GUIDO FUBINI

CLIFFORD PARALLELISM

IN

ELLIPTIC SPACES

LAUREA THESIS

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This treatise will study parallelism in spaces of positive constant curvature, which is an important element in the metric geometry of the spaces that were defined for the first time by Clifford ([1]). We thus introduce new line coordinates (viz., scroll parameters) that are given several geometric interpretations, and for which one can find a simple algorithm that permits one to treat them in a clear and rapid fashion without the necessity of carrying out extremely long and tedious calculations.

One can then give a proof of the principle of duality without appealing to the consideration of the absolute that first defines the angle between two skew lines, etc. The application of this principle to the theory of curves, while suggesting the introduction of a new element into it – namely, "Clifford torsion" – will prove a theorem of Prof. Bianchi ([2]) in a new and complete way; a modification of the Frenet formula for a spatial curve will, in my esteemed opinion, lead one to comparisons with flat space and new theorems. The application of the theory of parallels to the study of congruences will give directly the necessary and sufficient conditions for the differential quadratic form that defines a congruence to be compatible, it will give immediately a criterion for recognizing whether a congruence is *W*, as soon as one is given the fundamental formulas, and it will finally permit one to establish some new results for the density of a congruence.

If one applies these theorems to the theory of surfaces then one will obtain new geometric interpretations for the absolute curvature and the geodetic torsion that can be traced back to the theory of surfaces in curved spaces and the study of those representatives for the Euclidian sphere in itself for which corresponding parts have equal areas. One can therefore generalize some well-known formulas in Euclidian space for the theory of surfaces and triply-orthogonal systems, with the aid of which one can study – among other things – those congruences for which any deformation of the initial surface, which the rays of the congruence are assumed to be invariably linked with, is always arranged into ∞^1 Clifford rulings, and finally discover some curious results for the angle that is formed between linear elements that correspond to the surface and its planar image.

The application of these results to the theory of *W* surfaces will lead, among other things, to the study of noteworthy pairs of spherical elements, a study that can also be interpreted in the Euclidian metric, and then it is given the geometrical significance (albeit not a simple one) of the Lie transformation for the pseudo-spherical surface, which then solves the problem of determining those lattices on the Euclidian sphere that will divide it into equivalent, infinitesimal parallelograms, in a new way.

The study of the Riemannian images of parallel lines is a door to a new characteristic property of the (isocyclic) Demartres surface ([3]), a property that can meanwhile be interpreted with only the Euclidian metric, which leads to a new property of the isocyclic surface and the isothermal ruling (viz., the locus of binormals of a curve with constant torsion) ([4]) in spaces with constant curvature.

([1]) Cf., e.g., KLEIN, *Nicht-Euclidische Geometrie*.
([2]) BIANCHI, *Ann. di Matem.* (1896), 103. We shall denote this paper by A.
([3]) DEMARTRES, *Annales de l'École Normale Superieure*, **4** (1887).
([4]) BIANCHI. (A).

MAIN FORMULAS

§ 1. We suppose that certain special motions exist in elliptic space, for which, the distance between the initial and final position of an arbitrary point is constant, and which will be given the name of *scrolls*. In Euclidian space (where one finds the geodetically-representative images of curved space), these motions will correspond to the biaxial homographies whose axes are the two generators of the same ruled series as the absolute, and which will therefore make the generators of the other ruled series scroll on it. Moreover, conforming to the existence of two ruled series in a quadric, the scrolls of a space curve will divide into two entirely distinct systems: the right-handed scrolls and the left-handed ones.

Clifford defined parallelism of lines in a curved space by means of these special motions, and we will now give some fundamental properties of parallel lines that are easily deducible from each other and which can each serve as the definition of parallel lines. With Clifford, we therefore say that two or more lines are *parallel* when either:

α) The initial and final positions combine into the points of a system that is rigid relative to a scroll,

or

β) They support the same skew generators of the absolute,

or

γ) They are themselves the initial and final positions of a line that has been subjected to a scroll.

The existence of two types of scrolls proves the existence of two types of parallel lines: viz., right-handed and left-handed parallels. However, observe that if we are given just one scroll then we can deduce right-handed parallels, as well as left-handed ones, and that secondly that they will serve to generate parallel lines as in α) or γ).

We state the defining formulas for a scroll, in which, we suppose, as we always will from now on, that the curvature of the ambient space is equal to + 1, denote the Weierstrass coordinates of the initial and final positions of the same point by (x_i) and (x_i'), and let A, B, C, D and $(\alpha, \beta, \gamma, \delta)$ denote eight constants that are subject to the relations:

$$A^2 + B^2 + C^2 + D^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1.$$

For the scrolls of the first kind, we will have [1]:

[1] BIANCHI (A).

$$(1) \qquad \begin{cases} x_1' = Ax_1 - Bx_2 - Cx_3 - Dx_4, \\ x_2' = Bx_1 + Ax_2 - Dx_3 + Cx_4, \\ x_3' = Cx_1 + Dx_2 + Ax_3 - Bx_4, \\ x_4' = Dx_1 - Cx_2 + Bx_3 + Ax_4, \end{cases}$$

and for scrolls of the second kind:

$$(2) \qquad \begin{cases} x_1' = \alpha x_1 - \beta x_2 - \gamma x_3 - \delta x_4, \\ x_2' = \beta x_1 + \alpha x_2 + \delta x_3 - \gamma x_4, \\ x_3' = \gamma x_1 - \delta x_2 + \alpha x_3 + \beta x_4, \\ x_4' = \delta x_1 + \gamma x_2 - \beta x_3 + \alpha x_4. \end{cases}$$

§ 2. Along with the Weierstrass coordinates of a point and a plane, a (geodetic) line is defined by giving the coordinates (x_i) of any one of its points and the coordinates (ξ_i) of the plane normal to that line at (x_i); hence, in a manner that it is very convenient for some studies, although it is far from symmetric. The fundamental objective of the present treatise is the introduction of a new coordinate system that, we hope, will appear to be very appropriate to the nature of elliptic space in its applications.

Therefore, let a line be defined in the Weierstrass way, and let (x_i) and (ξ_i) be two conjugate points (at a distance of $\pi / 2$), such that $\sum_{i=1}^{4} x_i \xi_i = 0$. The scrolls that take the point (x_i) to the point (ξ_i) will have a right-hand and left-hand, and if we denote the constants that relate to the one by A, B, C, D and the constants that relate to the other by α, β, γ, δ, then we will meanwhile have:

$$A = \alpha = 0,$$

because

$$\sum x\,\xi = 0,$$

and therefore:

$$B^2 + C^2 + D^2 = \beta^2 + \gamma^2 + \delta^2 = 1.$$

We assume that the B, C, D, β, γ, δ (which are calculated immediately) are the coordinates of a line in elliptic space, and we give them the name of *scroll parameters* for the same line. As one sees immediately, they will be independent of the pair of conjugate points (x_i), (ξ_i) that is chosen on the line.

Meanwhile, observe that a line is distinguished when one is given the scroll parameters (and one will prove this effectively by calculation); in fact, two scrolls of different kinds are defined that leave the line fixed and which, in turn, define the four generators of the absolute that supports it, and will thus suffice to distinguish the line, in turn, from its polar line; this indeterminacy, which can also be useful when one studies, in turn, two polar figures, is increased even further when one considers the signs.

For the effective calculation of the six scroll parameters, one observes that for us (1) becomes:

$$(3)\qquad \begin{cases} \xi_1 = -Bx_2 - Cx_3 - Dx_4, \\ \xi_2 = Bx_1 + Cx_4 - Dx_3, \\ \xi_3 = -Bx_4 + Cx_1 + Dx_2, \\ \xi_4 = Bx_3 - Cx_2 + Dx_1. \end{cases}$$

If one solves these, while recalling that $\sum x^2 = \sum \xi^2 = 1$, $\sum x\,\xi = 0$, then one will get:

$$(4)\qquad \begin{cases} B = \xi_2 x_1 - x_2\xi_1 + \xi_4 x_3 - x_4\xi_3, \\ C = \xi_2 x_4 - \xi_4 x_2 + \xi_3 x_1 - x_3\xi_1, \\ D = \xi_3 x_2 - \xi_2 x_3 + \xi_4 x_1 - x_4\xi_1, \end{cases}$$

which is related precisely to:

$$(5)\qquad D^2 + B^2 + C^2 = 1.$$

Analogously, one has:

$$(3')\qquad \begin{cases} \xi_1 = -\beta x_2 - \gamma x_3 - \delta x_4, \\ \xi_2 = \beta x_1 + \delta x_4 - \gamma x_3, \\ \xi_3 = -\gamma x_4 - \delta x_1 + \beta x_2, \\ \xi_4 = \delta x_3 + \gamma x_2 - \beta x_1, \end{cases}$$

from which, one will get:

$$(4')\qquad \begin{cases} \beta = \xi_2 x_4 - \xi_4 x_3 + \xi_2 x_1 - \xi_1 x_2, \\ \gamma = \xi_4 x_2 - \xi_2 x_4 + \xi_3 x_1 - \xi_1 x_3, \\ \delta = \xi_4 x_1 - \xi_1 x_4 + \xi_2 x_3 - \xi_3 x_2, \end{cases}$$

with:

$$(5')\qquad \beta^2 + \gamma^2 + \delta^2 = 1.$$

If one takes (4) and (4′) to be the formulas that define the $B, C, D, \beta, \gamma, \delta$ then (3) and (3′) will give the coordinates (ξ_i) of the plane normal to our line at the point (x_i), if one denotes that point by (x_i).

It is easy then to recognize just what distinguishes the scroll parameters of two polar lines. For example, take the line that is normal to the plane (0, 1, 0, 0) at the point (1, 0, 0, 0), and the polar line that is normal to the plane (0, 0, 0, 1) at the point (0, 0, 1, 0).

For the one of them, one will have:

$$B = 1,\ C = D = 0, \qquad \beta = 1, \quad \gamma = \delta = 0,$$

and for the other:

$$B' = 1, \qquad C' = D' = 0, \quad \beta' = -1, \qquad \gamma' = \delta' = 0.$$

Therefore:

If one changes the signs of one of the two sets of scroll parameters of a line then one will obtain the polar line.

We will often see that the polar line plays the same role in elliptic space that the opposite direction does in flat space.

The simultaneous change of sign in all six scroll parameters does not alter the corresponding line, because that would be equivalent to changing (x_i) into $(-x_i)$, or (ξ_i) into $(-\xi_i)$, or exchanging the points (x_i), (ξ_i). (Cf., the final observations).

§ 3. However, calculating with B, C, D, β, γ, δ would be exhausting, so we shall introduce a simple algorithm that will then permit us to treat these new line coordinates with the maximum confidence and facility, and to pass from them to the usual formulas in Weierstrass coordinates. Therefore, observe that one can write:

$$(6)\qquad \begin{cases} B = \begin{vmatrix} \xi_2 & \xi_1 \\ x_2 & x_1 \end{vmatrix} + \begin{vmatrix} \xi_4 & \xi_3 \\ x_4 & x_3 \end{vmatrix}, \\ C = \begin{vmatrix} \xi_2 & \xi_4 \\ x_2 & x_4 \end{vmatrix} + \begin{vmatrix} \xi_3 & \xi_1 \\ x_3 & x_1 \end{vmatrix}, \\ D = \begin{vmatrix} \xi_3 & \xi_2 \\ x_3 & x_2 \end{vmatrix} + \begin{vmatrix} \xi_4 & \xi_1 \\ x_4 & x_1 \end{vmatrix}, \end{cases}$$

and analogously:

$$(6')\qquad \begin{cases} \beta = \begin{vmatrix} \xi_2 & \xi_1 \\ x_2 & x_1 \end{vmatrix} - \begin{vmatrix} \xi_4 & \xi_3 \\ x_4 & x_3 \end{vmatrix}, \\ \gamma = \begin{vmatrix} \xi_2 & \xi_4 \\ x_2 & x_4 \end{vmatrix} - \begin{vmatrix} \xi_3 & \xi_1 \\ x_3 & x_1 \end{vmatrix}, \\ \delta = \begin{vmatrix} \xi_3 & \xi_2 \\ x_3 & x_2 \end{vmatrix} - \begin{vmatrix} \xi_4 & \xi_1 \\ x_4 & x_1 \end{vmatrix}. \end{cases}$$

Let (t_1, t_2, t_3, t_4) and (d_1, d_2, d_3, d_4) then be two quadruples of variables, so we let $[td]_2$, $[td]_3$, $[td]_4$ denote three expressions that are formed from the t and d in precisely the same way that B, C, D are formed from $(\xi_1, \xi_2, \xi_3, \xi_4)$ and (x_1, x_2, x_3, x_4); analogously, let $[td]'_2$, $[td]'_3$, $[td]'_4$ denote the expressions that are formed from (t_i) and (d_i) just as β, γ, δ are formed from (ξ_i) and (x_i).

If we recall the development of the product of two matrices with two rows into the sum of the products of their corresponding minors, and the development of a determinant of fourth order into the sum of products of the second-order minors that are taken from the matrix that is formed from first of two rows for the complementary minors then we will easily obtain the following fundamental identity:

(7) $$\sum_i \{[td]_i [ef]_i\} = \begin{Vmatrix} t_1 & t_2 & t_3 & t_4 \\ d_1 & d_2 & d_3 & d_4 \end{Vmatrix} \cdot \begin{Vmatrix} e_1 & e_2 & e_3 & e_4 \\ f_1 & f_2 & f_3 & f_4 \end{Vmatrix} + \begin{vmatrix} t_1 & t_2 & t_3 & t_4 \\ d_1 & d_2 & d_3 & d_4 \\ e_1 & e_2 & e_3 & e_4 \\ f_1 & f_2 & f_3 & f_4 \end{vmatrix}.$$

If we let $(t\ d\ e\ f)$ denote the determinant of the right-hand side then this identity can be written:

(8) $$\sum_i \{[td]_i [ef]_i\} = \sum t\,e \sum d\,f - \sum d\,e + (t\ d\ e\ f).$$

If we would like to find the value of $(t\ d\ e\ f)$ as a function of $\sum t^2$, $\sum d^2$, …, $\sum td$, $\sum te$, …, $\sum de$, … then it will suffice that we take its square; we will then get a determinant in which the terms are all of the prescribed form. If we then extract the square root then we will have the value of $(t\ d\ e\ f)$, up to sign. We deduce from this that:

I. If $t_i = e_i$, $d_i = f_i$, $\sum t^2 = \sum d^2 = 1$, $\sum t\,d = 0$ then we will have:

$$\sum_i \{[td]_i [ef]_i\} = 1,$$

which could have been foreseen if one recalled (5) and (5′).

II. If the t, d, e, and f form four completely distinct quadruples and $\sum t^2 = \sum d^2 = \sum e^2 = \sum f^2 = 1$, while:

$$\sum td = \sum te = \sum tf = \sum de = \sum df = \sum ef = 0$$

then one will have:

$$\sum_i \{[td]_i [ef]_i\} = \pm 1,$$

and, without getting preoccupied with the sign, at the moment, it is enough to observe that it will change when two of the four quadruples are exchanged.

III. If $e_i = t_i$, but $d_i \neq f_i$ and $\sum ed = \sum ef = \sum df = 0$, while:

$$\sum e^2 = \sum d^2 = \sum f^2 = 1$$

then one will have:

$$\sum_i \{[td]_i [ef]_i\} = 0.$$

The ambiguity in sign that appears in II of these cases, and which must always appear from the way by which we calculated the determinant $(t\ d\ e\ f)$ as long as $(t\ d\ e\ f)$ is not zero, should not be the cause of any confusion, and that is because if one exchanges the symbols $[td]_i$ with the symbols $[td]'_i$ then one will get an identity that differs from (8) by only the sign of $(t\ d\ e\ f)$, as a simple calculation will reveal. Now, whereas we always consider the two types of parallelism and symbols simultaneously, it will suffice to perform the calculations with just one type of symbol – e.g., the unprimed symbols. We then get, in fact, terms with indeterminate sign. However, it will be quite pointless to determine this sign, because we must use one sign for parallelism in one sense and the opposite sign when we consider parallelism in the other sense.

We have already seen how, by means of (6) and (6′), one can calculate the scroll parameters of a line that is defined by means of just two of it its points (x_i) and (ξ_i) at a distance of $\pi / 2$. We would now like to show how, given the scroll parameters of a line, one can get back to the usual determination of that line. To that end, look for the coordinate of the point where the line whose scroll parameters are B, C, D, β, γ, δ meets, e.g., the plane $x_1 = 0$. If we set $x_1 = 0$ in (3) and (3′) and compare the values of ξ_2, ξ_3, ξ_4 that this produces then we will get:

$$x_2 : x_3 : x_4 = B + \beta : C + \gamma : D + \delta.$$

Since, if $x_1 = 0$ one must have $x_2^2 + x_3^2 + x_4^2 = 1$, one finally gets:

$$(9) \qquad x_1 = 0, \qquad x_2 = \frac{B+\beta}{\sqrt{2(1+B\beta+C\gamma+D\delta)}}, \qquad x_3 = \frac{C+\gamma}{\sqrt{2(1+B\beta+C\gamma+D\delta)}},$$

$$x_4 = \frac{D+\delta}{\sqrt{2(1+B\beta+C\gamma+D\delta)}}.$$

It is then easy to calculate the corresponding (ξ_i) by means of (3) or (3′).

§ 4. We now propose to study the geometric significance of the scroll parameters of a line. The fundamental property of them is that they are "invariant under parallelism," as is expressed by the following theorem:

If two lines have three equal – or equal, in the opposite sense – parameters with respect to the same triad then they will be parallel, in one sense or the other, depending upon whether the triad in question is the first or the second kind, respectively.

Indeed, in such a case there will exist a scrolling that scrolls both of the above in the same way. This theorem, which emerges immediately from our considerations, is fundamental for us. For that reason, it will not be wrong for us to establish it in a direct way, as a check of the calculations, and because we will then get other formulas that will be very useful in what follows.

