alpha_{i,i+1}$ simply as α_i. Then $\Pi = \{\alpha_i\}_{i=1}^{n-1}$. Moreover, $\alpha_{i,j} \in \mathcal{R}^+ \Leftrightarrow i < j$. One has

$$\alpha_{i,j} = \begin{cases} \sum_{k=i}^{j-1} \alpha_k & \text{if } i > j, \\ -\sum_{k=i}^{j-1} \alpha_k & \text{if } i < j. \end{cases}$$

Let $\mathfrak{g}_{\alpha_{i,j}} := \mathfrak{g}_{i,j} := \mathbb{C}e_{i,j}$ be the root space defined by $\alpha_{i,j} \in R$.

For $\alpha_i \in \Pi$, let P_{α_i} be the standard parabolic subgroup of GL_n with $\mathrm{Lie}(P_{\alpha_i}) = \mathfrak{b} \oplus \mathfrak{g}_{-\alpha_i} = \mathfrak{b} \oplus \mathfrak{g}_{i+1,i}$. Let M_{α_i} be the unipotent radical of P_{α_i} and $\mathfrak{m}_{\alpha_i} := \mathrm{Lie}(M_{\alpha_i}) = \bigoplus_{1\leqslant s<t\leqslant n,\ (s,t)\neq(i,i+1)} \mathfrak{g}_{s,t}$.

2.4. Let us return to the parametrization of the components of $\mathcal{F}_x$ in GL_n by standard Young tableaux. But first a few general remarks.

The group G operates diagonally on $\mathcal{F} \times \mathcal{F}$ and one version of the Bruhat's lemma says that the G-orbits are parameterized by the elements of the Weyl group $\mathcal{W}$ [23, p. 146]. More precisely, putting

$$O(w) := \bigl\{(gB, g'B) \in \mathcal{F} \times \mathcal{F} \mid g^{-1}g' \in BwB\bigr\},$$

we have a decomposition into G-orbits

$$\mathcal{F}\times\mathcal{F}=\coprod_{w\in\mathcal{W}}O(w).$$

If Y and Z are two irreducible subvarieties of $\mathcal{F}$, then there is a unique $O(w)$ such that $O(w)\cap Y\times Z$ is an open dense set of $Y\times Z$, and we say that Y and Z are in *relative position* with respect to w.

In GL_n the relative position can be interpreted as follows. If irreducible subvarieties Y and Z of $\mathcal{F}$ are in relative position with respect to w then for two generic flags $\mathcal{F}_1=(V_1,\dots,V_n)\in Y$ and $\mathcal{F}_2=(V_1',\dots,V_n')\in Z$ there exists a basis $\{v_i\}_{i=1}^n$ of $\mathbb{C}^n$ such that for any j: $1\leqslant j\leqslant n$ one has $\{v_i\}_{i=1}^j$ is a basis of V_j and $\{v_{w(i)}\}_{i=1}^j$ is a basis of V_j'.

2.5. Now we restrict to $\mathfrak{g}=\mathfrak{sl}_n$, then $\mathcal{N}$ is the variety of all nilpotent matrices, $\mathcal{F}$ is identified with the set of complete flags $\xi=(V_1\subset\cdots\subset V_n=\mathbb{C}^n)$ and $\mathcal{F}_x\cong\{\xi=(V_i)\in\mathcal{F}\mid x(V_i)\subset V_{i-1}\}$.

Recall notation from 1.2. Given $x\in\mathfrak{n}$ let $J(x)=\lambda$. By a slight abuse of notation we will not distinguish between the partition λ and its Young diagram. By R. Steinberg [26] and N. Spaltenstein [18] we have a parametrization of the irreducible components of $\mathcal{F}_x$ by the set $\mathbf{Tab}_\lambda$: Let $\xi=(V_i)\in\mathcal{F}_x$, then we get a sutured chain

$$\mathrm{St}(\xi):=\bigl(Y(x),Y(x|_{V_{n-1}}),\dots,Y(x|_{V_2}),Y(x|_{V_1})\bigr)$$

in the poset of Young diagrams (where $x|_{V_i}$ is the nilpotent endomorphism induced by x by restriction to the subspace V_i). Note that $J(x|_{V_{i+1}})$ differs from $J(x|_{V_i})$ by one corner box, put $i+1$ in it. It is easy to see that in such a way we get a standard Young tableau corresponding to the given chain. So we get a map $\mathrm{St}:\mathcal{F}_x\to\mathbf{Tab}_\lambda$. Then the collection $\{\mathrm{St}^{-1}(T)\}_{T\in\mathbf{Tab}_\lambda}$ is a partition of $\mathcal{F}_x$ into smooth irreducible subvarieties of the same dimension and $\{\overline{\mathrm{St}^{-1}(T)}\}_{T\in\mathbf{Tab}_\lambda}$ is the set of the irreducible components of $\mathcal{F}_x$. Let us denote $\mathcal{F}_\lambda:=\mathcal{F}_x$ if $J(x)=\lambda$ and the components of $\mathcal{F}_\lambda$ by $\mathcal{F}_T:=\overline{\mathrm{St}^{-1}(T)}$ where $T\in\mathbf{Tab}_\lambda$.

On the level of orbital varieties the construction is as follows. Consider the canonical projections $\pi_{1,n-i}:\mathfrak{n}_n\to\mathfrak{n}_{n-i}$ acting on a matrix by deleting the last i columns and the last i rows. Given $x\in\mathfrak{n}$ with $J(x)=\lambda$ for any $u\in\mathcal{O}_x\cap\mathfrak{n}$ set $J_n(u):=J(u)=\lambda$ and $J_{n-i}(u):=J(\pi_{1,n-i}(u))$ for any i: $1\leqslant i\leqslant n-1$. Exactly as in the previous construction we get a standard Young tableau corresponding to the chain $(J_n(u),\dots,J_1(u))$, so that $\mathrm{St}_1:\mathcal{O}_x\cap\mathfrak{n}\to\mathbf{Tab}_\lambda$. Again the collection $\{\mathrm{St}_1^{-1}(T)\}_{T\in\mathbf{Tab}_\lambda}$ is a partition of $\mathcal{O}_x\cap\mathfrak{n}$ into smooth irreducible subvarieties of the same dimensions and $\{\overline{\mathrm{St}_1^{-1}(T)}\cap\mathcal{O}_x\}_{T\in\mathbf{Tab}_\lambda}$ are the set of the irreducible components of $\mathcal{O}_x\cap\mathfrak{n}$. Let us denote $\mathcal{O}_\lambda:=\mathcal{O}_x$ if $J(x)=\lambda$ and orbital varieties associated to $\mathcal{O}_\lambda$ by $\mathcal{V}_T:=\overline{\mathrm{St}_1^{-1}(T)}\cap\mathcal{O}_\lambda$ where $T\in\mathbf{Tab}_\lambda$.

2.6. A general construction for orbital varieties by R. Steinberg (cf. [25]) is as follows. For $\alpha\in R$ let $\mathfrak{g}_\alpha$ denote the root space.

For $w \in \mathcal{W}$ consider the subspace

$$\mathfrak{n} \cap^{w} \mathfrak{n} := \bigoplus_{\alpha \in \mathcal{R}^+ \cap^w \mathcal{R}^+} \mathfrak{g}_\alpha$$

of $\mathfrak{n}$. Then $G.(\mathfrak{n} \cap^w \mathfrak{n})$ is an irreducible locally closed subvariety of $\mathcal{N}$. Since the nilpotent variety is a finite union of nilpotent orbits, it follows that there is a unique nilpotent orbit $\mathcal{O}_w$ such that $\overline{G.(\mathfrak{n} \cap^w \mathfrak{n})} = \overline{\mathcal{O}_w}$. By [25] $\mathcal{V}_w := \overline{B.(\mathfrak{n} \cap^w \mathfrak{n})} \cap \mathcal{O}_w$ is an orbital variety associated to $\mathcal{O}_w$ and the map $\varphi : w \mapsto \mathcal{V}_w$ is a surjection of $\mathcal{W}$ onto the set of all orbital varieties. According to the map φ, we decompose the Weyl group into the subsets $\mathcal{C}_w := \{v \in W \mid \mathcal{V}_v = \mathcal{V}_w\}$ which are called the *geometric cells* of $\mathcal{W}$.