Let a line be the intersection of two planes (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4), which, for simplicity, are assumed to be orthogonal. The absolute is defined by:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$$

so a system of its generators can be imagined to be defined by:

$$(\alpha) \qquad \begin{cases} (x_1 + ix_2) + \lambda(x_3 + ix_4) = 0, \\ (x_3 - ix_4) - \lambda(x_1 - ix_2) = 0, \end{cases}$$

where λ varies from generator to generator. Any point that belongs to two planes (a_i), (b_i) will satisfy:

$$(\beta) \qquad \sum a_i\, x_i = 0, \quad \sum b_i\, x_i = 0,$$

so that in order to find that generator of the series (α) that carries our line, it is sufficient to eliminate the (x_i) from the (α) and the (β); from that, we get:

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ 1 & i & \lambda & i\lambda \\ -\lambda & i\lambda & 1 & -i \end{vmatrix} = 0,$$

namely:

$$\lambda^2 \{[(a_1\, b_3 - b_1\, a_3) + (a_1\, b_4 - b_2\, a_4)] + i\, [(b_4\, a_1 - a_4\, b_1) + (b_2\, a_3 - a_2\, b_3)]$$
$$+ 2\, i\lambda\, (a_4b_3 - b_4a_3 + b_2a_1 - b_1a_2)$$
$$+ \{(a_4b_3 - b_4a_3 + b_2a_1 - b_1a_2) - i(a_4b_3 - b_4a_3 + b_2a_1 - b_1a_2)\} = 0.$$

In order for the line of intersection of the planes (a_i') and (b_i') to carry the same pair of generators, i.e., both parallels (in the sense that is determined by the ruled series (a)) of our line, one immediately deduces that one must have:

$$(a_1b_3 - b_1a_3 + a_2b_4 - b_2a_4) : (a_1b_4 - b_1a_4 + b_2\, a_3 - a_2b_3) : (a_1b_2 - b_1a_2 + a_4b_3 - a_3b_4)$$
$$= (a_1'b_3' - b_1'a_3' + a_2'b_4' - b_2'a_4') : (a_1'b_4' - b_1'a_4' + b_2'a_3' - a_2'b_3') : (a_1'b_2' - b_1'a_2' + a_4'b_3' - a_3'b_4')\,.$$

One proceeds analogously with the other series of generators of the absolute; the preceding formula not only proves out theorem, but gives an expression for the scrolling parameters of a line as functions of the coordinates of two perpendicular planes that pass through the line.

We must often find the trace upon a plane α of the parallel to a line that is drawn through its pole A, and call it the *Clifford image* of the line relative to that plane. If S_i and Z_i are the coordinates of the two traces then we will have, from (3) and (3′), and when one takes the point A to be the point (1, 0, 0, 0):

$$(10) \qquad \begin{cases} S_1 = 0,\ S_2 = B = [\xi\, x]_2,\ S_3 = C = [\xi\, x]_3,\ S_4 = D = [\xi\, x]_4, \\ Z_1 = 0,\ Z_2 = \beta = [\xi\, x]_2',\ Z_3 = \gamma = [\xi\, x]_3',\ Z_4 = \delta = [\xi\, x]_4'. \end{cases}$$

By means of this equivalence, and letting φ denote the distance between the two points S, Z that is defined by $\cos \varphi = \sum S_i\, Z_i$, (9) will become:

$$(9') \qquad x_1 = 0, \qquad x_i = \frac{S_i + Z_i}{2\cos\frac{\varphi}{2}} \quad [i = 2, 3, 4],$$

which will give this, by means of (3) or (3′):

$$(11) \qquad \xi_1 = -\cot\frac{\varphi}{2}, \qquad \xi_2 = \frac{1}{2\cot\frac{\varphi}{2}}\begin{vmatrix} S_3 & S_4 \\ Z_3 & Z_4 \end{vmatrix}, \qquad \xi_3 = \frac{1}{2\cot\frac{\varphi}{2}}\begin{vmatrix} S_4 & S_2 \\ Z_4 & Z_2 \end{vmatrix},$$

$$\xi_4 = \frac{1}{2\cot\frac{\varphi}{2}}\begin{vmatrix} S_2 & S_3 \\ Z_2 & Z_3 \end{vmatrix}.$$

One immediately verifies that:

The two Clifford images of the two lines that are polar to a plane and the traces on that plane of those lines are harmonically separated. One of these traces will bisect a segment that ends at the image.

§ 5. It will again be opportune for us to note that the scrolling parameters of a line, when also multiplied by an arbitrary factor, will satisfy:

$$B^2 + C^2 + D^2 - \beta^2 - \gamma^2 - \delta^2 = 0.$$

Therefore, we must always imagine that six quantities that are ruled by this relation will be the homogeneous coordinates of a line, in which one can fix, up to sign, a proportionality factor (in just one way) in such a manner that they will become the six scrolling parameters of line. The form of $B, C, D, \beta, \gamma, \delta$ gives the following theorem:

The scrolling parameters (invariants under parallelism) of a line are nothing but the Klein coordinates (appropriate sums and differences of the Plücker coordinates) of that line when one takes the fundamental tetrahedron to be a tetrahedron that is autopolar with respect to the absolute.

§ 6. A first noteworthy application of this coordinate is the definition of the angle between two arbitrary lines (which has, so far, not been the case for coplanar lines). We define the angle φ between two lines to be the angle φ that is defined by:

$$\cos\varphi = BB' + CC' + DD',$$

so the angle φ will generally be distinct from the preceding one, which can be defined by:

$$\cos \varphi = \beta\beta' + \gamma\gamma' + \delta\delta',$$

in which formula, we intend that (B, C, D, β, γ, δ) should be the parameters of one line and (B', C', D', β', γ', δ'), the homologous parameters of the other one. This definition emerges spontaneously from the following theorem:

The angle between a pair of coplanar lines and the angle between the parallels to it through an arbitrary point A will be equal, as long as the two parallels are drawn in the same direction.

In fact, there will then exist a scrolling (which will be right-handed or left-handed, according to the sense of the parallelism) that carries the common point of the first pair of lines to the point A and the first pair of lines to the pair of parallels through the point A.

The theorem can also be proved analytically: If the given lines are the lines (x, ξ), (x, η) then the angle φ between then will be defined by $\cos \varphi = \sum \xi\,\eta$; the angle φ' between the parallel lines that are drawn through the point (1, 0, 0, 0) is given by:

$$\cos \varphi' = \sum [x\ \xi]_i\,[x\ \eta]_i$$

or

$$\cos \varphi' = \sum [x\,\xi]'_i\,[x\,\eta]'_i$$

according to the direction of the parallelism. The identity (8) proves that one has $\cos \varphi' = \cos \varphi$ in both cases.

One then sees that the angle φ that is defined by:

$$\text{(12)} \qquad \begin{cases} \cos\varphi = BB' + CC' + DD', \\ \cos\varphi = \beta\beta' + \gamma\gamma' + \delta\delta' \end{cases}$$

is nothing but (10) for the *angles that are formed by the two pairs of parallels that are drawn through the point* (1, 0, 0, 0) *of the two lines, when they are given in one direction or the other, and from the preceding theorem, one sees that as a result of the fact that this parallel is drawn through the point* (1, 0, 0, 0), *one must draw that parallel through an arbitrary point in space without altering the determination of its angle.*

If the two lines are the lines (x, ξ) and (y, η) then we will have the following formulas for the angle between the two lines:

$$\cos \varphi = \sum [x\ \xi]_i\,[x\ \eta]_i \qquad \text{or} \qquad \cos \varphi = \sum [x\,\xi]'_i\,[x\,\eta]'_i,$$

depending upon whether the angle is measured using one sense of parallelism or the other; i.e., from the identity, one will have:

$$\cos \varphi = \cos \widehat{xy}\ \cos \widehat{\xi\eta} - \cos \widehat{\xi y}\,\cos \widehat{x\eta} \pm (x\,\xi\,y\,\eta),$$

in which $\widehat{xy}$, $\widehat{\xi\eta}$, $\widehat{\xi y}$, $\widehat{x\eta}$ mean the distances between the point (x_i) and the point (y_i), the point (ξ_i) and the point (η_i), etc. Without dwelling upon the geometric significance of $(x\ \xi\ y\ \eta)$, one observes that:

The determinant $(x\ \xi\ y\ \eta)$ *is zero, and the angle between the two lines admits only one determination if and only if the two lines are complementary.* (Cf., the final observation).

This theorem admits a noteworthy corollary, when it is applied to infinitely-close coplanar lines:

If we construct the Clifford images relative to an arbitrary plane of the generators of a ruling then the two lines thus-obtained will correspond in such a way that corresponding arc lengths will be equal if and only if the ruling is developable.

If one recalls (Bianchi A) that a ruling has zero curvature only if it is a Clifford ruling then we will see that this theorem is a counterpoint to the other one that:

A ruling will have zero curvature only if one of its Clifford images reduces to a point.

We immediately recognize a new meaning for the Klein coordinates: *They measure the angles that are formed between a line, in one direction or the other, and the edges of the reference tetrahedron, or, as one can say, an orthogonal triad of lines.*

Thus, the scrolling parameters are nothing but the projective coordinates of a line, and one can then define a linear complex with an equation:

$$l\,A + m\,B + n\,C = p\,\alpha + q\,\beta + r\,\gamma,$$

where l, m, n, p, q, r are constants, we see from (12) that this linear complex admits the following metric definition in an elliptic space:

The lines of a linear complex are those, and only those, lines for which $\dfrac{\cos\varphi}{\cos\psi}$ *is constant, where* φ *and* ψ *are the angles that any of them forms with a fixed line.*

With the preceding notation, that line will be the one whose parameters are:

$$\left(\frac{l}{\sqrt{l^2+m^2+n^2}}, \frac{m}{\sqrt{l^2+m^2+n^2}}, \frac{n}{\sqrt{l^2+m^2+n^2}}, \frac{p}{\sqrt{l^2+m^2+n^2}}, \text{etc.}\right).$$

In curved space, one will then have the theorem:

A linear complex always admits a line such that the helicoidal motion around it is referred to the same complex.

One will easily see that the locus of points such that the angle between parallels that are drawn through any of them to a fixed and constant line is a Clifford ruling, and one will then be able to generate the Clifford ruling by means of the helicoidal motions around a line. Prof. Bianchi gave a way of generating them in the paper that was cited above that was identical to this, in principle. However, when viewed in this different form, it can be interpreted projectively thus:

The projectivity that leaves a quadric fixed, along with two of its points A, B carries an arbitrary point of space to the points of a quadric that has the generators through A, B in common with the preceding.

Remark. It is not possible, I believe, to define parallelism in the plane that one would be led to define as a result of the parallelism of points under the law of duality; you can, however, define the parallelism of elements (as a result of the points and planes belong to them). *We say that the element* (A, α) *that is defined by the point A and the plane* α *is parallel to the element* (B, β) *when there exists a scrolling that takes A to B and* α *to* β.

The plane β is generated by the line that is drawn through B that is parallel to the line of α that passes through A.

The distance from A to B is equal to the angle between α and β.

The normal to α at A is parallel to the normal to β at B.

The more interesting thing in all of this is *the existence of dual figures that correspond with parallelism of the corresponding elements* and the subsequent proof of the principle of duality without *any consideration of the absolute.*

However, for the sake of brevity, I will prove their existence by starting with the absolute. Take a figure S, and consider the polar figure S' that an arbitrary scrolling carries in Σ. The figures S, Σ will be precisely two dual figures that correspond in the aforementioned way.

Clifford parallelism and the theory of curves.

§ 7. Prof. Bianchi (*loc. cit.*) proved the following formulas, which are generalizations of the Frenet formulas for a curve in a flat space:

$$\frac{dx_i}{d\sigma} = \xi_i, \qquad \frac{d\xi_i}{d\sigma} = \frac{\eta_i}{\rho} - x_i, \qquad \frac{d\eta_i}{d\sigma} = -\frac{\xi_i}{\rho} - \frac{\zeta_i}{\tau}, \qquad \frac{d\zeta_i}{d\sigma} = \frac{\eta_i}{\tau},$$

where σ, $1 / \rho$, $1 / \tau$ represent the arc length, the first curvature, and the second curvature of the generic point (x_i) of a curve, respectively, and where (ξ_i), (η_i), (ζ_i), denote the direction cosines of the tangent, principal normal, and binormal, respectively, of the curve at the point (x_i).

In the calculations that follow, as with everything in the rest of this treatise, we will perform the calculations with just one triad of scrolling parameters, while starting with what we saw in § 3. Draw the parallel through the point (1, 0, 0, 0) to the tangents that meet the polar plane; the arc length of the indicatrix of the tangents that is thus obtained is given by:

$$ds^2 = \sum \{d[x,\, \xi]_i\}^2 = \sum [dx, \xi]_i^2 + \sum [dx, d\xi]_i^2 + 2\sum [dx, \xi]_i [x, d\xi]_i\ .$$

Substitute the values for dx, $d\xi$ that were given above by Prof. Bianchi's formulas in this formula, and develop it, while recalling the identity (8) in § 3. One will get:

$$ds^2 = \frac{d\sigma^2}{\rho^2},$$

which is also valid for the parallelism in the other sense, because (§ 3) the missing terms will have double signs. That can be explained by observing that consecutive tangents are complementary, and recalling the theorem of § 6. Therefore:

The ratio of an arbitrary *one of the angles that formed between two consecutive tangents to the arc between the points of contact is equal to the curvature of the curve of corresponding points.*

On the contrary, we now consider a generic line that is normal to the point (x_i) of the curve and let its direction cosines be $(\eta_i \cos \varphi + \zeta_i \sin \varphi)$, where φ is constant. The Clifford image of the ruling that is formed from these lines have an arc length s that is defined by: $ds^2 = \sum \{d[\eta \cos\varphi + \zeta \sin\varphi, x]_i^2\}$, i.e., when one differentiates and recalls Prof. Bianchi's formula:

$$ds^2 = d\sigma^2 \sum \left[\sin\varphi \left\{ \left[\frac{\eta}{\tau}, x\right]_i + [\zeta\, \xi]_i \right\} + \cos\varphi \left\{ [\eta\, \xi]_i + \left[-\frac{\xi}{\rho} - \frac{\zeta}{\tau}, x \right] \right\} \right]^2$$
$$= \left\{ \frac{\cos^2\varphi}{\rho^2} + \left(\frac{1}{\tau} \pm 1\right)^2 \right\} d\sigma^2.$$

In this, by the identity that was cited above and with consideration of § 3, the double sign corresponds to the double sense of parallelism.

For $\varphi = \pi / 2$, one has $ds^2 = \left(\frac{1}{\tau} \pm 1\right)^2 d\sigma^2$; we will put:

$$\frac{1}{T} = \frac{1}{\tau} + 1, \qquad \frac{1}{T'} = \frac{1}{\tau} - 1,$$

in which we have, however, denoted both of the two expressions by $1 / T$, and we will call $1 / T$ and $1 / T'$ the two *Clifford torsions* of a curve at a point. We will then have:

The ratio of the angle between two consecutive binormals, measured in some sense, to the arc length between their feet is equal to the corresponding Clifford torsion. One then has a theorem of Prof. Bianchi (*loc. cit.*) – in a slightly different form – that results from this on a direct manner:

The necessary and sufficient condition for the binormals to a plane curve to be parallel in direction is that the corresponding Clifford torsion be zero.

In order for the line that emanates from the point (x_i) of a curve and has direction cosines ($a\,\xi_i + b\,\eta_i + c\,\zeta_i$) with $a^2 + b^2 + c^2 = 1$ (where a, b, c are constants) to generate a Clifford ruling by varying the point x_i, the arc length of one of its Clifford images must be equal to zero; i.e.:

$$\sum \{a\,d[x,\xi] + b\,d[x,\eta] + c\,d[x,\zeta]\}^2 = 0,$$

and with the usual procedures:

$$(\alpha) \ldots\ldots \qquad \left(\frac{a}{\rho} + \frac{c}{T}\right)^2 + b^2\left(\frac{1}{\rho^2} + \frac{1}{T^2}\right) = 0, \qquad \text{i.e.,} \qquad \frac{\rho}{T} = \text{const.}; \quad b = 0,$$

where $1 / T$ denotes the corresponding Clifford torsion. The curve is then a helix, but one that adds the new interpretation of the condition ρ / T = const., which says that there must be a *constant ratio of the first curvature to the Clifford torsion.*

If we then express the idea that our ruling has zero curvature then we will find that:

$$b = 0; \quad \left(\frac{a}{\rho} + \frac{c}{\tau} - c\right)\left(\frac{a}{\rho} + \frac{c}{\tau} + c\right) = 0,$$

which does not coincide with (α) for real elements. One then deduces the existence of *singular, imaginary rulings that are generated like the Clifford ruling of a line that is united invariably with the principal trihedron of a curve that does not have zero curvature, although one of its Clifford indicatrices is zero; their generators will therefore be tangent to the absolute.*

We add that, as a result, *all of the curves for which* ρ / τ = const. *are ones for which there exists a line that is united invariably with the principal trihedron will generate a developable.*

Indeed, it is enough to express (§ 6) the idea that the Clifford images of the ruling that is generated by this line will correspond, with equality of arc length, in order to find that, with the preceding notation:

$$\frac{b^2 + c^2}{\tau} + \frac{ac}{\rho} = 0.$$

§ 8. These preliminary considerations immediately suggest an idea that will serve to establish a very remarkable new form, in my opinion, of the formula of Prof. Bianchi that was cited above. In order to arrive at this formula, Prof. Bianchi examined the *direction cosines* of the tangent, principal normal, and binormal. We, in turn, will examine its *scrolling parameters*, which are denoted by:

$$(\alpha, \beta, \gamma, \alpha', \beta', \gamma'); \quad (\xi, \eta, \zeta; \xi', \eta', \zeta'); \quad (\lambda, \mu, \nu; \lambda', \mu', \nu').$$

However, one sees that:

$$(13) \quad \begin{cases} \alpha^2+\beta^2+\gamma^2 = \alpha'^2+\beta'^2+\gamma'^2 = \xi^2+\eta^2+\zeta^2 = \cdots = 1, \\ \alpha\beta+\beta\eta+\gamma\zeta = \alpha'\xi'+\beta'\eta'+\gamma'\zeta' = \xi\lambda+\eta\mu+\zeta\nu = \cdots = 0, \\ d\alpha^2+d\beta^2+d\gamma^2 = \dfrac{d\sigma^2}{\rho^2} = d\alpha'^2+d\beta'^2+d\gamma'^2, \\ d\lambda^2+d\mu^2+d\nu^2 = d\sigma^2\dfrac{1}{T^2}, \\ d\lambda'^2+d\mu'^2+d\nu'^2 = \dfrac{1}{T'^2}d\sigma^2; \quad \lambda\,d\alpha+\mu\,d\beta+\nu\,d\gamma = \lambda'\,d\alpha'+\cdots = 0. \end{cases}$$