Let $P_{\mathcal{V}_w}$ be the maximal standard parabolic subgroup of G stabilizing $\mathcal{V}_w$. Set $\tau(\mathcal{V}_w) := \{\alpha \in \Pi \colon P_\alpha.\mathcal{V}_w = \mathcal{V}_w\}$. Obviously, $P_{\mathcal{V}_w} = \langle P_\alpha \colon \alpha \in \tau(\mathcal{V}_w)\rangle$. Set $\tau(w) := \{\alpha \in \Pi \colon w^{-1}(\alpha) \in R^-\}$. By [5, §9] one has $\tau(\mathcal{V}_w) = \tau(w)$. In particular, $\tau(w) = \tau(y)$ for any $y \in \mathcal{C}_w$ and we can define $\tau(\mathcal{C}_w) := \tau(w)$.

Denote $R(w) := \{\alpha \in \mathcal{R}^+ \colon w^{-1}(\alpha) < 0\}$ and $S(w) := \{\alpha \in \mathcal{R}^+ \colon w^{-1}(\alpha) > 0\}$. Here is a very useful lemma

Lemma. *Fix a simple root α. Denote $l(\,)$ the length function*:

(1) *If $l(s_\alpha w) = l(w) + 1$, then $S(s_\alpha w) = s_\alpha(S(w)) - \{\alpha\}$.*
(2) *If $l(s_\alpha w) = l(w) - 1$, then $S(s_\alpha w) = s_\alpha(S(w)) \cup \{\alpha\}$.*
(3) *If $l(w s_\alpha) = l(w) + 1$, then $S(w s_\alpha) = S(w)) - \{w(\alpha)\}$.*
(4) *If $l(w s_\alpha) = l(w) - 1$, then $S(w s_\alpha) = S(w)) \cup \{w(-\alpha)\}$.*

Proof. If $l(s_\alpha w) = l(w) + 1$ and if $w = s_{i_1} \cdots s_{i_k}$ is a reduced expression for w then $s_\alpha w = s_\alpha s_{i_1} \cdots s_{i_k}$ is also a reduced expression for $s_\alpha w$, then by [23, p. 142] we have

$$R(w) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \ldots, s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})\} \tag{2.7}$$

and

$$R(s_\alpha w) = \{\alpha, s_\alpha(\alpha_{i_1}), s_\alpha s_{i_1}(\alpha_{i_2}), \ldots, s_\alpha s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})\}. \tag{2.8}$$

Therefore we get $R(s_\alpha w) = \{\alpha\} \cup s_\alpha(R(w))$; on the other hand, we have

$$\mathcal{R}^+ = R(s_\alpha w) \coprod S(s_\alpha w) = (\{\alpha\} \cup s_\alpha(R(w))) \coprod S(s_\alpha w) = R(w) \coprod S(w), \tag{2.9}$$

moreover, we have

$$s_\alpha(\mathcal{R}^+) = (\mathcal{R}^+ - \{\alpha\}) \cup \{-\alpha\} = s_\alpha(R(w)) \coprod s_\alpha(S(w)). \tag{2.10}$$

By (2.9) and (2.10) we deduce that $S(s_\alpha w) = s_\alpha(S(w)) - \{\alpha\}$. The other cases can be obtained in the same manner. □

2.7. Let us consider Steinberg's construction in $\mathfrak{sl}_n$. Here $\mathcal{W} = \mathbf{S}_n$ where we identify $s_{\alpha_i} := (i, i+1)$ (in the cyclic form). We write an element $w \in \mathbf{S}_n$ in a word form $w = [a_1, \dots, a_n]$ where $w(i) = a_i$. In what follows we denote $s_i := s_{\alpha_i}$.

Put $p_w(i) := w^{-1}(i)$ to be its position in the word w. By [6, 2.3] one has

Proposition. *For any* $w \in \mathbf{S}_n$

$$\mathfrak{n} \cap {}^w\mathfrak{n} = \bigoplus_{\substack{1 \leqslant i < j \leqslant n \\ p_w(i) < p_w(j)}} \mathfrak{g}_{i,j}.$$

In particular, $\tau(w) = \{\alpha_i\colon\ p_w(i) > p_w(i+1)\}$.