By using a procedure that is identical to the one that followed for the Frenet formula in flat space, this formula will suffice to give following formulas:

$$(14) \quad \begin{cases} \dfrac{d\alpha}{d\sigma} = \dfrac{\xi}{\rho}; \quad \dfrac{d\xi}{d\sigma} = -\dfrac{\alpha}{\rho}-\dfrac{\lambda}{T}; \quad \dfrac{d\lambda}{d\sigma} = \dfrac{\xi}{T}, \\ \dfrac{d\alpha'}{d\sigma} = \dfrac{\xi'}{\rho}; \quad \dfrac{d\xi'}{d\sigma} = -\dfrac{\alpha'}{\rho}-\dfrac{\lambda'}{T}; \quad \dfrac{d\lambda'}{d\sigma} = \dfrac{\xi'}{T'}, \end{cases}$$

and analogous ones for $\beta, \gamma, \beta', \gamma', \eta, \zeta$, etc. However, there would be a sign ambiguity in the right-hand side of (14) that would stem from the fact that only $1/\rho$, $1/T$ appear in (13), and we extract the square root, so we would be uncertain whether to take $1/\rho$, $1/T$ or $-1/\rho$, $-1/T$. However, formula (14) is easily verified by starting with Prof. Bianchi's formula. As for:

$$\frac{d\alpha}{d\sigma} = \frac{\xi}{\rho}; \qquad \frac{d\alpha'}{d\sigma} = \frac{\xi'}{\rho},$$

it is sufficient to recall the effective values of our parameters, and the proof will follow immediately. Now, observe that, e.g.:

$$(\gamma) \qquad \frac{d\xi}{d\sigma} = -\frac{\alpha}{\rho}-\frac{\lambda}{T}.$$

Recall that:

$$\xi = \eta_1\,x_2 - \eta_2\,x_1 + \eta_3\,x_4 - \eta_4\,x_3\,.$$

One deduces from Prof. Bianchi's formulas, recalling that $\alpha = [x\ \xi]_1$ and that $\lambda = [x\ \zeta]_1$, that:

$$\frac{d\xi}{d\sigma} = -\frac{\alpha}{\rho} - \frac{\lambda}{T} + (\eta_1\ \xi_2 - \eta_2\ \xi_1 + \eta_3\ \xi_4 - \eta_4\ \xi_3).$$

Now, $\eta_1\ \xi_2 - \eta_2\ \xi_1 + \eta_3\ \xi_4 - \eta_4\ \xi_3$ is nothing but a scrolling parameter of the line that is polar to the binormal, and therefore, by a theorem that was proved already (§ 2), is equal to $\pm\lambda$, according to the sense of parallelism; (γ) is then proved ([1]). One proves the other formula (14) in an analogous way.

One can deduce other results from (14), which individually approaches the Frenet formulas for flat space, that are worthy of note, in my opinion.

As for the integration of each of the two groups (14) of formulas, which reduces to a Ricatti equation, one gets:

The effective construction of a curve for which the curvature and torsion are known as functions of the arc length reduces to the integration of two Ricatti equations.

Thus, if the two curves correspond point-by-point with parallelism in one direction of the principal triehdron and for which (ρ, T, σ) and (ρ_1, T_1, σ_1) are the first curvature, the corresponding Clifford torsion, and the arc length, resp., at corresponding points, then one will have:

$$\frac{\rho}{\rho_1} = \frac{T}{T_1} = \frac{d\sigma}{d\sigma_1};$$

(as Prof. Bianchi observed for flat space) this therefore permits one to *reduce the construction of a curve for which one is given intrinsic equations to a curve for which one has* ρ = const *or* T = const.

The analogy between (14) and the Frenet formulas immediately gives some theorems for which the proof repeats, step-by-step, what one proves for the analogues in flat space. Therefore, e.g.:

If two curves have parallel principal normals at corresponding points then the angle between their corresponding tangents will be constant, and the curvature of one of them will be a linear function of that of the other one. (This can then be of service in the study of the Bertrand curve in curved space.)

One can thus deduce the things that are done in all of the theory of helices, etc., in a different way. What seems important to me to observe is that often the calculations are performed more simply in curved space than they are in flat space. If we, e.g., would like to find the evolute of a curve then it will suffice that we look for those times when the ruling that is generated by a normal to the curve with direction cosines (ξ_i cos $\varphi + \lambda_i$ sin

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([1]) Here, we have used (λ, μ, ν) in order to denote both of the triads (λ, μ, ν), (λ', μ', ν'), since that will certainly not cause any confusion; one must just remember not to confuse the ξ with the ξ_i .

φ) (where φ is a function of σ) generates a developable – i.e., when the two Clifford images of the ruling correspond with equality of arc lengths *s*. Now, one has:

$$ds^2 = \sum [d\xi \cos\varphi + d\lambda \sin\varphi - \xi \sin\varphi\, d\varphi + \lambda \cos\varphi\, d\varphi, x]_i^2 .$$

If we start with the usual observations of § 3, using (14) and recalling that $1 / T$ admits two determinations then we will see that is suffices to annul the terms with double signs in the preceding expression in order to find the desired condition, and by calculation, we will then get: $\varphi = \int \frac{d\sigma}{\tau}$.

A more noteworthy result, for which, we will see some applications in what follows, is given by the following proposition:

Any curve C in elliptic space will correspond to two curves C′, C″ in flat space that correspond to C (and therefore also to each other) point-by-point, with equality of arc length and first curvature, while the torsions at corresponding points will differ by a constant. Conversely, two curves C′, C″ in flat space that correspond point-by-point with equality of arc length and first curvature, and whose torsions differ by a constant ± 2 *at corresponding points will give, without quadrature, a curve C in elliptic space that has curvature* + 1 *that will correspond, point-by-point, with equality of arc length and first curvatures and that will have Clifford torsions at a point that are the torsions of C′ and C″ at corresponding points.*

From (14), the first part of this theorem is obvious; we prove the second part. If we let (α, β, γ), (ξ, η, ζ), (λ, μ, ν) denote the direction cosines of the tangent, principal normal, and binormal at a point of C' and let $(\alpha', \beta', \gamma')$, (ξ', η', ζ'), (λ', μ', ν') denote the cosines of the correspond line for C'', let s, $1 / \rho$ denote the arc length and curvature of C' and C'' (at corresponding points), and let $1 / T$ and $1 / T'$ denote the corresponding torsions then we will have:

$$(a) \qquad \begin{cases} \dfrac{d\alpha}{ds} = \dfrac{\xi}{\rho}; \quad \dfrac{d\alpha}{ds} = -\dfrac{\alpha}{\rho} - \dfrac{\lambda}{T}; \quad \dfrac{d\lambda}{ds} = \dfrac{\xi}{T}, \text{etc.} \\[2ex] \dfrac{d\alpha'}{ds} = \dfrac{\xi'}{\rho}; \quad \dfrac{d\alpha'}{ds} = -\dfrac{\alpha'}{\rho} - \dfrac{\lambda'}{T'}; \quad \dfrac{d\lambda'}{ds} = \dfrac{\xi'}{T'}, \text{etc.} \end{cases}$$

for the Frenet formulas in flat space.

Since $\alpha^2 + \beta^2 + \gamma^2 = \alpha'^2 + \beta'^2 + \gamma'^2 = \xi^2 + \eta^2 + \zeta^2 = \ldots = 1$, we can imagine that the $(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$, $(\xi, \eta, \zeta; \xi', \eta', \zeta')$, and $(\lambda, \mu, \nu; \lambda', \mu', \nu')$ are the scrolling parameters of three lines of the space curve. Since one has $d\alpha^2 + d\beta^2 + d\gamma^2 = d\alpha'^2 + d\beta'^2 + d\gamma'^2$, from (α), the lines $(\alpha, \beta, \gamma; \alpha', \beta', \gamma')$ will describe (§ 6) a developable – i.e., one that envelops of curve C – and since, from (α):

$$\lambda\, \alpha + \mu\, \beta + \nu\, \gamma = \lambda'\, \alpha' + \mu'\, \beta' + \nu'\, \gamma' = \lambda\, d\alpha + \mu\, d\beta + \nu\, d\gamma = \lambda'\, d\alpha' + \mu'\, d\beta' + \nu'\, d\gamma' = 0,$$

the line (λ, μ, ν ; λ', μ', ν') is precisely the binormal to that curve at a generic point, and since, from (α):

$$\xi\,\alpha + \eta\,\beta + \zeta\,\gamma = \xi'\,\alpha' + \eta'\,\beta' + \zeta'\,\gamma' = \xi\,\lambda + \eta\,\mu + \zeta\,\nu = \xi'\,\lambda' + \eta'\,\beta' + \zeta'\,\gamma' = 0,$$

the line (ξ, η, ζ ; ξ', η', ζ') is precisely the principal normal to the curve C at its generic point.

These last arguments could also be made if one knew something about the torsions of C' and C''; however, in such a case, one could say, at most, that if σ is the arc length of C then one will have $d\sigma = ds$. If we, in turn, suppose that $\frac{1}{T}-\frac{1}{T'}$ = const., and if, for greater simplicity, we suppose that $\frac{1}{T}-\frac{1}{T'} = \pm 2$ then we will see immediately that $d\sigma = ds$. In fact, we will see that in a space with curvature + 1, the arc length will be defined by:

$$d\sigma = \pm\frac{1}{2}\{(\lambda\,d\xi + \mu\,d\eta + \nu\,d\zeta) - (\lambda'\,d\xi' + \mu'\,d\eta' + \nu'\,d\zeta')\}.$$

The fact that the constant difference $\frac{1}{T}-\frac{1}{T'}$ is $\pm$ 2 is no loss of generality. If $\frac{1}{T}-\frac{1}{T'}$ were a constant that is distinct from $\pm$ 2 then one would have, as one easily infers, a curve C in an elliptic space with a curvature that is different from + 1. (The rest of the time, one can always go from such a pair of curves to a pair of curves for which one has $\frac{1}{T}-\frac{1}{T'} = \pm 2$ by a similitude).

We prove that if C corresponds to C', C'' with equality of arc length then when (α) is compared to (14), that will prove our theorem completely.

A corollary that will be of great utility is the following one:

Any pair of curves in flat space that have constant, but distinct, torsions and correspond with equality of arc length and first curvature will correspond to a curve with constant torsion in curved space, and vice versa.

Moreover, the theorem that one can find all of the curves in flat space that have constant torsion by quadrature will appear in a new light; from the theorem that we just proved, one deduces that:

The problem of finding the curves of constant torsion in flat space, and that of finding all of the plane curves in elliptic space are equivalent. Therefore, since the solution of one of them is immediate, the other one will be solved completely.

We finally observe the generalization to space curves that was recently obtained by Prof. Razzaboni for the transformations ([1]) of the curves with constant torsion can be interpreted for these theorems on the Euclidian metric as a transformation of those pairs of curves with constant, but distinct, torsions that correspond with equality of arc length and first curvature.

On scrolled surfaces.

§ 9. By way of example, we would like to state a very simple theorem about scrolling surfaces – i.e., the ones that can be generated by a continuous scrolling of a curve, and which (Bianchi A) thus admit a second similar generation:

The necessary and sufficient condition for a surface to be scrolled along u = const. *and* v = const *is that the tangents to the* u = const. *along* v = const. *be parallel and therefore merely the tangents to* v = const. *along a* u = const. *Therefore, one can set* $E = G = 1$ *in the quadratic form that defines the surface and make* $F = \cos\sigma$, *so one must have* $D' = \sin\sigma$; *conversely, if* $E = G = 1$, $F^2 + D'^2 = 1$ *then the surface will be scrolled along the u and v.*

The linear element of the Clifford image of the ruling that is defined by the tangent to a v = const. along a u = const. is given (we denote the partial differentials with respect to v by d) by:

$$ds^2 = \sum\left[d\left[x, \frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}\right]_i\right]^2$$

$$= dv^2 \sum\left\{\left[\frac{\partial x}{\partial v}, \frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}\right]_i + \left[x, \frac{1}{\sqrt{E}}\frac{\partial^2 x}{\partial u\,\partial v}\right]_i + \left[x, -\frac{1}{2}\frac{\frac{\partial E}{\partial v}}{\sqrt{E^3}}\frac{\partial x}{\partial u}\right]_i\right\}^2 .$$

The expression in the right-hand side must be zero.

Developing this, while noting the identity, and recalling the formula that gives the second derivative of (x_i) as a function of the first derivative and the direction cosines of the normal, and noting that:

([1]) BIANCHI, *Giornale di Battaglia*, 1884.

$$\begin{vmatrix} x_1 & x_2 \quad x_3 & x_4 \\ \dfrac{\partial x_1}{\partial u} & \cdots\cdots & \dfrac{\partial x_4}{\partial u} \\ \dfrac{\partial x_1}{\partial v} & \cdots\cdots & \dfrac{\partial x_4}{\partial v} \\ \dfrac{\partial^2 x_1}{\partial u\,\partial v} & \cdots\cdots & \dfrac{\partial^2 x_4}{\partial u\,\partial v} \end{vmatrix} = D' \begin{vmatrix} x_1 & x_2 \quad x_3 & x_4 \\ \dfrac{\partial x_1}{\partial u} & \cdots\cdots & \dfrac{\partial x_4}{\partial u} \\ \dfrac{\partial x_1}{\partial v} & \cdots\cdots & \dfrac{\partial x_4}{\partial v} \\ \xi_1 & \cdots\cdots & \xi_4 \end{vmatrix} = \pm D' \sqrt{EG - F^2}\,,$$

one finally obtains, with the usual meaning for the double sign:

$$(D' \pm \sqrt{EG - F^2}\,)^2 + \frac{\left(F\dfrac{\partial E}{\partial v} - E\dfrac{\partial G}{\partial u}\right)^2}{4E(EG - F^2)} = 0;$$

i.e:

$$D' \pm \sqrt{EG - F^2} = \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} = 0.$$

Analogously, one will find that $\begin{Bmatrix} 12 \\ 2 \end{Bmatrix} = 0$. One can thus make $E = G = 1$, and then $D'^2 = 1 - F^2$. The first of these three formulas proves that the surface results from scrolling; the last one proves the second part of our theorem.

On ray congruences.

§ 10. The congruence of lines in curved space was studied by Fibbi in one of his papers that was published in *Annali della Scuola Normale Superiore*, Tomo VII, 1895. Without going into the particular cases, we will study those consequences that one can infer from the consideration of plane figures that are generated by drawing the parallel to the rays of a congruence through the point (1, 0, 0, 0) that meets the polar plane. Let (x_i) denote the generic point of the surface that is chosen to be the initial one of the congruence, and let (ξ_i) be the plane through it that is normal to the corresponding rays. Fibbi sets:

$$\begin{Vmatrix} \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ dx_1 & dx_2 & dx_3 & dx_4 \end{Vmatrix}^2 = E\,du^2 + 2\,F\,du\,dv + G\,dv^2,$$

$$\begin{Vmatrix} x_1 & x_2 & x_3 & x_4 \\ d\xi_1 & d\xi_2 & d\xi_3 & d\xi_4 \end{Vmatrix}^2 = E'du^2 + 2\,F'du\,dv + G'dv^2,$$

$$\sum dx_i\,d\xi_i = e\,du^2 + (f + f')\,du\,dv + g\,dv^2.$$

We will have for the linear element of the planar image above:

$$ds^2 = \sum [d(x, \xi)_i]^2 = \sum [dx, \xi]_i^2 + \sum [x, d\xi]_i^2 + 2\sum [dx, \xi]_i [x, d\xi]_i$$
$$= \sum dx^2 - (\sum \xi\, dx)^2 + (\sum x\, d\xi)^2 \pm 2\,(x, dx, \xi, d\xi).$$

Let $(x, dx, \xi, d\xi)$ denote the usual determinant, whose rows are (x_1, x_2, x_3, x_4), $(dx_1, \ldots)$, $(\xi_1, \ldots)$, $(d\xi_1, \ldots)$; the double sign is due to the usual reason. When $(x, dx, \xi, d\xi) = 0$, the angle between two consecutive generators will have just one determination. Therefore (§ 6), $(x, dx, \xi, d\xi) = 0$ is, *as Fibbi recognized directly, the equation of the developable of the congruence.* The part of the preceding formula that has constant sign is, with Fibbi's notations:

$$(\alpha) \qquad (E + E')\, du^2 + 2\,(F + F')\, du\, dv + (G + G')\, dv^2.$$

The part that has the variable sign is, without the factor of ± 2, equal to:

$$(b) \qquad \left\{ \begin{aligned} &\sqrt{\begin{vmatrix} E\,du^2 + 2F\,du\,dv + G\,dv^2 & e\,du^2 + 2(f + f')\,du\,dv + g\,dv^2 \\ e\,du^2 + 2(f + f')\,du\,dv + g\,dv^2 & E'\,du^2 + 2F'\,du\,dv + G'\,dv^2 \end{vmatrix}} \\ &= \frac{(Ef - Fe)\,du^2 + (Eg - F(f' - f) - Ge)\,du\,dv + (Fg - Gf')\,dv^2}{\sqrt{EG - F^2}} \\ &= \frac{(E'f' - F'e)\,du^2 + (E'g + F'(f' - f) - G'e)\,du\,dv + (F'g - G'f)\,dv^2}{\sqrt{E'G' - F'^2}}. \end{aligned} \right.$$

(α) and (β) are two quadratic forms that are completely independent of the surface that is chosen to be the initial one.

§§ 11. Theorem: *The only equations that must be satisfied by the forms (α) and (β) in order for the Fibbi form (which is already linked by simple algebraic equations* that Fibbi himself had noticed) *to correspond, in reality, to a congruence, are the ones that say that their sums and differences must be forms with curvature* + 1. (Recall the numerical factor that multiples β.)

This theorem, which permits one to *generalize the equations of Gauss and Codazzi to the congruence* are deduced by recalling that for one line – and therefore, also the ∞^2 straight lines of a congruence – one can give the planar images arbitrarily; that will then define the congruence.

The determination the points of a plane in curved space (and the Euclidian sphere) for which one is given the linear element reduces to the integration of a Riccati equation. Therefore:

Given the form (α), (β) of a congruence or a Fibbi form that satisfies the preceding conditions, the integration of two Ricatti equations will suffice to determine the congruence effectively.

The traces on the representative plane of the lines of the congruence are obtained (§ 4) by halving the segments that connect corresponding points of the two plane images; if, as usual, one lets (Y_1, Y_2, Y_3, O) and (Z_1, Z_2, Z_3, O) denote the corresponding points of these images and lets φ denote the distance between them then the linear element of the plane that is referred to that trace will be:

$$\sum\left[d\left(\frac{Y_i+Z_i}{\sqrt{2(1+\cos\varphi)}}\right)\right]^2 = \sum\left(\frac{dY_i+dZ_i}{2\cos\frac{\varphi}{2}}+\frac{Y_i+Z_i}{4}\frac{\sin\frac{\varphi}{2}}{\cos^2\frac{\varphi}{2}}d\varphi\right)^2 .$$

One will note this immediately when, in addition to linear elements of the planar images, one knows φ and the derivatives of φ with respect to u, v, u', v', where one imagines the (u, v) to be the coordinates that define (Y_i), the (u', v') to be the ones that define (Z_i), and all four of them are imagined in that distinct derivation. Indeed:

$$\frac{\partial\cos\varphi}{\partial u} = \sum Z\frac{\partial Y}{\partial u}; \qquad \frac{\partial\cos\varphi}{\partial u'} = \sum Z\frac{\partial Y}{\partial u'}, \text{ etc.}$$

$$2\sum dY\,dZ = d^2\sum Y\,Z - \sum(Y\,d^2Z + Z\,d^2Y).$$

This last equation reduces immediately once one recalls the formula that gives d^2Z, d^2Y in terms of Y, Z, and their first differentials.

If might be interesting to observe that when the congruence is W, it will suffice to know the linear element of the plane that is referred to that trace (when the u = const., v = const. are the developables). Indeed, with a geodetic representation of the face on Euclidian space, the representative plane will correspond to the plane at infinity. Such a linear element will become the linear element of the Euclidian sphere, referred to the spherical images of the developables. Using the notation of Prof. Bianchi (*Lezioni*, etc., Chap. 10, §§ 149, 150), one must have:

$$D_1 : D_1'' = D_2 : D_2'', \qquad \text{i.e., set} \qquad \rho = e^{\tau},$$

$$(\gamma) \qquad \frac{\partial\tau}{\partial u}\frac{\partial\tau}{\partial v}+\begin{Bmatrix}12\\1\end{Bmatrix}\frac{\partial\tau}{\partial u}+\begin{Bmatrix}12\\1\end{Bmatrix}\frac{\partial\tau}{\partial v}+\begin{Bmatrix}12\\1\end{Bmatrix}\begin{Bmatrix}12\\2\end{Bmatrix}-\begin{Bmatrix}22\\1\end{Bmatrix}\begin{Bmatrix}11\\2\end{Bmatrix} = 0,$$

while Guichard's equation, when subtracted from the preceding, becomes:

$$(\delta) \qquad \frac{\partial^2 \tau}{\partial u\,\partial v} = \begin{Bmatrix}12\\1\end{Bmatrix}\begin{Bmatrix}12\\2\end{Bmatrix} - \begin{Bmatrix}22\\1\end{Bmatrix}\begin{Bmatrix}11\\2\end{Bmatrix} - \frac{\partial \begin{Bmatrix}12\\1\end{Bmatrix}}{\partial u} - \frac{\partial \begin{Bmatrix}12\\2\end{Bmatrix}}{\partial v} - F\,.$$

The latter gives a result in form $\tau = M(u, v) + \int \varphi(u)\,du + \int \psi(v)\,dv$, where M is known, and φ and ψ are to be determined. If one substitutes this in the preceding equation then one will get $\psi(v)$ as a function of $\varphi(u)$; by differentiating that with respect to (u), one will get an equation for $\varphi(u)$ that takes the form:

$$A\,\varphi^2 + B\,\varphi + C + D\,\varphi' = 0, \quad \text{with} \quad A, B, C \text{ known.}$$

When this equation is repeatedly differentiated with respect to v, that will give the means to determine φ, and therefore τ and the congruence. Without entering into a detailed discussion, we further observe that if we suppose that $F = 0$ then the preceding equations will become:

$$\frac{\partial^2 \tau}{\partial u\,\partial v} = -\frac{\partial^2 \log\sqrt{EG}}{\partial u\,\partial v}; \qquad \frac{\partial \tau}{\partial u}\frac{\partial \tau}{\partial v} + \frac{\partial \log\sqrt{E}}{\partial v}\frac{\partial \tau}{\partial u} + \frac{\partial \log\sqrt{G}}{\partial u}\frac{\partial \tau}{\partial v} = 0.$$

By altering the parameters in u = const., v = const., one can arrange that $\rho = 1/\sqrt{EG}$ satisfies both of them; if one then sets:

$$\tau = -\log\sqrt{EG} + \int \varphi(u)\,du + \int \psi(v)\,dv$$

then one will have:

$$\varphi\,\psi = \varphi\frac{\partial \log\sqrt{G}}{\partial v} + \psi\frac{\partial \log\sqrt{E}}{\partial u} + \frac{\partial(\log\sqrt{E}, \log\sqrt{G})}{\partial(u, v)} = 0,$$

and since $\varphi = \psi = 0$ is a solution, E will be a function of G. *This is a theorem of Weingarten for the W surface*. However, here we observe that if the equation:

$$1 = \frac{1}{\varphi(u)}\frac{\partial \log\sqrt{G}}{\partial u} + \frac{1}{\psi(v)}\frac{\partial \log\sqrt{E}}{\partial v}$$

is soluble then one will get the other W-congruence from the same linear element; e.g., for $E = 1$, $G = \sin^2 u$.

§ 12. We return to curved space and resolve the question of knowing whether a congruence is W when one is given the fundamental forms – or what amounts to the same thing – the linear elements of its Clifford image planes. Thus, it will suffice that we recall that a line is defined by its scrolling parameters, which, as we know, are nothing but the projective coordinates of the line. Now (Darboux, *Leçons*, t. 3, page 345), the coordinates of a line that describes a *W*-congruence are solutions of the same second-order partial differential equation, such that if (α, β, γ, α_1, β_1, γ_1) are the scrolling parameters of a generic line of the congruence then one must have:

$$0 = \begin{vmatrix} \alpha & \beta \;\; \gamma \;\; \alpha_1 \;\; \beta_1 & \gamma_1 \\ \dfrac{\partial \alpha}{\partial u} & \cdots\cdots\cdots\cdots & \dfrac{\partial \gamma_1}{\partial u} \\ \dfrac{\partial \alpha}{\partial v} & \cdots\cdots\cdots\cdots & \dfrac{\partial \gamma_1}{\partial u} \\ \dfrac{\partial^2 \alpha}{\partial u^2} & \cdots\cdots\cdots\cdots & \dfrac{\partial^2 \gamma_1}{\partial u^2} \\ \dfrac{\partial^2 \alpha}{\partial u\, \partial v} & \cdots\cdots\cdots\cdots & \dfrac{\partial^2 \gamma_1}{\partial u\, \partial v} \\ \dfrac{\partial^2 \alpha}{\partial v^2} & \cdots\cdots\cdots\cdots & \dfrac{\partial^2 \gamma_1}{\partial v^2} \end{vmatrix}.$$

We now observe that $\alpha^2 + \beta^2 + \gamma^2 = 1$ and that $d\alpha^2 + d\beta^2 + d\gamma^2$ is known, so it is the linear element of one of the planar images of a congruence.

One can therefore conceive of α, β, γ as the coordinates of a variable point on a Euclidian sphere, for which one knows the linear element as a function of u, v; one can therefore express the second derivatives of the α, β, γ as functions of their first derivatives, the α, β, γ themselves, and the coefficients of that linear element, and analogously for α', β', γ'. Substitute these values for the second derivatives of (α, β, γ, α', β', γ'), develop their determinant, and form the sum of the products that are obtained by multiplying the third-order minors that belong to the matrix that is formed from the first three columns with the complementary minors. If we denote the linear elements of the two planar images by $e\, du^2 + 2\, f\, du\, dv + g\, dv^2$ and $e'\, du^2 + 2\, f'\, du\, dv + g'\, dv^2$, and set $\Delta = \sqrt{eg - f^2}$, $\Delta' = \sqrt{e'g' - f'^2}$ (which are supposed to be non-zero) then we will easily find the values of the third-order determinants above.

One will have, e.g.:

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \dfrac{\partial \alpha}{\partial u} & \cdots & \cdots \\ \dfrac{\partial \alpha}{\partial v} & \cdots & \cdots \end{vmatrix} = \Delta, \qquad \begin{vmatrix} \alpha & \beta & \gamma \\ \dfrac{\partial \alpha}{\partial v} & \cdots & \cdots \\ \dfrac{\partial^2 \alpha}{\partial u^2} & \cdots & \cdots \end{vmatrix} = -\begin{Bmatrix} 11 \\ 1 \end{Bmatrix} \Delta, \qquad \text{etc.}$$

Let Δ denote the determinant:

$$\begin{vmatrix} \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} & \begin{Bmatrix} 11 \\ 2 \end{Bmatrix} & e \\ \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} & \begin{Bmatrix} 12 \\ 2 \end{Bmatrix} & f \\ \begin{Bmatrix} 22 \\ 1 \end{Bmatrix} & \begin{Bmatrix} 22 \\ 2 \end{Bmatrix} & g \end{vmatrix},$$

and let $(-A)$ denote the determinant that one obtains by switching the signs in the last column; analogously, set A' and $(-A')$ equal to the corresponding determinants for the second linear element. One then sees immediately that the condition for our congruence to be W is obtained by making *the expression that is obtained by adding to* A *the sum of the terms in* $(-A)$, *multiplied by the complementary minors of the corresponding terms in* A', *and the expression that one deduces from that by exchanging* e *and* e', f *and* f', g *and* g'.

We would now like to present another application of our principles to the theory of congruences, and precisely to the concept of the "density" of a congruence at a point. In order to define that density at a point P, Fibbi proceeded in the following manner:

On the plane p through P that is normal to the corresponding ray of our congruence, it refers to an infinitesimal rotation $d\omega$ around P, and on the line of the congruence that emanates from the point C of $d\omega$, it refers to the point D that is conjugate with respect to the absolute of the point C. The line that connects the point P to that point D will determine an infinitesimal area $r^2\, d\omega'$ on a sphere of infinitesimal radius r and center at P. The ratio $d\omega' / d\omega$ is what Fibbi called the "density" of the congruence at the point P. Along with the "density" that is defined in the Fibbi way, we introduce a new element that we call the "Clifford density" of a congruence, which is perhaps better adapted to the intrinsic nature of elliptic space, and will, in any event, take us to one of the most important theorems of this present treatise. Through any point A of elliptic space, draw the parallel – à la Clifford – to the line of the congruence that emanates from the points of $d\omega$; it will determine an infinitesimal element $d\omega''$ in the polar plane A; the ratio $d\omega'' / d\omega$ (which will naturally have two determinations) will measure the "Clifford density" (right-handed or left-handed) of the congruence at the point for us; the arithmetic mean of these two densities will measure what we call the *absolute Clifford density* of the congruence at the point P. We proceed with the effective calculation, observe that with no loss of generality, we can suppose that the point P is the point (1, 0, 0, 0), and that the plane p through P that is normal to the ray of the congruence that passes through P is the plane (0, 0, 0, 1). We take two points P', P'' on $d\omega$ that are infinitely close to P, and consider the planes π', π'' through P', P'' that are normal to the corresponding rays of the congruence. If we recall the relations that link the coordinates of a point and those of a plane, and the condition for a point to belong to a plane, then we will see that, up to higher-order infinitesimals, one can set:

$$P' = (1, dx_2, dx_3, 0),$$
$$P'' = (1, \delta x_2, \delta x_3, 0),$$

$$\pi' = (0, d\xi_2, d\xi_3, 1),$$
$$\pi'' = (0, \delta\xi_2, \delta\xi_3, 1),$$

where d, δ are differential symbols.

If we point the parallel in the first direction then we will get, from the usual formulas, that the Clifford images of the rays of P, P', P'' will have the coordinates:

$$(1, 0, 0, 0),$$
$$(-1 - dx_2\, d\xi_3 + dx_2\, d\xi_2, - dx_2 + d\xi_2 , - dx_3 - d\xi_2, 0),$$
$$(-1 - \delta x_2\, \delta\xi_3 + \delta x_2\, \delta\xi_2, - \delta x_2 + \delta\xi_2 , - \delta x_3 - \delta\xi_2, 0),$$

respectively, when one takes the point (0, 0, 0, 1) to be the point through which the parallel is drawn. Up to negligible infinitesimals, these three images will then have the coordinates:

$$(-1, 0, 0, 0),$$
$$(- 1, - dx_2 + dx_3, - dx_3 - dx_2, 0),$$
$$(- 1, - \delta x_2 + \delta x_3, - \delta x_3 - \delta x_2, 0),$$

and the area $d\omega''$ of the triangle they span will be given by:

$$36\, d\omega''^2 = \begin{Vmatrix} -1 & 0 & 0 \\ -1 & -dx_2 + d\xi_3 - dx_3 - d\xi_2 & 0 \\ -1 & -\delta x_2 + \delta\xi_3 - \delta x_3 - \delta\xi_2 & 0 \end{Vmatrix}^2 ,$$

i.e., by:

$$6\, d\omega'' = \pm \left\{ \begin{vmatrix} d\xi_3 & dx_3 \\ \delta\xi_3 & \delta x_3 \end{vmatrix} + \begin{vmatrix} d\xi_2 & dx_2 \\ \delta\xi_2 & \delta x_2 \end{vmatrix} + \begin{vmatrix} d\xi_3 & d\xi_3 \\ \delta\xi_3 & \delta\xi_2 \end{vmatrix} + \begin{vmatrix} dx_3 & dx_3 \\ \delta x_3 & \delta x_2 \end{vmatrix} \right\}.$$

It is easy to verify that *(with a suitable choice of sign)*:

$$6\, d\omega = \begin{vmatrix} dx_3 & dx_2 \\ \delta x_3 & \delta x_2 \end{vmatrix}$$

and

$$6\, d\omega' = \begin{vmatrix} d\xi_3 & d\xi_2 \\ \delta\xi_3 & \delta\xi_2 \end{vmatrix}.$$

Therefore, calculate the sum:

$$\begin{vmatrix} d\xi_3 & dx_3 \\ \delta\xi_3 & \delta x_3 \end{vmatrix} + \begin{vmatrix} d\xi_2 & dx_2 \\ \delta\xi_2 & \delta x_2 \end{vmatrix}.$$

Setting:

$$d\xi_i = \frac{\partial\xi_i}{\partial u} du + \frac{\partial\xi_i}{\partial v} dv , \qquad \delta\xi_i = \frac{\partial\xi_i}{\partial u} \delta u + \frac{\partial\xi_i}{\partial v} \delta v , \qquad \text{etc.,}$$

one sees immediately that this sum is equal to:

$$(du\ \delta v - \delta u\ dv)\left(\frac{\partial x_2}{\partial v}\frac{\partial \xi_2}{\partial u}+\frac{\partial x_3}{\partial v}\frac{\partial \xi_3}{\partial u}-\frac{\partial x_2}{\partial u}\frac{\partial \xi_2}{\partial v}-\frac{\partial x_3}{\partial u}\frac{\partial \xi_3}{\partial v}\right),$$

and recall that everything will remain unaltered upon changing the sense of the parallelism, unless this sum is considered to have the altered sign, as one proves by a simple calculation. Now (as is known for the calculation of the coordinates of P', P'', π', π''), we have:

$$dx_1 = dx_4 = d\xi_1 = d\xi_4 = 0;$$

therefore, with Fibbi's notation, which one also recalls, one sees that the preceding sum can be written:

$$\pm (f - f')\ (du\ \delta v - \delta u\ dv),$$

where, from what was said, the double sign corresponds to the double sense of parallelism. One thus has:

$$(\alpha) \qquad 6\ d\omega'' = 6\ d\omega + 6\ d\omega' \pm (f - f')\ (du\ \delta v - \delta v\ du).$$

When one is given $d\omega$, since:

$$E\ G - F^2 = \left(x, \xi, \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}\right)^2$$

(where, as usual, the symbols in parentheses denote a IV$^{\text{th}}$-order determinant (§ 3)), and since $dx_1 = dx_4 = dx_1 = dx_4 = 0$, the formula that was noted above will become:

$$(\beta) \qquad 6\ d\omega = \sqrt{EG - F^2}\ (du\ \delta v - \delta u\ dv).$$

(α) and (β) give the following theorem:

One of the two Clifford densities of a congruence at a point will differ from the corresponding Fibbi density by the curvature of the ambient space plus:

$$\pm (f - f')\ \frac{1}{\sqrt{EG - F^2}},$$

up to a numerical factor.

The absolute density of a congruence at a point is equal to the curvature of the ambient space, augmented by the Fibbi density at that point.

The necessary and sufficient condition for the two Clifford densities to be equal is that $f = f'$, i.e., that the congruence be normal (which is a theorem that will soon be recast in a more opportune form).

One finds the generalization to an arbitrary congruence of the fact that two curvatures are defined for a surface, and we can say that:

The relative curvature and the absolute curvature of a surface at a point P are nothing but the Fibbi density and absolute density, resp., of the corresponding normal congruence at the point P. This last density is then equal to the right density and the left density of that congruence at the point P.

On the theory of surfaces.

§ 14. In the preceding paragraphs we have already obtained some theorems about surfaces that we will now prove in a direct manner without appealing to the general formulas that were just found on the theory of congruences.

With the usual notation, let:

$$E\,du^2 + 2\,F\,du\,dv + G\,dv^2, \qquad D\,du^2 + 2\,D'\,du\,dv + D''\,dv^2$$

be the two fundamental forms of a surface (Bianchi A); let (x_i) and (ξ_i) be the coordinates of a generic point and its corresponding tangent plane. We will have:

$$ds'^2 = \sum (d[x,\,\xi]_i)^2 = \sum ([dx,\,\xi]_i)^2 + \sum ([x,\,d\xi]_i)^2 + 2\sum [dx,\,\xi]_i\,[x,\,d\xi]_i$$

for the linear element of the Clifford image of the corresponding normal congruence.