2.8. Let us describe the geometric cells in the case $G = \mathrm{GL}_n$. In The Robinson–Schensted correspondence gives the bijection from the ordered pairs of standard Young tableaux of the same shape onto the $\mathbf{S}_n$ (cf. [3], for example). Let us denote it by $\mathrm{RS} : \coprod_{\lambda \vdash n} \mathbf{Tab}_\lambda \times \mathbf{Tab}_\lambda \to \mathbf{S}_n$ and describe it in short. Let (T, T') be the pair of Standard Young tableaux of the same shape. Remove the number n (and the cell that contains it) from T'. Then take the number which is in the same position in T as n was in T' and move it up one row to displace the largest number in that row that is smaller than it; use the displaced number to displace a number in the next higher row according to the same rule, and so on, until a number r_n, is displaced from the first row; set $\mathrm{RS}(T, T')(n) = r_n$. Note that the two new tableaux of size $n-1$ are again of the same shape and the second tableau is standard. Repeat the process to get $\mathrm{RS}(T, T')(n-1) = r_{n-1}$ and so on. Repeating this procedure n times we get the required element $\mathrm{RS}(T, T')$. We will write it in a word form $\mathrm{RS}(T, T') = [r_1, \dots, r_n]$.

$\mathbf{S}_n$ is decomposed into Young cells where a *Young cell* corresponding to $T \in \mathbf{Tab}_\lambda$ is defined by $\mathcal{C}_T := \{\mathrm{RS}(T, T')\colon\ T' \in \mathbf{Tab}_\lambda\}$. By [25, §5] one has (cf. [11, p. 201], for example).

Theorem. *Let* $w = \mathrm{RS}(T, T')$ *where* T, T' *are of shape* λ. *Then*

(1) $\mathcal{O}_w = \mathcal{O}_\lambda$;
(2) $\mathcal{V}_w = \mathcal{V}_T$;
(3) $\mathcal{C}_w = \mathcal{C}_T$.

2.9. Note also that the two constructions we gave in GL_n coincide, namely (cf., for example, [12, 3.4]). Moreover, we can notice that the geometric cells coincide with the Young cells.

Proposition. *Let* $x \in \mathfrak{n} \cap \mathcal{O}_\lambda$ *and* $T \in \mathbf{Tab}_\lambda$. *Then for any* $w = \mathrm{RS}(T, T')$ *one has* $B.(\mathfrak{n} \cap {}^w\mathfrak{n}) \cap \mathrm{St}_1^{-1}(T)$ *is dense in* $B.(\mathfrak{n} \cap {}^w\mathfrak{n})$.

2.10. Let us mention a few well-known combinatorial facts concerning Robinson–Schensted procedure.

Let $\mathcal{C}^r_T := \{\mathrm{RS}(T', T)\colon\ T' \in \mathbf{Tab}_\lambda\}$. Let $\mathcal{C}_\lambda := \{w \in \mathbf{S}_n\colon\ \mathcal{O}_w = \mathcal{O}_\lambda\}$ Obviously, $\mathcal{C}_\lambda = \coprod_{T\in\mathbf{Tab}_\lambda} \mathcal{C}_T = \coprod_{T\in\mathbf{Tab}_\lambda} \mathcal{C}^r_T$.

Given $T \in \mathbf{Tab}_\lambda$ put $r_T(j)$ to be the number of the row j belongs to and $c_T(j)$ to be the number of the column j belongs to.

Proposition.

(1) $\tau(\mathcal{C}_T) = \{\alpha_i\colon\ r_T(i) < r_T(i+1)\}$.
(2) $(\mathcal{C}^r_T) = \{w^{-1}\colon\ w \in \mathcal{C}_T\}$.
(3) *Let* $w = \mathrm{RS}(T, T')$ *and let* λ *be the shape of* T. *If* $ws_i \in \mathcal{C}_\lambda$ *(respectively* $s_i w \in \mathcal{C}_\lambda$*) for some* i *then* $ws_i \in \mathcal{C}_T$ *(respectively* $s_i w \in \mathcal{C}^r_{T'}$*).*

Proof. We give a short proof for the completeness.

(1) The first result is a straightforward corollary of RS algorithm and of Proposition 2.7.

(2) The second result is a straightforward corollary of the Robinson–Schensted theorem (cf. [9, 5.1.4], for example) claiming $(\mathrm{RS}(T, T'))^{-1} = \mathrm{RS}(T', T)$.