Develop this with the usual identity that relates to determinants with two equal rows; one will have:

$$ds'^2 = = \sum dx^2 + \sum d\xi^2 \pm 2 \begin{vmatrix} dx_1 & dx_2 & dx_3 & dx_4 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \\ x_1 & x_2 & x_3 & x_4 \\ d\xi_1 & d\xi_2 & d\xi_3 & d\xi_4 \end{vmatrix},$$

where it is easy to verify that the double sign is due to the double sense of parallelism. Now, (Bianchi, *loc. cit.*), one has:

$$\frac{\partial \xi_i}{\partial u} = \frac{FD' - GD}{EG - F^2}\frac{\partial x_i}{\partial u} + \frac{FD - ED'}{EG - F^2}\frac{\partial x_i}{\partial v},$$

$$\frac{\partial \xi_i}{\partial v} = \frac{FD'' - GD}{EG - F^2}\frac{\partial x_i}{\partial u} + \frac{FD' - ED''}{EG - F^2}\frac{\partial x_i}{\partial v}.$$

Using this formula and recalling that:

$$\begin{vmatrix} \frac{\partial x_1}{\partial u} & \cdots\cdots & \frac{\partial x_4}{\partial u} \\ \frac{\partial x_1}{\partial v} & \cdots\cdots & \frac{\partial x_4}{\partial v} \\ x_1 & \cdots\cdots & x_4 \\ \xi_1 & \cdots\cdots & \xi_4 \end{vmatrix}^2 = EG - F^2,$$

one obtains that:

$$\begin{vmatrix} \frac{\partial x_1}{\partial u} & \cdots\cdots & \frac{\partial x_4}{\partial u} \\ x_1 & \cdots\cdots & x_4 \\ \xi_1 & \cdots\cdots & \xi_1 \\ \frac{\partial \xi_1}{\partial u} & \cdots\cdots & \frac{\partial \xi_4}{\partial u} \end{vmatrix} = \pm \frac{FD - ED'}{\sqrt{EG - F^2}},$$

$$\begin{vmatrix} \frac{\partial x_1}{\partial u} & \cdots\cdots & \frac{\partial x_4}{\partial u} \\ x_1 & \cdots\cdots & x_4 \\ \xi_1 & \cdots\cdots & \xi_1 \\ \frac{\partial \xi_1}{\partial v} & \cdots\cdots & \frac{\partial \xi_4}{\partial v} \end{vmatrix} = \pm \frac{FD' - ED''}{\sqrt{EG - F^2}}.$$

Analogously:

$$\begin{vmatrix} \frac{\partial x_1}{\partial v} & \cdots\cdots & \frac{\partial x_4}{\partial v} \\ x_1 & \cdots\cdots & x_4 \\ \xi_1 & \cdots\cdots & \xi_1 \\ \frac{\partial \xi_1}{\partial v} & \cdots\cdots & \frac{\partial \xi_4}{\partial v} \end{vmatrix} = \mp \frac{FD'' - GD'}{\sqrt{EG - F^2}},$$

$$\begin{vmatrix} \frac{\partial x_1}{\partial v} & \cdots\cdots & \frac{\partial x_4}{\partial v} \\ x_1 & \cdots\cdots & x_4 \\ \xi_1 & \cdots\cdots & \xi_1 \\ \frac{\partial \xi_1}{\partial u} & \cdots\cdots & \frac{\partial \xi_4}{\partial u} \end{vmatrix} = \mp \frac{FD' - GD}{\sqrt{EG - F^2}},$$

where the upper (lower) signs must be taken consistently. If we develop the value of ds'^2 with the formula that we just obtained then we will finally have:

$$ds'^2 = e\, du^2 + 2 f\, du\, dv + g\, dv^2,$$

where, with the usual notation for surfaces:

$$e = E + E' \pm 2\,\frac{FD - ED'}{\sqrt{EG - F^2}},$$

$$f = F + F' \pm \frac{GD - ED'}{\sqrt{EG - F^2}},$$

$$g = G + G' \mp \frac{FD'' - GD'}{\sqrt{EG - F^2}}.$$

Once again, one must use the upper (lower) signs consistently, and the sign ambiguity is due to the double sense of parallelism. Furthermore, note that:

The part of ds'^2 that has a constant sign and the part that has a variable sign can serve as the individual forms for a system of parallel surfaces.

The part with constant sign is obviously the *sum of squares of the corresponding linear elements on the two polar surfaces.* As for the part with a variable sign, it is easily verified that, also in elliptic space, *the geodetic torsion of a curve at a point A* (i.e., the torsion of the geodesic that is tangent at *A*) *is given by* $\frac{1}{T} + \frac{d\sigma}{ds}$ (where T is the torsion, s is the arc length, σ is the angle between the normal to the surface and the principal normal of that curve), *which is zero for the line of curvature and is also given by:*

$$\frac{(FD - ED')\,du^2 + (GD - ED'')\,du\,dv + (GD' - FD'')\,dv^2}{\sqrt{EG - F^2}\,(E\,du^2 + 2F\,du\,dv + G\,dv^2)}.$$

One then has that the geodetic torsion of an element of the curve is equal – minus a numerical factor – to the variable part of the squares of the Clifford linear elements of the surface (i.e., to the difference between the squares of two image arcs), divided by the square of the length of that element.

Now, let *u*, *v* be lines of curvature of the surface; recalling the Codazzi equations, one gets, this case:

$$e = \frac{E}{\sin^2 w_2} = E\left(1 + \frac{1}{r_2^2}\right),$$

$$f = \sqrt{EG}\;(\cot w_1 - \cot w_2) = \pm\sqrt{EG}\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

$$g = \frac{G}{\sin^2 w_1} = G\left(1+\frac{1}{r_1^2}\right),$$

where r_1, r_2 are the curvature rays of the surface.

These formulas are sufficiently important for us that it would not be wrong for us to derive them in a different way that will have the advantage of showing how much the concept of the scrolling parameters of a line conforms to the intrinsic nature of elliptic space.

§ 15. Form the u = const. and the v = const. of an orthogonal system and let (X_1, Y_1, Z_1), (X_2, Y_2, Z_2), (X_3, Y_3, Z_3) be the scrolling parameters (in a certain direction) of the tangent to v = const., the tangent to u = const., and the normal to the surface, resp., at its generic point (x_i). It is easy to write the effective expression for the surface, and if one takes the derivative with respect to u, v, while recalling the relations between the parameters of the polar line and the formula that gives second derivative of x_i and the first derivative of ξ_1, ξ_2, ξ_3, ξ_4 , then one will obtain the following set of formulas:

$$dX_1 = \left[-\frac{1}{2\sqrt{EG}}\frac{\partial E}{\partial v}X_2 + \frac{D}{\sqrt{E}}X_3\right]du + \left[\frac{1}{2\sqrt{EG}}\frac{\partial G}{\partial u}X_2 + \left(\frac{D'}{\sqrt{E}} \pm \sqrt{G}\right)X_3\right]dv,$$

$$dX_2 = \left[\frac{1}{2\sqrt{EG}}\frac{\partial E}{\partial v}X_1 + \left(\frac{D'}{\sqrt{G}} \mp \sqrt{E}\right)X_3\right]du + \left[-\frac{1}{2\sqrt{EG}}\frac{\partial G}{\partial u}X_1 + \frac{D''}{\sqrt{G}}X_3\right]dv,$$

$$dX_3 = \left[\left(\pm\sqrt{E} - \frac{D'}{\sqrt{G}}\right)X_2 - \frac{D}{\sqrt{E}}X_1\right]du + \left[\left(\mp\sqrt{G} - \frac{D'}{\sqrt{E}}\right)X_1 - \frac{D''}{\sqrt{G}}X_2\right]dv,$$

with the usual consideration regarding the sign. These formulas are perfectly analogous to the corresponding ones in Euclidian space that one deduces by replacing D' with $D' \pm \sqrt{EG}$.

The effective determination of a surface that is given the fundamental forms reduces to the integration of a system of two total differential equations that are each reducible to a Ricatti equation.

Thus, in curved space, as in Euclidian space, one will have:

$$D\,du^2 + 2\,D'du\,dv + D''dv^2 = -\sum(\sqrt{E}X_1du + \sqrt{G}X_2dv)\,dX_3\,.$$

Therefore, in order for a parameter line $\lambda X_1 + \mu X_2 + \nu X_3$ (λ, μ, ν constants) through the point (x_i) to generate a developable when one moves along a v = const. the spherical images of the generating line must becomes equal; i.e., if $D' = F = 0$ then

$$\lambda\left(\mu D+\nu\frac{\frac{\partial E}{\partial v}}{2\sqrt{G}}\right)=0,$$

and by moving along u = const. instead one will then have that:

$$\mu\left(\lambda D''+\nu\frac{\frac{\partial G}{\partial u}}{2\sqrt{E}}\right)=0.$$

One can solve another question with the aid of the preceding set of formulas: Determine the congruence in which rays that are dragged along by an arbitrary deformation of an initial surface, to which one imagines that they are invariably linked, always defines ∞^1 Clifford rulings – or, what amounts to the same thing – a congruence for which one of the Clifford images is degenerate. If the u, v are the lines normal to the rays that are traced out on the initial surface Σ and those of their orthogonal trajectories, respectively, then the scrolling parameters of a generic ray of the congruence will be:

$$X=\cos\varphi\, X_1+\sin\varphi\, X_3,\qquad Y=\cos\varphi\, Y_1+\sin\varphi\, Y_3,\qquad Z=\cos\varphi\, Z_1+\sin\varphi\, Z_3,$$

where φ is a function of u, v. The linear element of the Clifford image that is obtained in the sense in which one calculates the X, Y, Z is given by:

$$ds'^2=\left[\sum\left(\frac{\partial X}{\partial u}\right)^2\right]du^2+2\sum\frac{\partial X}{\partial u}\frac{\partial X}{\partial v}\,du\,dv+\sum\left(\frac{\partial X}{\partial v}\right)^2 dv^2;$$

now, it is easy to calculate, precisely with the set of formulas in this paragraph, that:

$$\sum\left(\frac{\partial X}{\partial u}\right)^2=\frac{D^2}{E}+\left(\frac{\partial\varphi}{\partial u}\right)^2+2\frac{D}{\sqrt{E}}\frac{\partial\varphi}{\partial u}+\left(\frac{\cos\varphi}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}+\sin\varphi\frac{D'\mp\sqrt{EG}}{\sqrt{G}}\right)^2,$$

$$\sum\left(\frac{\partial X}{\partial v}\right)^2=\left(\frac{\partial\varphi}{\partial v}+\frac{D'\mp\sqrt{EG}}{\sqrt{E}}\right)^2+\left(\frac{\cos\varphi}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}-\sin\varphi\frac{D''}{\sqrt{G}}\right)^2,$$

$$\begin{aligned}\sum\frac{\partial X}{\partial u}\frac{\partial X}{\partial v}&=\left(\frac{D}{\sqrt{E}}+\frac{\partial\varphi}{\partial u}\right)\left(\frac{D'\pm\sqrt{EG}}{\sqrt{E}}+\frac{\partial\varphi}{\partial v}\right)\\&\quad-\left(\frac{\cos\varphi}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}+\sin\varphi\frac{D'\mp\sqrt{EG}}{\sqrt{EG}}\right)\left(\frac{\cos\varphi}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}-\sin\varphi\frac{D''}{\sqrt{G}}\right).\end{aligned}$$

In order for the corresponding Clifford image to be degenerate, one must have:

$$\left[\sum\left(\frac{\partial X}{\partial u}\right)^2\right]\left[\sum\left(\frac{\partial X}{\partial v}\right)^2\right]-\left(\sum\frac{\partial X}{\partial u}\frac{\partial X}{\partial v}\right)^2 = 0;$$

i.e., one must have that (the square of):

$$\left(\frac{D}{\sqrt{E}}+\frac{\partial\varphi}{\partial u}\right)\left(\frac{\cos\varphi}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}-\sin\varphi\frac{D''}{\sqrt{G}}\right)$$
$$+\left(\frac{D'\pm\sqrt{EG}}{\sqrt{E}}+\frac{\partial\varphi}{\partial v}\right)\left(\frac{\cos\varphi}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}+\sin\varphi\frac{D'\mp\sqrt{EG}}{\sqrt{EG}}\right)$$

is zero.

If one develops this, recalling that if K is the curvature of the initial surface, then:

$$\frac{DD''}{\sqrt{EG}}-\frac{D'\pm\sqrt{EG}}{\sqrt{E}}\frac{D'\mp\sqrt{EG}}{\sqrt{G}} = \sqrt{EG}\,K,$$

and one will have:

$$-\sin\varphi\,\frac{\partial\varphi}{\partial u}\frac{D''}{\sqrt{G}}+\frac{\cos\varphi}{E}D\frac{\partial\sqrt{G}}{\partial u}+\frac{\cos\varphi}{\sqrt{E}}\frac{\partial\varphi}{\partial u}\frac{\partial\sqrt{G}}{\partial u}+\frac{\cos\varphi}{\sqrt{G}}\frac{\partial\varphi}{\partial v}\frac{\partial\sqrt{E}}{\partial v}$$

$$+\sin\varphi\,\frac{\partial\varphi}{\partial v}\frac{D'\mp\sqrt{EG}}{\sqrt{G}}+\frac{\cos\varphi}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\frac{D'\pm\sqrt{EG}}{\sqrt{E}}-\frac{\cos\varphi}{\sqrt{E}}\frac{\partial\varphi}{\partial u}\frac{\partial\sqrt{G}}{\partial u}-\sin\varphi K\sqrt{EG} = 0.$$

One multiplies this by D and replaces D, D'' with their values that one deduces from the penultimate formula; the result must be identically zero in D, D'. Therefore, one will have, in the first place:

$$\frac{\cos\varphi}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u} = 0.$$

For $\varphi = \pi / 2$, one recognizes immediately that $K = 0$, and one has the *normals to a surface of zero curvature.* For $\cos\varphi \neq 0$, one must have that G is a function of only v; since the coefficient of D' must also be zero, one will have $\partial\varphi / \partial u = 0$. Analogously, one finally obtains:

$$\frac{\partial\log\cos\varphi}{\partial v} = \frac{\partial\log\sqrt{E}}{\partial v},$$

$$\frac{\cos\varphi}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\frac{\partial\varphi}{\partial v}-\sin\varphi K\sqrt{EG}\pm\sqrt{EG}\left(\frac{\cos\varphi}{\sqrt{EG}}\frac{\partial\sqrt{E}}{\partial v}-\frac{\sin\varphi}{\sqrt{G}}\frac{\partial\varphi}{\partial v}\right) = 0.$$

If φ is constant (which we can always assume to be zero) then one can set $E = G = 1$ and obtain *the line that is inclined with a constant angle to a surface* Σ *with zero curvature and normal to the common point to* Σ *of the geodetic of a system (of* ∞^1 *parallel geodesics) that pass through that point.*

If φ is not constant then one will set $\varphi = v$, $E = \cos^2 \varphi = \cos^2 v$, and the last equation will become successively:

$$\frac{\sin v \cos v}{\sqrt{G}} + \sin v \frac{\partial}{\partial v}\left(\frac{\sin v}{\sqrt{G}}\right) \pm 2 \sin v \cos v = 0,$$

$$\frac{\partial}{\partial v}\left(\frac{\sin v}{\sqrt{G}}\right) + \cot v\left(\frac{\sin v}{\sqrt{G}}\right) \pm 2 \cos v = 0.$$

From this:

$$\frac{\sin v}{\sqrt{G}} = \pm \frac{1}{\sin v}(2 \sin v \cos v \, dv + C),$$

$$\sqrt{G} = \pm \frac{2\sin^2 v}{\cos 2v + C},$$

where C is a constant.

One then obtains a surface of rotation (or deformation) for the initial surface that becomes, in the limit, precisely one of the Weingarten surfaces that present themselves in the direct study for Euclidian space.

After this study, observe that now it is sufficient to set:

$$D' = 0$$

in the set of formulas in this paragraph and to take the sum $\sum dX_3^2$ in order to obtain the linear element of § 14 once more, as desired.

Moreover, one must observe that the term in $du\,dv$ that has the double sign – which can seem surprising on first glance when one recalls the analogous situations for Euclidian and hyperbolic spaces – *is, in turn, something that is predictable "a priori," because* u = const., v = const. *are precisely the developables of the congruences of normals to the surface* (§ 11).