(3) If $l(ws_i) = l(w)+1$, by Lemma 2.6(3), one has $\mathfrak{n}\cap^{ws_i}\mathfrak{n} \subset \mathfrak{n}\cap^{w}\mathfrak{n}$ so that $\bar{\mathcal{V}}_{ws_i} \subset \bar{\mathcal{V}}_w$. On the other hand, by equidimensionality of orbital varieties associated to $\mathcal{O}_\lambda$ one has $\dim \mathcal{V}_w = \dim \mathcal{V}_{ws_i}$. Thus $\mathcal{V}_w = \mathcal{V}_{ws_i}$, i.e. $w, ws_i \in \mathcal{C}_T$. Now if $l(ws_i) = l(w) - 1$ then $w = ys_i$ where $y = ws_i$ and $l(w) = l(y) + 1$ so that by the previous $\mathcal{V}_w = \mathcal{V}_y$.

The result for $w, s_i w$ is obtained by applying (2). □

For a tableau T we put $\tau(T) := \{\alpha_i\colon\ r_T(i) < r_T(i+1)\}$. By the proposition above one has $\tau(T) = \tau(\mathcal{C}_T)$.

2.11. In [26] R. Steinberg gives also a very beautiful interpretation of the relative position between the irreducible components of $\mathcal{F}_\lambda$ by the Robinson–Schensted correspondence. Let $T, T' \in \mathbf{Tab}_\lambda$ and let $\mathcal{F}_T, \mathcal{F}_{T'}$ be the corresponding components of $\mathcal{F}_\lambda$. Then by [26] the relative position between the irreducible components $\mathcal{F}_T$ and $\mathcal{F}_{T'}$ is exactly $\mathrm{RS}(T, T')$.

2.12. Recall the Bruhat–Tits decomposition of the flag manifold:

$$\mathcal{F} = \coprod_{w\in\mathbf{S}_n} X_w.$$

Where $X_w := B.(w(\xi_0))$ is the B-orbit of the flag $w(\xi_0)$ where ξ_0 is the canonical flag. It is well known that X_w is an affine space called the *Schubert cell* (associated to w) and its closure $\overline{X}_w$ is called a *Schubert variety* (cf. [23, p. 149]).

Let C be an irreducible subvariety of $\mathcal{F}$, then there is a unique Schubert cell X_w such that $X_w \cap C$ in an open dense subset of C. We will call the element w the *position* of C in the flag manifold $\mathcal{F}$ (w.r.t. $(\mathfrak{h}, \mathfrak{b})$).

2.13. Note also the following straight connection between Steinberg's construction and relative position:

Theorem. *Let $T \in \mathbf{Tab}_\lambda$ and let $w = \mathrm{RS}(T, T')$. Then for a general element $x \in \mathcal{V}_T \cap B.(\mathfrak{n} \cap^w \mathfrak{n})$ the position of the irreducible component $\mathcal{F}_{T'}$ of the Springer fiber $\mathcal{F}_x$ is given by w.*

Proof. Let $x \in \mathcal{V}_T \cap B.(\mathfrak{n} \cap^w \mathfrak{n})$ be in a general position. Let $\mathcal{F}_{T'}$ be an irreducible component of the Springer fiber $\mathcal{F}_x$ above x, and denote w its position. Then $X_w \cap \mathcal{F}_{T'}$ is an open dense subset of $\mathcal{F}_{T'}$ and by the Bruhat–Tits decomposition any element $\xi = gB \in X_w \cap \mathcal{F}_{T'}$ can be written as $g = bn_w b'$ where n_w is a representative of w in $\mathrm{Norm}_G(\mathfrak{h})$ and we can assume that $b' = e$. By (1.1) we have

$$\begin{aligned} gB \in \mathcal{F}_x &\Leftrightarrow x \in g.\mathfrak{n} \\ &\Leftrightarrow x \in bn_w.\mathfrak{n} \\ &\Leftrightarrow b^{-1}xb \in \mathfrak{n} \cap^w \mathfrak{n} \\ &\Leftrightarrow x \in B.\big(\mathfrak{n} \cap^w \mathfrak{n}\big). \end{aligned}$$

Note that by [25, Corollary 3.9.] $x \in \mathcal{V}_T \cap B.(\mathfrak{n} \cap^w \mathfrak{n})$ being in a general position is equivalent to choose gB in a general position in $\mathcal{F}_{T'}$.

Because of the fact that x is in a general position in $\mathcal{V}_T$ we may assume $x \in \mathfrak{n}(T)$ by 2.9, so we get $\xi_0 \in \mathcal{F}_T$. Now the key point is to observe that we can choose x generically in $\mathfrak{n}(T)$ such that ξ_0 is also in general position in $\mathcal{F}_T$, and the proof is complete. □

Remarks.

(1) Thus, the Young cell corresponding to T describes generically the different positions of the irreducible components of the Springer fiber above the orbital variety $\mathcal{V}_T$.
(2) The last theorem is a natural generalization of a result obtained in [15]: Let $\mathcal{B} = (e_1, \dots, e_n)$ a base of $\mathbb{C}^n$ such that $E_i = \langle e_1, \dots, e_i \rangle$. A nilpotent element x is said to be adapted to $\mathcal{B}$ if the matrix of x in $\mathcal{B}$ is a Jordan matrix with decreasing block sizes. Let $T_{\max}$ denote the standard tableau obtained by filling first line of the Young diagram $Y(\lambda)$ with the integers $\{1, \dots, \lambda_1\}$, the second one with the integers $\{\lambda_1 + 1, \dots, \lambda_1 + \lambda_2\}$, and so on.... Then we have $\xi_0 \in \mathcal{F}_{T_{\max}}$, moreover, we have shown that the irreducible component $\mathcal{F}_{T_{\max}}$ contain a dense orbit under the centralizer of x, this property is not true in general (cf. [25, Remark 5.7. (d)]). As it was explained in [14], the choice of x in the Jordan form is done to have a computation of the Springer fiber easier.

3. Some intersections of codimension one

3.1. In this section we start to consider the orbital varieties (respectively components of Springer fiber) of codimension 1.

For this last section we give a very simple sufficient condition for two orbital varieties associated to $\mathcal{O}_x$ (respectively two components of $\mathcal{F}_x$) to intersect in codimension 1.

Proposition.

(1) *If* $\alpha_k \in \tau(w)$ *then* $\overline{B.^{s_k}(\mathfrak{n}\cap^w\mathfrak{n})} \subseteq \overline{\mathcal{V}}_w$, $\overline{B.^{s_k}(\mathfrak{n}\cap^w\mathfrak{n})} \cap \mathcal{O}_w \subseteq \mathcal{V}_w$ *and* $\operatorname{codim}_{\mathcal{V}_w} \overline{B.^{s_k}(\mathfrak{n}\cap^w\mathfrak{n})} \cap \mathcal{O}_w \leqslant 1$.
(2) *If* $\alpha_k \notin \tau(w)$ *and* $\mathcal{O}_w = \mathcal{O}_{s_k w}$ *then* $\operatorname{codim}_{\mathcal{V}_w} \mathcal{V}_w \cap \mathcal{V}_{s_k w} = 1$.

Proof. (1) If $\alpha_k \in \tau(w)$, denote n_{s_k} a representative of s_k in $\mathrm{Norm}_G(\mathfrak{h})$; we have by 2.6 that $\overline{\mathcal{V}}_w$ is P_{α_k}-stable, and since $n_{s_k} \in P_{\alpha_k}$ we have ${}^{s_k}(\mathfrak{n}\cap^w\mathfrak{n}) = n_{s_k}.(\mathfrak{n}\cap^w\mathfrak{n}) \subseteq \overline{P_{\alpha_k}.(\mathfrak{n}\cap^w\mathfrak{n})} = \overline{B.(\mathfrak{n}\cap^w\mathfrak{n})} = \overline{\mathcal{V}}_w$, so we deduce that $\overline{B.^{s_k}(\mathfrak{n}\cap^w\mathfrak{n})} \subseteq \overline{\mathcal{V}}_w$. On the over hand, we have $\overline{B.^{s_k}(\mathfrak{n}\cap^w\mathfrak{n})} \subseteq \overline{P_{\alpha_k}.^{s_k}(\mathfrak{n}\cap^w\mathfrak{n})} = \overline{P_{\alpha_k}.(\mathfrak{n}\cap^w\mathfrak{n})} = \overline{\mathcal{V}}_w$, and since $\operatorname{codim}_{P_{\alpha_k}} B = 1$ we get

$$\operatorname{codim}_{\overline{\mathcal{V}}_w} \overline{B.^{s_k}\big(\mathfrak{n}\cap^w\mathfrak{n}\big)} \leqslant 1. \tag{3.1}$$