§ 16. We now pose the following question:

Given the two planar images of a congruence (in bijective correspondence), *how do we know whether the congruence is normal?*

Meanwhile, the analysis that one makes when u = const., v = cont. define the lines of curvature of a surface shows that *a necessary condition is that the images must correspond with equality of the areas.* We now prove that, conversely: *If the images are* *such that two corresponding parts are equivalent then congruence will either be normal or dual to a normal congruence* (this last property is not contrary to the generality of the result, since dual congruences have the same Clifford images). (Cf., § 13). In fact, if the u = const. and the v = const. are the developables of our congruences (i.e., the lines that correspond with equality of arc length on the Clifford images and prove to be real) then the part of the linear element of the Clifford images that has variable sign will reduce to at most the term in $du\,dv$, and if the two images correspond in the manner that is assumed then the term in "$du\,dv$" will be zero, or in the part with constant sign, or the one in which the sign is variable. In this last case, the congruence will have an indeterminate developable – i.e., one that is formed from the lines that are normal to a plane; in the other case, $F + F' = 0$. Now, let:

$$ds^2 = A\,du^2 + 2\,B\,du\,dv + C\,dv^2, \qquad D\,du^2 + 2\,D'du\,dv + D''dv^2$$

be the fundamental forms of one of the focal sheets (*falde focali*) of the focal point (x_1, x_2, x_3, x_4) of the congruence, and let $\xi_i = \frac{1}{\sqrt{A}}\frac{\partial x_i}{\partial u}$ be the direction cosines of the ray through x_i . One will have $E = F = 0$, and in order to have $F + F' = 0$, one must have:

$$F' = \sum \frac{\partial \xi}{\partial u}\frac{\partial \xi}{\partial v} - \left(\sum x\frac{\partial \xi}{\partial u}\right)\left(\sum x\frac{\partial \xi}{\partial v}\right)$$

$$= \sum \frac{\partial}{\partial u}\left[\frac{1}{\sqrt{A}}\frac{\partial x}{\partial u}\right]\frac{\partial}{\partial v}\left[\frac{1}{\sqrt{A}}\frac{\partial x}{\partial u}\right] - \left(\sum \xi\frac{\partial x}{\partial u}\right)\left(\sum \xi\frac{\partial x}{\partial v}\right) = 0.$$

Now:

$$\left(\sum \xi\frac{\partial x}{\partial u}\right) = \sqrt{A}\,; \qquad \left(\sum \xi\frac{\partial x}{\partial v}\right) = \frac{B}{\sqrt{A}}\,; \qquad \left(\sum \xi\frac{\partial x}{\partial u}\right)\left(\sum \xi\frac{\partial x}{\partial v}\right) = B,$$

so one will therefore have:

$$\frac{1}{4A^2}\frac{\partial A}{\partial u}\frac{\partial A}{\partial v} + \frac{1}{A}\sum \frac{\partial^2 x}{\partial u^2}\frac{\partial^2 x}{\partial u\,\partial v} - B = 0.$$

Recalling that $D' = 0$, one will have, given that (X_1, X_2, X_3, X_4) are the direction cosines of the normal to the surface, that:

$$\frac{\partial^2 x}{\partial u^2} = \begin{Bmatrix}11\\1\end{Bmatrix}\frac{\partial x}{\partial u} + \begin{Bmatrix}11\\2\end{Bmatrix}\frac{\partial x}{\partial v} - Ax + DX,$$

where the Christoffel symbols are referred to the form:

$$A\,du^2 + 2\,B\,du\,dv + C\,dv^2.$$

One gets:

$$\sum \frac{\partial^2 x}{\partial u^2}\frac{\partial^2 x}{\partial u\,\partial v} = \frac{1}{2}\begin{Bmatrix}11\\1\end{Bmatrix}\frac{\partial A}{\partial v}+\begin{Bmatrix}11\\2\end{Bmatrix}\frac{\partial C}{\partial u} + A\,B,$$

so one finally has (since $A \neq 0$):

$$-\frac{\partial A}{\partial u}\frac{\partial A}{\partial v}+2A^2\left[\begin{Bmatrix}11\\1\end{Bmatrix}\frac{\partial A}{\partial v}+\begin{Bmatrix}11\\2\end{Bmatrix}\right]\frac{\partial C}{\partial u} = 0,$$

and since:

$$A\,C - B^2 \neq 0, \qquad A \neq 0,$$

one will have:

$$\begin{Bmatrix}12\\2\end{Bmatrix}\begin{Bmatrix}11\\2\end{Bmatrix} = 0.$$

If $\begin{Bmatrix}11\\2\end{Bmatrix} = 0$ then the v = const. will be geodetic and the congruence will be normal; if $\begin{Bmatrix}12\\2\end{Bmatrix} = 0$ then one will have $\begin{Bmatrix}12\\1\end{Bmatrix} = 0$ for the dual surface, and the dual congruence will be normal.

It is therefore enough to see that, as is permitted, the lines of equal length under such a correspondence of a Euclidian sphere with itself are real (in which case, one certainly has $A \neq 0$, as one supposes); in fact, if the two linear elements are referred to the common *real* orthogonal system then it will assume the form $\bar{E}\,du^2+\bar{G}\,dv$, $\bar{E}'\,du^2+\bar{G}'\,dv$, and the lines in question will be given by:

$$(\bar{E}-\bar{E}')\,du^2+(\bar{G}-\bar{G}')\,dv^2 = 0.$$

Since $\bar{E}\,\bar{G} = \bar{E}'\,\bar{G}'$, the differences $\bar{E}-\bar{E}'$, $\bar{G}-\bar{G}'$ cannot have the same sign, and one will have that $\bar{E}$, $\bar{G}$, $\bar{E}'$, $\bar{G}'$ are positive; therefore, these lines will certainly be real. (This proof of the reality of these lines was cordially communicated to me by Prof. Bianchi.)

§ 17. We have given the general conditions that the Fibbi form must satisfy in order for it to correspond to a real congruence. Therefore, it would not be inopportune to verify them for the congruence of normals to a surface, at least, when one chooses u = const. and v = const. to be the lines of curvature. In fact, we express the ideas that the complex of terms appear in the expression for the curvature of:

$$e\,du^2 + 2\,f\,du\,dv + g\,dv^2,$$

and contain "*f*" linearly, and its derivative is zero; whatever must happen in our case, because under an exchange of sign of *f*, the element will again remain a spherical element. We get:

$$\frac{\partial}{\partial u}\left[\frac{\frac{1}{r_1}-\frac{1}{r_2}}{1+\frac{1}{r_1 r_2}}\right]\frac{\partial \log\left[E\left(1+\frac{1}{r_2^2}\right)\right]}{\partial v}-\frac{\partial}{\partial v}\left[\frac{\frac{1}{r_1}-\frac{1}{r_2}}{1+\frac{1}{r_1 r_2}}\right]\frac{\partial \log\left[E\left(1+\frac{1}{r_2^2}\right)\right]}{\partial u}$$

$$+\ 2\frac{\partial}{\partial v}\left[\frac{\frac{\partial}{\partial u}\left[\sqrt{EG}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]}{\sqrt{EG}\left(1+\frac{1}{r_1 r_2}\right)}\right] = 0.$$

The third term in this sum is:

$$\frac{\frac{1}{r_1}-\frac{1}{r_2}}{1+\frac{1}{r_1 r_2}}\left\{\frac{\partial^2 \log E}{\partial u\,\partial v}+2\frac{\partial^2 \log\left[\sqrt{G}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]}{\partial u\,\partial v}\right\}$$

$$+\ \frac{\partial}{\partial v}\left[\frac{\frac{1}{r_1}-\frac{1}{r_2}}{1+\frac{1}{r_1 r_2}}\right]\left\{\frac{\partial^2 \log E}{\partial u}+2\frac{\partial^2 \log\left[\sqrt{G}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right]}{\partial u}\right\}.$$

Combining the terms that contain *E* and the ones that contain *G*, the preceding equality then becomes:

$$\frac{\partial}{\partial u}\left[\frac{\frac{1}{r_1}-\frac{1}{r_2}}{1+\frac{1}{r_1 r_2}}\frac{\partial \log E}{\partial v}\right]+2\frac{\partial}{\partial v}\left[\frac{\frac{1}{r_1}-\frac{1}{r_2}}{1+\frac{1}{r_1 r_2}}\frac{\partial \log\left\{\sqrt{G}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right\}}{\partial u}\right]$$

$$+\ \frac{\partial}{\partial u}\left(\frac{\frac{1}{r_1}-\frac{1}{r_2}}{1+\frac{1}{r_1 r_2}}\right)\frac{\partial \log\left(1+\frac{1}{r_2^2}\right)}{\partial v}-2\frac{\partial}{\partial v}\left(\frac{\frac{1}{r_1}-\frac{1}{r_2}}{1+\frac{1}{r_1 r_2}}\right)\frac{\partial \log\left(1+\frac{1}{r_2^2}\right)}{\partial u} = 0.$$

If we replace $\frac{\partial \log E}{\partial v}$, $\frac{\partial \log G}{\partial u}$ with the values that are given by the Codazzi formula then we will obtain an identity in $\frac{1}{r_1}$, $\frac{1}{r_2}$.

The terms that were removed thus prove to be zero, so in order to see that the curvature of the element is + 1, it is sufficient to see that:

$$1 = \frac{-1}{2\sqrt{EG}\left(1+\frac{1}{r_1 r_2}\right)}\left\{\frac{\partial}{\partial u}\left[\frac{1}{\sqrt{EG}\left(1+\frac{1}{r_1 r_2}\right)}\frac{\partial\left[G\left(1+\frac{1}{r_1^2}\right)\right]}{\partial u}\right]\right.$$

$$\left.+\frac{\partial}{\partial v}\left[\frac{1}{\sqrt{EG}\left(1+\frac{1}{r_1 r_2}\right)}\frac{\partial\left[E\left(1+\frac{1}{r_1^2}\right)\right]}{\partial v}\right]\right\}.$$

If we replace $\frac{\partial}{\partial u}\left(\frac{1}{r_1}\right)$, $\frac{\partial}{\partial v}\left(\frac{1}{r_2}\right)$ with the values that they get from the Codazzi equations then the preceding equation will become the Gauss equation:

$$1 + \frac{1}{r_1 r_2} = -\frac{1}{\sqrt{EG}}\left\{\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{G}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\frac{\partial\sqrt{E}}{\partial v}\right)\right\}.$$

Here, we have two equations, instead of three (viz., two Codazzi and one Gauss), due to the fact that it is implicit that $f + f' = 0$, which expresses the idea that corresponding parts of the two images are equivalent.

§ 18. The angle that two corresponding elements (to two polar surfaces) AA', BB' form between themselves is measured immediately when one thinks (§ 2) that the direction BB'' that is conjugate to BB' at B is precisely the line that is dual to AA', and is therefore parallel to AA' in the two senses. Therefore, the angles of AA' with BB' are *equal or supplementary* to those of BB' with BB''.

If AA' is tangent to a line of curvature for A then it will be normal to BB' in both senses (and will therefore meet BB'). We would now like to study what happens for the angle φ between the corresponding elements on a surface and on the planar image (constructed in a certain sense). Take the lines u, v to be the lines of curvature, and measure φ in the same sense by which the planar image was constructed. For the singularity of the result, one performs the calculations in two ways, one of which will be given in the rational form cos φ, while the other in the form sin φ. Let (Y_1, Y_2, Y_3, 0) be the image point of the point (x_i) of the surface relative to the plane $x_4 = 0$, and let (X_i) be the direction cosines of the parallel through (Y) to the element that emanates from (x) with the direction cosine $\left(\frac{dx}{ds}\right)$; if we denote the corresponding arc length of the planar image by ds then:

$$\cos\varphi = \frac{\sum X\,dY}{d\sigma} = \frac{1}{ds\,d\sigma}\begin{vmatrix} dY_1 & Y_1 & [dx,x]_1 \\ dY_2 & Y_2 & [dx,x]_2 \\ dY_3 & Y_3 & [dx,x]_3 \end{vmatrix}$$

$$\cos^2\varphi = \frac{1}{ds^2\,d\sigma^2}\begin{vmatrix} d\sigma^2 & 0 & \sum[dx,x]\,d[\xi,x] \\ 0 & 1 & \sum[dx,x]\,d[\xi,x] \\ \sum[dx,x]\,d[\xi,x] & \sum[dx,x]\,d[\xi,x] & ds^2 \end{vmatrix},$$

and, recalling the usual identity:

$$\cos\varphi = 1 - \left(\frac{\sum(dx,x)\,d(\xi,x)}{ds\,d\sigma}\right)^2 = 1 - \left(\frac{\sum[dx,x][x,d\xi]}{d\sigma\,ds}\right)^2 = 1 - \left(\frac{\sum dx\,d\xi}{ds\,d\sigma}\right)^2.$$

which makes:

$$\sin\varphi = \pm\frac{\sum dx\,d\xi}{ds\,d\sigma}.$$

We now avail ourselves of the notations and formulas in the set of § 15 in order to find a rational form for cos φ.

The third formula of this set gives:

$$\frac{dX_3}{d\sigma} = \left(\pm\sqrt{E}\,X_2 - \frac{D}{\sqrt{E}}\,X_1\right)\frac{du}{d\sigma} + \left(\mp\sqrt{G}\,X_1 - \frac{D''}{\sqrt{G}}\,X_2\right)\frac{dv}{d\sigma}.$$

The scrolling parameters of the line that emanates from the point (u, v) of the surface to the point ($u + du$, $v + dv$) are:

$$\sqrt{E}\,\frac{du}{ds}\,X_1 + \sqrt{G}\,\frac{dv}{ds}\,X_2.$$

One has, with the one notations:

$$\cos\varphi = \frac{1}{ds\,d\sigma}\begin{vmatrix} dX_3 & X_3 & \sqrt{E}\,du\,X_1 + \sqrt{G}\,dv\,X_2 \\ dY_3 & Y_3 & \sqrt{E}\,du\,Y_1 + \sqrt{G}\,dv\,Y_2 \\ dZ_3 & Z_3 & \sqrt{E}\,du\,Z_1 + \sqrt{G}\,dv\,Z_2 \end{vmatrix}.$$

One splits the right-hand side into two determinants and replaces dX_3, dY_3, dZ_3 with their values; one gets:

$$\cos\varphi = \frac{1}{ds\,d\sigma}\left\{\sqrt{E}\,du\left(\pm\sqrt{E}\,du - \frac{D''}{\sqrt{G}}\,dv\right)\begin{vmatrix} X_2 & X_3 & X_1 \\ Y_2 & Y_3 & Z_1 \\ Z_2 & Z_3 & Z_1 \end{vmatrix}\right.$$

$$\left. + \sqrt{G}\,dv\left(-\frac{D}{\sqrt{E}}\,du \mp \sqrt{G}\,dv\right)\begin{vmatrix} X_1 & X_2 & X_3 \\ Y_1 & Y_2 & Y_2 \\ Z_1 & Z_2 & Z_3 \end{vmatrix}\right\}.$$

One of these two determinants is equal to + 1, while the other one is equal to – 1, so:

$$\cos\varphi = \pm\frac{1}{ds\,d\sigma}\left(E\,du^2 \pm \sqrt{EG}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)du\,dv + G\,dv^2\right).$$

From the formulas that give $\cos\varphi$, $\sin\varphi$ (which are immediately seen to be equivalent), one has:

The asymptotes are characterized by the fact that the tangent at a point is parallel to the tangent at the corresponding point on the image curve.

The lines of curvatures are characterized by the fact that they result from the displacement of equal angles in two planar images.

One will then have that $\cos\varphi = 1$ for the asymptotes, so one sees that this property is quite different from the analogue for Euclidian space.

§ 19. Note that:

$$e\,g - f^2 = E\,G\left(1 + \frac{1}{r_1\,r_2}\right)^2.$$

Therefore:

The ratio of the areas of two corresponding infinitesimal elements in the planar image and the surface (around the corresponding points A′, A, resp.) is equal to the *curvature of the quadratic form that the linear element of the surface is endowed with at the point A; i.e., to the absolute curvature of the surface.*

As a result, the relative curvature is given by the ratio of the infinitesimal elements of two dual surfaces.

As one sees, we have a fact that is analogous to the one that is presented for the torsion of the curve; that, combined with the facts that were enumerated in these paragraphs, will give rise to the observation that the property of the parallels in Euclidian space seems, in many cases, to split into two classes, one of which has the property that it is preserved in hyperbolic space, and the other, the property that it is it preserved in elliptic space. One then observes:

The angle between the image lines of the lines of curvature is given by the complement of $\pm (w_1 - w_2)$, *depending on the sense of the parallelism.*

We again explicitly point out a result that was mentioned elsewhere:

The planar image is degenerate for the surfaces with zero curvature, and for them alone. One could predict the first part of this theorem by noting that the asymptotes of such a surface have torsion ± 1.

Thus, since the mean of the squares of the torsions of the asymptotes at A is equal to the curvature relative to the surface at A, *the mean of the squares of the Clifford torsions is equal to the absolute curvature.*

The orthogonal systems of the surface that are preserved in the planar image are given by:

$$\begin{vmatrix} E\,du & G\,dv \\ \dfrac{E}{\sin^2 w_2}du \pm \sqrt{EG}\left(\dfrac{1}{r_1}-\dfrac{1}{r_2}\right)dv & \dfrac{G}{\sin^2 w_1}dv \pm \sqrt{EG}\left(\dfrac{1}{r_1}-\dfrac{1}{r_2}\right)du \end{vmatrix} = 0;$$

i.e., by:

$$\pm \sqrt{EG}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)(E\,du^2 - G\,dv^2) + E\,G\,du\,dv\left(\frac{1}{\sin^2 w_1}-\frac{1}{\sin^2 w_2}\right) = 0.$$

One has $w_1 + w_2 = 0$ for the minimal surface, so:

For the surfaces with minimal area, the asymptotes form an orthogonal system on the surface such that they are preserved in the planar images.

§ 20. Now, following the advice of Prof. Bianchi, we shall apply the preceding result to the W surface.

Let:

$$ds^2 = E\,du^2 + G\,dv^2$$

be the linear element of such a surface, when u, v are the lines of curvature; since the Codazzi formulas are identical for our space and elliptic space, the Weingarten formula will also apply here. If we thus set:

$$E = \frac{1}{\beta^2}, \qquad G = \frac{1}{\theta'^2(\beta)},$$

with

$$\frac{1}{r_2} = \theta(\beta), \qquad \frac{1}{r_1} = \theta(\beta) - \beta\,\theta'(\beta),$$

then we will get the linear element of the planar image:

$$e = \frac{1+\theta^2(\beta)}{\beta^2}, \qquad f = \pm 1, \qquad g = \frac{1+[\theta(\beta)-\beta\,\theta'(\beta)]^2}{\theta'^2(\beta)};$$

i.e.:

f = const. *for a W surface, and "e" is a function of "g";* the determination of the *W* surface is thus reduced to the search for all such linear elements of the elliptic plane or *Euclidian sphere.*

Conversely, if one satisfies this condition then one can write down the preceding formulas, and then, by Weingarten's observation, the Codazzi equations will be satisfied, and by a previous calculation, the Gauss equation will be satisfied.

However, we would like to examine this result more precisely, and if we observe the remarkable fact that of the two conditions "f = const." and "e is a function of g," one of them is a consequence of the other one when one already knows that:

$$e\,du^2 + 2\,f\,du\,dv + g\,dv^2$$

is the linear element of one of the images of a surface, when referred to the lines of curvature. In fact, since u, v are the image lines of the lines of curvature, the spherical element:

$$e\,du^2 + 2\,f\,du\,dv + g\,dv^2$$

must continue to have curvature + 1 when one changes the sign of f. Recalling that f is constant, and subtracting one of the equations that express the idea that the curvature of the form:

$$e\,du^2 \pm 2\,f\,du\,dv + g\,dv^2$$

is equal to + 1 from the other one, one will get (setting $f = 1$, for simplicity):

$$\frac{\partial}{\partial u}\left(\frac{1}{e\sqrt{eg-1}}\frac{\partial e}{\partial v}\right) - \frac{\partial}{\partial v}\left(\frac{1}{e\sqrt{eg-1}}\frac{\partial e}{\partial u}\right) = 0;$$

i.e.:

$$\frac{\partial(e\sqrt{eg-1})}{\partial u}\frac{\partial e}{\partial v}-\frac{\partial(e\sqrt{eg-1})}{\partial v}\frac{\partial e}{\partial u} = 0,$$

so

$$\frac{\partial(eg)}{\partial u}\frac{\partial e}{\partial v}-\frac{\partial(eg)}{\partial v}\frac{\partial e}{\partial u} = 0,$$

namely:

$$\begin{vmatrix} \dfrac{\partial e}{\partial u} & \dfrac{\partial g}{\partial u} \\ \dfrac{\partial e}{\partial v} & \dfrac{\partial g}{\partial v} \end{vmatrix} = 0.$$

Therefore, our theorem can be stated in the following form:

The search for the W surfaces in elliptic space reduces to the search for the linear elements of the Euclidian sphere for which f is constant and the curvature remains equal to + 1 *when one alters the sign of f.*

This explains the origin of the condition in Weingarten's theorems in Euclidian space that *"e is a function of g."*

Moreover, we see that:

A necessary and sufficient condition for a surface to be W is that it one can make:

$$\sqrt{EG}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) = \text{const.} \neq 0.$$

If f is a non-zero constant and e, g are functions of u or v that do not change value when one switches "u" with "– u" or "v" with "– v" then the linear element will correspond to the planar image of a W surface.

In fact, if one switches u with $-u$ (or v with $-v$) and e and g do not change in value then that will show that f changes only in sign, and one thus finds oneself in the presence of two forms with curvature + 1 that differ by only the constant sign of f.

We must now resolve a question that was already posed on other occasions, namely, that of constructing a surface for which one is given the planar images by quadrature; in fact, the process is different from the one that one follows in Euclidian space, but also much simpler.

Let $(Y_1, Y_2, Y_3, 0)$ and $(Z_1, Z_2, Z_3, 0)$ be two corresponding points of the two planar images in the plane $x_4 = 0$. If one chooses the plane $x_4 = 0$ to be the initial surface of the corresponding normal congruence then one will have (§ 4) for a generic ray of that congruence:

$$x_1 = \frac{Y_1 + Z_1}{\sqrt{2(1+\sum YZ)}}, \quad x_2 = \frac{Y_2 + Z_2}{\sqrt{2(1+\sum YZ)}}, \quad x_3 = \frac{Y_3 + Z_3}{\sqrt{2(1+\sum YZ)}}, \quad x_4 = 0,$$

$$\xi_1 = \frac{Y_3 Z_2 - Y_2 Z_3}{\sqrt{2(1+\sum YZ)}}, \quad \xi_2 = \frac{Y_1 Z_3 - Y_3 Z_1}{\sqrt{2(1+\sum YZ)}}, \quad \xi_3 = \frac{Y_2 Z_1 - Y_1 Z_2}{\sqrt{2(1+\sum YZ)}}, \quad \xi_4 = -\sqrt{\frac{1+\sum YZ}{2}}.$$

In order for this congruence to be normal, one can set:

$$dw = -\sum \xi_i \, dx_i,$$

$$w = \int \frac{\begin{vmatrix} d(Y_1+Z_2) & d(Y_2+Z_2) & d(Y_3+Z_3) \\ Y_1 & Y_2 & Y_3 \\ Z_1 & Z_2 & Z_3 \end{vmatrix}}{2\left(1+\sum YZ\right)},$$

and for the generic point (x) of one of the ∞^1 corresponding surfaces, one will then have:

$$X_i = x_i \cos w + \xi_i \sin w,$$

where an arbitrary additive constant enters into w.

We would now like to interpret this fact in a Euclidian metric. Take a point (x_i) in elliptic space, and set $x = x_1 / x_4$, $y = x_2 / x_4$, $z = x_3 / x_4$, where the x, y, z form a tri-rectangular trihedron. A point for which $x_4 = 0$ will represent a point of the plane at infinity, and the values of x_1, x_2, x_3 will give the corresponding direction cosines. Observe that the metric on the plane at infinity relates to the conic $x^2 + y^2 + z^2 = 0$, so it will coincide with the analogue of the plane $x_4 = 0$ in curved space, which is referred to the conic $x_1^2 + x_2^2 + x_3^2 = 0$. Thus, the pair of elements $e\,du^2 \pm 2\,f\,du\,dv + g\,dv^2$ of the plane $x_4 = 0$ corresponds to the same pair of linear elements for the sphere in flat space, if, for the moment, we make a point A in the plane $x_4 = 0$ correspond to that point of the sphere that determines the direction that corresponds to A under the projectivity above. The absolute is then changed into the imaginary sphere:

$$x^2 + y^2 + z^2 + 1 = 0,$$

so we finally have:

Given a pair of spherical elements $e\,du^2 \pm 2\,f\,du\,dv + g\,dv^2$ *with f constant, the lines that are parallel to the ray that is determined by the midpoint of one of the arcs that are terminated by a pair of corresponding points A, A′ and pass through a point B that is placed upon the diameter that is normal to that arc and at a distance of* $\tan \varphi / 2$ *(where* φ *is the distance between the points A, A′) from the center of the sphere (where* φ *is the distance between the points A, A′) will generate a W congruence in which focal planes are anti-polar with respect to that sphere.*

§ 21. Returning to curved space, we will give some simple examples of these theorems that will also lead to interesting consequences.

The problem of determining the developables in elliptic space is equivalent to that of determining those of our spherical elements for which $\theta(\beta) = \beta$, namely, $f = g = 1$; i.e., the elements $e\,du^2 + 2\,du\,dv + dv^2$. Set $u' = u$, $v' = u + v$. This element will become the other one:

$$(e - 1)\,du'^2 + dv'^2,$$

namely, the one that relates to the surface canals (*canali*) in flat space that, as one will deduce from the Weingarten formulas, or those of Codazzi, have constant e or g, provided that one chooses the parameter of the corresponding line of curvature suitably. What is known of the rest is that all developables in elliptic space are known.

Another, much more interesting, case is the one for which $w_1 - w_2$ = const., since the evolutes of such a surface will be complementary pseudo-spherical surfaces. Since the angle of the spherical images of the lines of curvature must (§ 19) be constant, the problem of determining such surfaces is identical with the problem of determining the spherical elements of the form:

$$e\,d\alpha^2 + 2\,d\alpha\,d\beta + d\beta^2$$

that have constant K, i.e., one of the form:

$$e^{-2\tau}\,d\alpha^2 + 2\cos\sigma\,d\alpha\,d\beta + e^{2\tau}\,d\beta^2,$$

where σ is constant [complement of $\pm(w_1 - w_2)$].

The geodetic torsion of an element of such a surface is proportional to $\dfrac{d\alpha\,d\beta}{ds^2}$; one observes, moreover, that our result can be stated:

In order to find all systems of lines that divides the sphere into ∞^2 equivalent infinitesimal parallelograms, it is sufficient to find the Clifford images of the more general pseudo-spherical, normal congruence in curved space.

By comparing the results that were obtained for these spherical linear elements with the ones that were obtained by Prof. Bianchi in his article in t. XVIII of *Annali di matematica* (1890), one will obtain some consequences that seem noteworthy to me.

Prof. Bianchi proved that any spherical element:

$$(\alpha)\qquad ds^2 = H_1^2 du^2 + H_2^2 dv^2$$

with

$$(\beta)\qquad \tan\frac{\sigma}{2}\frac{\partial}{\partial v}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial v}\right) = \cot\frac{\sigma}{2}\frac{\partial}{\partial u}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial v}\right),$$

where σ is constant, is the linear element of the spherical image of a Euclidian, pseudo-spherical congruence that is referred to the lines that correspond to the asymptotes of the focal sheet, and that if one sets:

$$\tau = \int\left(\tan\frac{\sigma}{2}\frac{1}{H_2}\frac{\partial H_1}{\partial v}du + \cot\frac{\sigma}{2}\frac{1}{H_1}\frac{\partial H_2}{\partial u}dv\right)$$

then one can set:

$$(\gamma) \qquad \begin{cases} e^{\tau}\left(H_1\sin\dfrac{\sigma}{2}du + H_2\cos\dfrac{\sigma}{2}dv\right) = \sin\sigma\, d\alpha, \\ e^{-\tau}\left(H_1\sin\dfrac{\sigma}{2}du - H_2\cos\dfrac{\sigma}{2}dv\right) = \sin\sigma\, d\beta, \end{cases}$$

namely:

$$(\delta) \qquad \begin{cases} H_1 du = \cos\dfrac{\sigma}{2}(e^{-\tau}d\alpha + e^{\tau}d\beta), \\ H_2 dv = \sin\dfrac{\sigma}{2}(e^{-\tau}d\alpha - e^{\tau}d\beta), \end{cases}$$

so the element (a) can also be written:

$$(\varepsilon) \qquad ds^2 = e^{-2\tau}d\alpha^2 + 2\cos\sigma\, d\alpha\, d\beta + e^{2\tau}d\beta^2,$$

where α = const., β = const. are the orthogonal trajectories of the planar images of the developables of the congruence, and conversely, any element (ε) can be put into the form (α) when (β) is true. Moreover, if 2θ, 2ω are the angles between the asymptotes of the two focal sheets of the pseudo-spherical congruence above then one will have that:

$$(\zeta) \qquad H_1 = \frac{\cos(\theta+\omega)}{\cos\dfrac{\sigma}{2}}, \qquad H_2 = \frac{\cos(\theta-\omega)}{\sin\dfrac{\sigma}{2}},$$

where:

$$(\eta) \qquad \frac{\partial(\theta-\omega)}{\partial u} = \tan\frac{\sigma}{2}\cos(\theta+\omega), \qquad \frac{\partial(\theta+\omega)}{\partial v} = -\cot\frac{\sigma}{2}\cos(\theta-\omega).$$

We now have to make the observation that when the coefficient of $du\,dv$ is constant, and the coefficient of du^2 is a function of the coefficient of dv^2, the linear element will remain spherical when one changes the sign of f.

One then poses the following question:

What geometric relation exists between the two W congruences that are determined by the method of Prof. Bianchi by starting with the element (ε) *and the element:*

$$(\varepsilon') \qquad e^{-2\tau}d\alpha^2 - 2\cos\sigma\, d\alpha\, d\beta + e^{2\tau}d\beta^2$$

that can be deduced by changing the sign of $\cos\sigma$*?*

The element (ε') is deduced from (ε) by changing σ into $\pi - \sigma$. Thus, the element (α') that is deduced from (ε') in the same way that (α) is deduced from (ε) will be:

$$(\alpha') \qquad ds^2 = E\, du'^2 + G\, dv'^2,$$

and this will give rise to the relations:

$$(\delta') \qquad \begin{cases} \sqrt{E}\, du' = \sin\dfrac{\sigma}{2}(e^{-\tau}d\alpha + e^{\tau}d\beta), \\ \sqrt{G}\, dv' = \cos\dfrac{\sigma}{2}(e^{-\tau}d\alpha - e^{\tau}d\beta), \end{cases}$$

$$(\zeta') \qquad \sqrt{E} = \frac{\cos(\theta_1+\omega_1)}{\sin\dfrac{\sigma}{2}}, \qquad \sqrt{G} = \frac{\cos(\theta_1-\omega_1)}{\cos\dfrac{\sigma}{2}},$$

where the angles θ_1, ω_1 satisfy:

$$(\eta') \qquad \frac{\partial(\theta_1+\omega_1)}{\partial v'} = -\tan\frac{\sigma}{2}\cos(\theta_1-\omega_1), \qquad \frac{\partial(\theta_1-\omega_1)}{\partial u'} = \cot\frac{\sigma}{2}\cos(\theta_1+\omega_1).$$

When (δ') is compared to (δ), that will say that u' is a function of only u and v' is a function of only v; if one compares the values of:

$$e^{-\tau}d\alpha \pm e^{\tau}d\beta$$

that are obtained from (δ), (δ'), and recalls that (ζ), (ζ') then one will get:

$$(\theta) \qquad \begin{cases} \cot\dfrac{\sigma}{2}\cos(\theta_1+\omega_1)\,du' = \tan\dfrac{\sigma}{2}\cos(\theta+\omega)\,du, \\ \tan\dfrac{\sigma}{2}\cos(\theta_1-\omega_1)\,dv' = \cot\dfrac{\sigma}{2}\cos(\theta-\omega)\,dv. \end{cases}$$

Use the given values of (θ) in the right-hand sides of (η'), and therefore replace $\tan\dfrac{\sigma}{2}\cos(\theta+\omega)$ and $\cot\dfrac{\sigma}{2}\cos(\theta-\omega)$ with the values that they get from (η):

$$(\kappa) \qquad \begin{cases} \dfrac{\partial(\theta_1+\omega_1)}{\partial v'} = \dfrac{\partial(\theta+\omega)}{\partial v}\dfrac{dv}{dv'}, \\ \dfrac{\partial(\theta_1-\omega_1)}{\partial u'} = \dfrac{\partial(\theta-\omega)}{\partial u}\dfrac{du}{du'}. \end{cases}$$

It is therefore natural to set:

$$\theta_1(u', v') = \theta(u, v)\,; \quad \omega_1(u', v') = \omega(u, v)$$

which gives:

$$(\lambda) \qquad u' = u \tan^2 \frac{\sigma}{2}, \qquad v' = v \cot^2 \frac{\sigma}{2},$$

for the (θ), so:

$$(\mu) \qquad \begin{cases} \theta_1(u', v') = \theta(u, v) = \theta\left(u' \cot^2 \dfrac{\sigma}{2}, v' \tan^2 \dfrac{\sigma}{2}\right), \\ \omega_1(u', v') = \omega(u, v) = \omega\left(u' \cot^2 \dfrac{\sigma}{2}, v' \tan^2 \dfrac{\sigma}{2}\right). \end{cases}$$

As for the rest of them, one verifies immediately that these values of θ_1 and ω_1 will satisfy (η'), and therefore, by Prof. Bianchi's theorem and from (ζ'), one will get the spherical element (α') that one deduces from (ε') in the same way as (ε) is deduced from (α).

(μ) gives precisely the following theorem:

The Clifford images of a pseudo-spherical, normal congruence in curved space, referred to the developables, admit linear elements that are the linear elements of the spherical images of two pseudo-spherical congruences in plane space, referred to the orthogonal trajectories of the developables (which therefore correspond on the congruence); the focal sheets of one of the two congruences are Lie transforms of the focal sheets of the other one, and the Lie transformation by which one passes from one to the other is determined immediately once one is given one of the two congruences.

The geometry of elliptic space then gives a geometric interpretation of an arbitrary Lie transformation that is applied to a pseudo-spherical surface when it is imagined to be the focal sheet of a suitable pseudo-spherical congruence. Moreover, the most general Bäcklund transformation for flat space is thus obtained from the only complementary transformation in elliptic space, while the Lie transformation comes about by the fact of the double sense of parallelism. Moreover, in a more correct language, we see a doubling of the Bäcklund transformation into a complementary transformation and a Lie transformation.

We also note that for a pseudo-spherical congruence in flat space, one will have a spherical linear element:

$$e^{-2\tau}\,du^2 + 2\cos\sigma\,du\,dv + e^{2\tau}\,dv^2.$$

The only solution to a Riccati equation will suffice to determine the associated element:

$$e^{-2\tau}\,du^2 - 2\cos\sigma\,du\,dv + e^{2\tau}\,dv^2,$$

and therefore the most general pseudo-spherical, normal congruence in curved space. Thus:

Given a pseudo-spherical congruence in flat space, with the single solution of a Ricatti equation, one gets two complementary pseudo-spherical surfaces in curved space and another pseudo-spherical congruence in flat space.

Conversely, suppose that we are given a surface *S* in elliptic space for which $w_2 - w_1$ = const. – i.e., a pseudo-spherical, normal congruence – and that we know the bisecting lines of the planar images of the developables. We will then have an orthogonal system of lines such that the double system of its isogonal trajectories, under a certain angle, divides the sphere into equivalent infinitesimal parallelograms. By quadrature, one will obtain (Bianchi, *loc. cit.*, § 33) a cyclic orthogonal system on the sphere such that the axes of its circles form a Ribacour congruence with a pseudo-spherical generator; one then deduces a pseudo-spherical generator and one of its infinitely small deformations, and therefore a pseudo-spherical congruence.

Therefore, given a normal, pseudo-spherical congruence in curved space – i.e., a surface in curved space – for which one has:

$$w_1 - w_2 = \text{const.},$$

one will deduce two pseudo-spherical congruences in Euclidian space, and therefore a tetrad of pseudo-spherical surfaces in that space, as long as one knows the bisecting lines of the planar images of the developables.

§ 22. The fourth formula in § 15 ultimately gives another consequence.

Let:

$$ds^2 = H_1^2 d\rho_1^2 + H_2^2 d\rho_2^2 + H_3^2 d\rho_3^2$$

be the linear element of curved space, referred to a triply-orthogonal system, and let (X_1, X_2, X_3), (Y_1, Y_2, Y_3), (Z_1, Z_2, Z_3) denote the scrolling parameters of the normals to ρ_1 = const., ρ_2 = const., ρ_3 = const,., respectively. We immediately get for the four formulas above:

$$\left.\begin{aligned}
\frac{\partial X_k}{\partial \rho_k} &= -\frac{1}{H_i}\frac{\partial H_k}{\partial \rho_k} X_i - \frac{1}{H_k}\frac{\partial H_k}{\partial \rho_l} X_l, \\
\frac{\partial X_k}{\partial \rho_i} &= \frac{1}{H_k}\frac{\partial H_i}{\partial \rho_k} X_i \pm H_i X_k, \\
\frac{\partial X_k}{\partial \rho_l} &= \frac{1}{H_k}\frac{\partial H_l}{\partial \rho_k} X_l \mp H_l X_l
\end{aligned}\right\} \qquad (i \neq k \neq l).$$

In these formulas, *which are deduced immediately from the relations that couple* H_1, H_2, H_3, the double sign is attributed to the double sense of parallelism, and in order to fix that, one will then recall that one takes the upper or lower sign according to whether (*i k l*) is an even or odd permutation (odd or even), respectively.

It then results that $H_i^2 = \left(\sum X_i \frac{\partial X_k}{\partial \rho_i}\right)^2$, etc., so one will have:

Two triply-orthogonal systems that correspond point-by-point with parallelism in one sense of the fundamental trihedron will be equal to each other.

On the Riemannian representation of parallel lines and on isocyclic surfaces.