Since $\mathcal{V}_w = \overline{B.(\mathfrak{n}\cap^w\mathfrak{n})} \cap \mathcal{O}_w$, we deduce in particular that $(\mathfrak{n}\cap^w\mathfrak{n}) \cap \mathcal{O}_w \neq \emptyset$ and $(\mathfrak{n}\cap^w\mathfrak{n}) \subseteq \overline{\mathcal{O}_w}$, so ${}^{s_k}(\mathfrak{n}\cap^w\mathfrak{n}) \cap \mathcal{O}_w \neq \emptyset$ and ${}^{s_k}(\mathfrak{n}\cap^w\mathfrak{n}) \subseteq \overline{\mathcal{O}_w}$. The subvariety $\overline{B.^{s_k}(\mathfrak{n}\cap^w\mathfrak{n})}$ is irreducible and is contained in the nilpotent variety, there is a unique nilpotent orbit $\mathcal{O}$ such that $\overline{B.^{s_k}(\mathfrak{n}\cap^w\mathfrak{n})} \cap \mathcal{O}$ is open and dense in $\overline{B.^{s_k}(\mathfrak{n}\cap^w\mathfrak{n})}$, and by the analysis did before we necessary have $\mathcal{O} = \mathcal{O}_w$ and $\overline{B.^{s_k}(\mathfrak{n}\cap^w\mathfrak{n})} \cap \mathcal{O}_w \subseteq \mathcal{V}_w$, and with (3.1) we get

$$\begin{aligned}\operatorname{codim}_{\overline{\mathcal{V}}_w} \overline{B.^{s_k}(\mathfrak{n}\cap^w\mathfrak{n})} &= \operatorname{codim}_{\overline{\mathcal{V}}_w} \overline{B.^{s_k}\big(\mathfrak{n}\cap^w\mathfrak{n}\big)} \cap \mathcal{O}_w \\ &= \operatorname{codim}_{\mathcal{V}_w} \overline{B.^{s_k}\big(\mathfrak{n}\cap^w\mathfrak{n}\big)} \cap \mathcal{O}_w \leqslant 1.\end{aligned} \tag{3.2}$$

(2) If $\alpha_k \notin \tau(w)$ (i.e. $l(w) = l(s_k w) - 1$) then by Lemma 2.6(2) we get $\mathfrak{n}\cap^w\mathfrak{n} = \mathfrak{g}_{k,k+1} \oplus^{s_k} (\mathfrak{n}\cap^{s_k w}\mathfrak{n})$, then $\overline{B.^{s_k}(\mathfrak{n}\cap^{s_k w}\mathfrak{n})} \subseteq \overline{B.(\mathfrak{n}\cap^w\mathfrak{n})} = \overline{\mathcal{V}}_w$, and as before we have also $\overline{B.^{s_k}(\mathfrak{n}\cap^{s_k w}\mathfrak{n})} \cap \mathcal{O}_w \subseteq \mathcal{V}_w$. If $\mathcal{O}_{s_k w} = \mathcal{O}_w$ then with the analysis did in (1) for the case $\alpha_k \in \tau(s_k w)$ we have $\overline{B.^{s_k}(\mathfrak{n}\cap^{s_k w}\mathfrak{n})} \cap \mathcal{O}_w \subseteq \mathcal{V}_{s_k w}$, therefore $\overline{B.^{s_k}(\mathfrak{n}\cap^{w s_k}\mathfrak{n})} \cap \mathcal{O}_w \subset \mathcal{V}_w \cap \mathcal{V}_{s_k w}$ and since $\mathcal{V}_w \neq \mathcal{V}_{s_k w}$ with (3.2) we get

$$\operatorname{codim}_{\mathcal{V}_{s_k w}} \overline{B.^{s_k}\big(\mathfrak{n}\cap^w\mathfrak{n}\big)} \cap \mathcal{O}_{s_k w} = \operatorname{codim}_{\mathcal{V}_w} \overline{B.^{s_k}\big(\mathfrak{n}\cap^w\mathfrak{n}\big)} \cap \mathcal{O}_w = 1. \qquad \square \tag{3.3}$$