§ 23. The formulas that give the transformation from Riemann coordinates to Weierstrass coordinates are the following ones:

$$x_1 = \frac{\frac{1}{4}-(x^2+y^2+z^2)}{\frac{1}{4}+(x^2+y^2+z^2)}\,;\; x_2 = \frac{x}{\frac{1}{4}+x^2+y^2+z^2}\,;\; x_3 = \frac{y}{\frac{1}{4}+x^2+y^2+z^2}\,;\; x_1 = \frac{z}{\frac{1}{4}+x^2+y^2+z^2}\,.$$

A generic plane is represented by the sphere of Euclidian space:

$$x^2 + y^2 + z^2 + a_1\,x + a_2\,y + a_3\,z = \tfrac{1}{4},$$

in which, a_1, a_2, a_3 are arbitrary constants, and all of these spheres intersect in a great circle of the sphere:

$$x^2 + y^2 + z^2 = \tfrac{1}{4}.$$

A system of generators of the sphere $x^2 + y^2 + z^2 + \frac{1}{4} = 0$ is given by:

$$(\alpha) \qquad \begin{cases} x+\dfrac{i}{2}+\lambda(y-iz)=0, \\[2mm] y+iz-\lambda(x-\dfrac{i}{2})=0. \end{cases}$$

If one is given two spheres:

$$(\beta) \qquad \begin{cases} x^2+y^2+z^2+a_1x+a_2y+a_3z=\frac{1}{4}, \\ x^2+y^2+z^2+a_1'x+a_2'y+a_3'z=\frac{1}{4} \end{cases}$$

then in order to find which pairs of generators of the sphere $x^2 + y^2 + z^2 + \frac{1}{4} = 0$ support the circle (β) (viz., the image of a line in curved space), one subtracts $x^2 + y^2 + z^2 + \frac{1}{4} = 0$ from (β), and then eliminates x, y, z from the equations thus obtained, as well as from (α). One will arrive at:

$$\begin{vmatrix} -1 & a_1 & a_2 & a_3 \\ -1 & b_1 & b_2 & b_3 \\ i & 1 & \lambda & -i\lambda \\ i\lambda & -\lambda & 1 & i \end{vmatrix} = 0$$

for the determination of λ.

From this, one deduces that:

In order for the circle (β) *and the circle:*

$$x^2 + y^2 + z^2 + a_1'x + a_2'y + a_3'z - \tfrac{1}{4} = 0,$$

$$x^2 + y^2 + z^2 + b_1'x + b_2'y + b_3'z - \tfrac{1}{4} = 0$$

to meet in the same pair of generators of:

$$x^2 + y^2 + z^2 + \tfrac{1}{4} = 0,$$

one must have:

$$\begin{vmatrix} -1 & a_1 \\ -1 & b_1 \end{vmatrix} + \begin{vmatrix} b_2 & b_3 \\ a_2 & a_3 \end{vmatrix} : \begin{vmatrix} -1 & a_2 \\ -1 & b_2 \end{vmatrix} + \begin{vmatrix} b_3 & b_1 \\ a_3 & a_1 \end{vmatrix} : \begin{vmatrix} -1 & a_3 \\ -1 & b_3 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & a_1' \\ -1 & b_1' \end{vmatrix} + \begin{vmatrix} b_2' & b_3' \\ a_2' & a_3' \end{vmatrix} : \begin{vmatrix} -1 & a_2' \\ -1 & b_2' \end{vmatrix} + \begin{vmatrix} b_3' & b_1' \\ a_3' & a_1' \end{vmatrix} : \begin{vmatrix} -1 & a_3' \\ -1 & b_3' \end{vmatrix} + \begin{vmatrix} b_1' & b_2' \\ a_1' & a_2' \end{vmatrix} .$$

Therefore (§ 4):

In the conformal representation of curved space, parallel lines are represented by circles that intersect in the same pair of skew generators of the image sphere of the absolute.

§ 23. From that, and a theorem of Prof. Bianchi that was cited above, one deduces that:

The surface that is generated by a circle that moves – with or without deformation – and always intersects the same pair of skew generators of a sphere $x^2 + y^2 + z^2 + 1 = 0$ *will admit the family of these circles as its family of isothermal curves.*

This theorem can be generalized; indeed, one has:

All of the circular surfaces in a flat space that admit the family of circular generators as their family of isothermal curves can be obtained in the conformal representation on a flat space of the spaces with constant curvature as images of the isothermal lines of the latter space – i.e., (Bianchi A) *as images of the rulings generated by the binormals to a curve of constant torsion.*

Having proved this theorem, one then has immediately (because under conformal representations of a non-Euclidian space in a flat space, the circles go to circles and the families of isothermal curves go to isothermal curves) that: *All of the circular surfaces in an arbitrary space of constant curvature that admit circles as families of isothermal curves are deduced, with conformal representations, from the ruled locus of binormals to a curve with constant torsion in a space that has constant curvature, moreover.*

In a beautiful article of Demartres ([1]), it is proved that the point P of intersection of the line that is common to the planes of two consecutive circles and the line that joins the intersection points of one of the these circles and the projection of the other circle onto the plane of the first one is a fixed point in space. Then, take a moving trihedron, whose origin is the center of any generic one of these circles and whose x-axis passes through the point P, in such a way that the coordinates of the point P are $(\alpha, 0, 0)$. Demartres also proved that if R is the radius of the circle then binomial $\alpha^2 - R^2$ is a constant, so all of the rest of the Demartres discussion can be avoided with a very simple consideration: Indeed, consider the sphere T with center P and radius $\sqrt{R^2 - \alpha^2}$. Since the equation of the corresponding circle with respect to the moving trihedron is:

$$x^2 + y^2 = R^2, \quad z = 0,$$

one verifies immediately that this circle meets our sphere at diametrically-opposite points. Now, if we represent a space of constant curvature on a flat space in a conformal manner in such a way that the sphere T represents the absolute then the circles in question will correspond to lines in curved space, and our circular surface will have a ruling for its image in curve space, for which the lines forms an isothermal family; this is what we would like to prove. With Demartres, we call such surfaces *isocyclic* surfaces; we will then have:

The problem of constructing the isocyclic surfaces in spaces of constant curvature (or, in particular, in flat space) coincides with the problem of determining all of the curves with constant torsion in a space of constant curvature.

It then remains for us to resolve two questions: One of them is to find the effective formulas that permit one to pass from one problem to the other. The other one is to interpret this theorem when it is applied to flat space with just the Euclidian metric; naturally, that is the more interesting question of the two.

Therefore, let the isocyclic surface Σ in flat space be defined by the form:

$$ds^2 = E\,(du^2 + dv^2), \qquad D\,du^2 + 2\,D'\,du\,dv + D''\,dv^2,$$

and let u = const. be the constituent circles of the usual isothermal family. The absolute curvature $1 / \rho_u$ of the u = const. will then be a function of only u; the torsion of the u = const. will always be zero, and if one lets σ denote the angle between the principal normal to u = const. and the normal to the surface at generic point then one will have:

([1]) *Annales de l'École Normale Supérieure,* t. IV, 1887, page 145, *et seq.*

$$\frac{\cos\sigma}{\rho_u} = \frac{D''}{E}, \qquad \frac{\sin\sigma}{\rho_u} = -\frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial u},$$

which, from what we said, gives the derivative with respect to v:

$$\text{(1)} \qquad \frac{D''}{E}\frac{\partial\sigma}{\partial v} = -\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial u}\right),$$

$$\text{(2)} \qquad \frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial u}\frac{\partial\sigma}{\partial v} = \frac{\partial}{\partial v}\left(\frac{D''}{E}\right),$$

with which, the assumed property gives:

$$\text{(3)} \qquad \frac{\partial}{\partial v}\left(\frac{D''^2 + \left(\frac{\partial\sqrt{E}}{\partial u}\right)^2}{D^2}\right) = 0, \qquad \frac{D'}{E} = \frac{1}{\sqrt{E}}\frac{\partial\sigma}{\partial v},$$

while, for the Codazzi and Gauss equations, one has:

$$\text{(4)} \qquad D\,D'' - D'^2 = -\frac{E}{2}\left(\frac{\partial^2 \log E}{\partial u^2} + \frac{\partial^2 \log E}{\partial v^2}\right),$$

$$\text{(5)} \qquad \frac{\partial D}{\partial v} - \frac{\partial D'}{\partial u} - \frac{1}{2}\frac{\partial \log E}{\partial v}D - \frac{1}{2}\frac{\partial \log E}{\partial v}D'' = 0,$$

$$\text{(6)} \qquad \frac{\partial D''}{\partial u} - \frac{\partial D'}{\partial u} - \frac{1}{2}\frac{\partial \log E}{\partial u}D - \frac{1}{2}\frac{\partial \log E}{\partial u}D'' = 0.$$

The surface Σ' that is imagined in curved space will have the linear element:

$$ds'^2 = \frac{E}{\lambda^2}(du^2 + dv^2),$$

where λ is determined by observing that the u = const are geodetic, so:

$$\text{(7)} \qquad \frac{\partial \log\lambda}{\partial u} = \frac{\partial \log\sqrt{E}}{\partial u}.$$

The second fundamental form of Σ' is, as one calculates immediately:

$$D_1\, du^2 + 2\, D_1'\, du\, dv + D_1''\, dv^2 =$$

$$= -\,\frac{\lambda(D\,du^2 + 2D'du\,dv + D''dv^2) + 2E(xX + yY + zZ)(du^2 + dv^2)}{\lambda^2}.$$

If one sets $Xx + Yy + Zz$ equal to the value that it gets from [1] $\rho_{12} - D'W$ and recalls that one can assume that $\lambda - 2\rho$ is constant then one can deduce [recalling (1) and (3)] that:

$$D_1\, du^2 + 2\, D_1'\, du\, dv + D_1''\, dv^2 = \frac{D'' - D}{\lambda}du^2 - \frac{D'}{\lambda}du\,dv\,.$$

The u = const. then reduce to asymptotes – i.e., to lines, precisely. In order to find λ without quadrature, one uses the Codazzi and Gauss formulas; one will get λ / D' = const., and the Codazzi and Gauss formulas will reduce to:

$$(\alpha) \qquad \frac{\partial}{\partial u}\left(\frac{D'}{\sqrt{E}}\right) = 0,$$

$$(\beta) \qquad \frac{\partial \log \dfrac{D'' - D}{\sqrt{E}}}{\partial v} = 0,$$

$$(\gamma) \qquad \left(\frac{D'}{\sqrt{E}}\right)^2 \left\{\left(\frac{D'}{\sqrt{E}}\right)^2 + \frac{\partial^2 \log\left(\dfrac{D'}{\sqrt{E}}\right)}{\partial v^2}\right\} = \text{const.}$$

From (2), (3), (5), the first two of these equations are consequences of each other. If one could therefore prove one of the two directly, along with the third one, then one could prove our theorem, *and in a new way*.

(α) gives, from (3):

$$\frac{\partial^2 \sigma}{\partial u\, \partial v} = 0.$$

Therefore: *Consider a quadrangle in an isocyclic surface in flat space that is defined by two circles and two orthogonal trajectories for the system of circles. Calculate the values of σ (angle between the normal to the surface at a point with the plane of the circle that passes through that point) of the four vertices of the quadrangle; the sum of the values that the aforementioned angle take on at two opposite vertices is equal to the sum of the values that they take on at the other two vertices.*

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[1] BIANCHI (*Lezioni*, Chap, V, page 114).

However, a result that is far more noteworthy can be deduced from (α), (β).
It shows us that the binomial differential:

$$(D'' - D)\, du = 2\, D'\, dv$$

admits $1/\sqrt{E}$ as an integrating factor, namely, that:

$$D_1\, du + 2\, D_1'\, dv = \frac{D'' - D}{\lambda} du - 2\frac{D'}{\lambda} dv\, du - 2\, dv$$

admits $\lambda/\sqrt{E}$ as an integrating factor (which one sees once one is given the linear element of the isothermal ruling). Therefore:

One finds the asymptotes of the corresponding isothermal ruling on any isocyclic surface with only quadratures.

The asymptotes to any ruled locus of binormals to a curve of constant torsion are determined by quadrature.

This last theorem has an elegant geometric explanation: One knows that the asymptotes on any ruling are determined by means of a Riccati equation; it is therefore sufficient to know one asymptote, since the other one will be determined by quadrature.

If we compare the construction that was given by Darboux (t. III, Chap. XIV) of the conformal Euclidian image of a surface in curved space with the construction that Demartres gave for the isocyclic surface then one will obtain the following theorem, which permits one to construct exactly one – and therefore all – of the asymptotes to an isothermal ruling with just a quadrature:

A characteristic property of the ruling that is defined by the binormals to a curve with constant torsion is that the developable that is defined by the planes that are tangent to the ruling and the absolute will have an asymptote of the ruling as its edge of regression.

The last problem to be solved is that of interpreting the results that were just obtained for the isocyclic surface in flat space in the Euclidian metric. If one recalls the theorem of § 8, which I arrived at precisely in order to resolve this question, then one will have immediately:

The problem of finding the isocyclic surfaces in flat space is equivalent to that of finding those pairs of curves with constant, but distinct, torsion in that space that correspond point-by-point with equality of arc lengths and first curvature. The Razzaboni transformation for them will give a transformation of the isocyclic surfaces.

Varied and supplementary observations.

In addition to the questions and examples that were treated in the present treatise, other problems can be posed. For example, that of giving the typical forms for the linear elements of the Clifford images of particular congruences (e.g., pseudo-spherical, etc.). All of the questions that result in elliptic space are much simpler to treat than the corresponding ones in flat space, due to the property that was proved above that the Clifford images determine a congruence. We have therefore solved only the most important cases of knowing the normal congruences and *W*-congruences – i.e., their images – but with processes that are indirect and greatest simplicity; we have, however, always assumed that the Clifford images were non-degenerate. If one of them can degenerate then in order for the congruence to be normal it will be necessary that the other one degenerates, as well, and one will have the congruence of normals to a surface of zero curvature, as we already know. However, one can say more:

The congruence will be W if and only if just one of the Clifford images reduces to a curve C, or C is a line, or the lines of the other corresponding image to the points of C are geodetically parallel.

If both of the two Clifford images are degenerate then the congruence is W and is normal to a surface with zero curvature.

This last theorem gives a new *characteristic* projective property of the normal congruences to a surface with zero curvature, while so far it was proved that having degenerate images is a property that distinguishes these congruences only for normal congruences.

These theorems are proved immediately: If α, β, γ in § 12 are functions of only "*u*" then the equivalence of § 12 becomes:

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \dfrac{\partial\alpha}{\partial u} & \dfrac{\partial\beta}{\partial u} & \dfrac{\partial\gamma}{\partial u} \\ \dfrac{\partial^2\alpha}{\partial u^2} & \dfrac{\partial^2\beta}{\partial u^2} & \dfrac{\partial^2\gamma}{\partial u^2} \end{vmatrix} \begin{vmatrix} \dfrac{\partial\alpha_1}{\partial v} & \dfrac{\partial\beta_1}{\partial v} & \dfrac{\partial\gamma_1}{\partial v} \\ \dfrac{\partial^2\alpha_1}{\partial u\,\partial v} & \dfrac{\partial^2\beta_1}{\partial u\,\partial v} & \dfrac{\partial^2\gamma_1}{\partial u\,\partial v} \\ \dfrac{\partial^2\alpha_1}{\partial v^2} & \dfrac{\partial^2\beta_1}{\partial v^2} & \dfrac{\partial^2\gamma_1}{\partial v^2} \end{vmatrix} = 0.$$

If the first of these two determinants is zero then α, β, γ will be coupled by a linear relation, and the curve *C* will be a line; if the second one is zero then one soon recognizes from the procedures in § 12 that the v = const. prove to be geodetically parallel.

Finally, if α, β, γ are functions of only v – i.e., both of the Clifford images reduce to a line – then it is quite clear that the corresponding congruence will be *W*, since the second of the two preceding determinants will be annulled; indeed, the congruence will be properly normal, as one sees from an argument that is analogous to the ones in § 16, and as one can also convince oneself geometrically.

Something else to note in the present treatise is, perhaps, the definition of the angle between two skew lines; I therefore believe that it is not pointless to give them another equivalent definition that is independent of any concept of parallelism.

Let a, b be two lines and, without diminishing the generality, let a be the line that connects the point (1, 0, 0, 0) with the point (0, 0, 0, 1), and let the common perpendiculars to a and b be the line β that goes from the point (1, 0, 0, 0) to the point (0, 1, 0, 0) and the line γ that goes from (0, 0, 0, 1) to the point (0, 0, 1, 0). The line b removes segments of length φ, f from the lines β, γ by starting with the points (1, 0, 0, 0) and (0, 0, 0, 1), respectively; the line b will be the line that connects the point ($\cos \varphi$, $\sin \varphi$, 0, 0) to the point (0, 0, $\sin f$, $\cos f$). It is then easy to construct the scrolling parameters of a, b, and if one lets w denote the angle between these two lines then one will have, as one sees immediately, $\cos w = \cos (\varphi \pm f)$ according to the sense in which the angle is measured. Therefore:

The cosine of the angle between the two skew lines is equal to the cosine of the sum or difference between their minimum and maximum distances, according to the sense in which it is measured.

It then follows immediately from the theorem that was cited many times above that the angle between two lines will admit just one determination when and only when the two lines are coplanar.

We would expressly like to note that in all of this treatise the question of the orientation of a line was always left untouched, and therefore that of the precise determination of the angle between two skew lines; a greater degree of precision was always useless for us, and could be easily established, moreover.

I must also add that I had already completed the present treatise when Prof. Bianchi informed to me that Study (*Ueber Nicht-Euclidische und Linen-Geometrie;* Greifswald, 1900, pages 73-79) has treated Clifford parallelism. In those pages, Study, starting from the purely geometric viewpoint, stated and then gave some very simple corollaries to the following two theorems:

The totality of polar pairs of lines in curved space can be referred to the totality of all pairs of lines that are formed from a line of a fixed star and a line of another fixed star in flat space. The rotations of one or the other of these stars will correspond to scrolling in one or the other direction in flat space.

The totality of oriented lines in elliptic space can be imagined to be bijectively referred to the pairs of points of a Euclidian sphere, in such a way that the motions of one or the other of the images correspond to scrolling in one sense or the other.

The application of our principles to hyperbolic space leads to complicated formulas in imaginary numbers. The direct study of Lobatschewsky parallelism would not be quite so symmetric, since in hyperbolic space the two senses according to which one can draw parallel lines are not distinct from each other, as they are in elliptic space.