Actually we can also deduce the last result from the work of J. Tits: Let $x \in \mathfrak{n}$ a nilpotent element. Consider an element $\xi = gB \in \mathcal{F}_x$, by the Bruhat–Tits decomposition we write $g = bn_w b'$ and we can assume that $b' = e$. Write $w = s_1 \cdots s_k$, where s_i is the reflexion with respect to the simple root $\alpha_i \in S$ and k is minimal (i.e. $w = s_1 \cdots s_k$ is a reduced expression for w, in particular we have $w(\alpha_k) < 0$). Denote $g_1 = bn_{w'}$ where $w' := s_1 \cdots s_{k-1}$, and P_k the minimal parabolic subgroup containing B associated to the simple root α_k. Then the projective line $g_1 P_k B$ in $\mathcal{F}$ joins the two points gB and $g_1 B$, moreover, J. Tits showed that $g_1 P_k B$ lies in $\mathcal{F}_x$ (cf. [21, p. 377] or [24, Proposition 1, p. 131]). In particular, if w corresponds to the position of the irreducible component $\mathcal{F}_T$, then $\mathcal{F}_T$ is a union of projective lines of type α_k, i.e. the natural projection $\pi_k : G/B \to G/P_k$ induces a structure of $\mathbb{P}^1$-bundle on $\mathcal{F}_T$ (see e.g. [20, Lemme 1.11.]).

Consider the morphism

$$\pi_w : X_w \cap \mathcal{F}_T \to \mathcal{F}_x,\ gB \mapsto g_1 B \tag{3.4}$$

which consists to "flat" the irreducible component $\mathcal{F}_T$ under the direction α_k, then $\overline{\mathrm{Im}(\pi_w)}$ is an irreducible subvariety of codimension 1 in $\mathcal{F}_T$. In particular if w' is the position of an other irreducible component $\mathcal{F}_{T'}$, then $\mathcal{F}_T$ and $\mathcal{F}_{T'}$ have an intersection of codimension 1.

If $\mathcal{O}_w = \mathcal{O}_{s_k w}$, then by Proposition 2.10(3) there exist $T, T', T'' \in \mathbf{Tab}_\lambda$ such that $w = \mathrm{RS}(T, T'')$ and $s_k w = \mathrm{RS}(T', T'')$. By the last proposition we have

$$\mathrm{codim}_{\mathcal{V}_T}(\mathcal{V}_T \cap \mathcal{V}_{T'}) = \mathrm{codim}_{\mathcal{V}_{T'}}(\mathcal{V}_T \cap \mathcal{V}_{T'}) = 1.$$

By Proposition 2.2, we also have

$$\mathrm{codim}_{\mathcal{F}_T}(\mathcal{F}_T \cap \mathcal{F}_{T'}) = \mathrm{codim}_{\mathcal{F}_{T'}}(\mathcal{F}_T \cap \mathcal{F}_{T'}) = 1.$$

This is coherent with the description we did just above with the work of J. Tits: Indeed by Theorem 2.13, w^{-1} and $w^{-1}s_k$ are exactly the positions of the irreducible components $\mathcal{F}_T$ and $\mathcal{F}_{T'}$ above the orbital variety $\mathcal{V}_{T''}$.

Remarks.

(1) Thus, if there exists $T'' \in \mathbf{Tab}_\lambda$ such that $\mathrm{RS}(T'', T') = \mathrm{RS}(T'', T)s_k$ for some s_k, then $\mathcal{F}_T$ and $\mathcal{F}_{T'}$ have an intersection in codimension one.

(2) The computation in low rank cases and the full picture in hook case described in [27] gives an impression that $\mathrm{codim}_{\mathcal{F}_T}(\mathcal{F}_T \cap \mathcal{F}_{T'}) = 1$ if and only if there exists $T'' \in \mathbf{Tab}_\lambda$ such that $\mathrm{RS}(T', T'') = s_k\mathrm{RS}(T, T'')$ for some s_k. However this is not true in general as we show in [13]. The problem of defining all possible pairs $T, T' \in \mathbf{Tab}_\lambda$ such that $\mathrm{codim}_{\mathcal{F}_T}(\mathcal{F}_T \cap \mathcal{F}_{T'}) = 1$ in terms of Young tableaux only is very tricky.

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