§30 The Countability Axioms

Recall the definition we gave in §21.

Definition. A space X is said to have a ***countable basis at x*** if there is a countable collection $\mathcal{B}$ of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathcal{B}$. A space that has a countable basis at each of its points is said to satisfy the ***first countability axiom***, or to be ***first-countable***.

We have already noted that every metrizable space satisfies this axiom; see §21.

The most useful fact concerning spaces that satisfy this axiom is the fact that in such a space, convergent sequences are adequate to detect limit points of sets and to check continuity of functions. We have noted this before; now we state it formally as a theorem:

Theorem 30.1. *Let X be a topological space.*

(a) *Let A be a subset of X. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is first-countable.*

(b) *Let $f : X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if X is first-countable.*

The proof is a direct generalization of the proof given in §21 under the hypothesis of metrizability, so it will not be repeated here.

Of much greater importance than the first countability axiom is the following:

Definition. If a space X has a countable basis for its topology, then X is said to satisfy the ***second countability axiom***, or to be ***second-countable***.

Obviously, the second axiom implies the first: if $\mathcal{B}$ is a countable basis for the topology of X, then the subset of $\mathcal{B}$ consisting of those basis elements containing the point x is a countable basis at x. The second axiom is, in fact, much stronger than the first; it is so strong that not even every metric space satisfies it.

Why then is this second axiom interesting? Well, for one thing, many familiar spaces do satisfy it. For another, it is a crucial hypothesis used in proving such theorems as the Urysohn metrization theorem, as we shall see.

EXAMPLE 1. The real line $\mathbb{R}$ has a countable basis—the collection of all open intervals (a, b) with rational end points. Likewise, $\mathbb{R}^n$ has a countable basis—the collection of all products of intervals having rational end points. Even $\mathbb{R}^\omega$ has a countable basis—the collection of all products $\prod_{n \in \mathbb{Z}_+} U_n$, where U_n is an open interval with rational end points for finitely many values of n, and $U_n = \mathbb{R}$ for all other values of n.

EXAMPLE 2. In the uniform topology, $\mathbb{R}^\omega$ satisfies the first countability axiom (being metrizable). However, it does not satisfy the second. To verify this fact, we first show that if X is a space having a countable basis $\mathcal{B}$, then any discrete subspace A of X must be countable. Choose, for each $a \in A$, a basis element B_a that intersects A in the point a

alone. If a and b are distinct points of A, the sets B_a and B_b are different, since the first contains a and the second does not. It follows that the map $a \to B_a$ is an injection of A into $\mathcal{B}$, so A must be countable.

Now we note that the subspace A of $\mathbb{R}^\omega$ consisting of all sequences of 0's and 1's is uncountable; and it has the discrete topology because $\bar{\rho}(a, b) = 1$ for any two distinct points a and b of A. Therefore, in the uniform topology $\mathbb{R}^\omega$ does not have a countable basis.

Both countability axioms are well behaved with respect to the operations of taking subspaces or countable products:

Theorem 30.2. *A subspace of a first-countable space is first-countable, and a countable product of first-countable spaces is first-countable. A subspace of a second-countable space is second-countable, and a countable product of second-countable spaces is second-countable.*

Proof. Consider the second countability axiom. If $\mathcal{B}$ is a countable basis for X, then $\{B \cap A \mid B \in \mathcal{B}\}$ is a countable basis for the subspace A of X. If $\mathcal{B}_i$ is a countable basis for the space X_i, then the collection of all products $\prod U_i$, where $U_i \in \mathcal{B}_i$ for finitely many values of i and $U_i = X_i$ for all other values of i, is a countable basis for $\prod X_i$.

The proof for the first countability axiom is similar. ∎

Two consequences of the second countability axiom that will be useful to us later are given in the following theorem. First, a definition:

Definition. A subset A of a space X is said to be ***dense*** in X if $\bar{A} = X$.

Theorem 30.3. *Suppose that X has a countable basis. Then:*

(a) Every open covering of X contains a countable subcollection covering X.

(b) There exists a countable subset of X that is dense in X.

Proof. Let $\{B_n\}$ be a countable basis for X.

(a) Let $\mathcal{A}$ be an open covering of X. For each positive integer n for which it is possible, choose an element A_n of $\mathcal{A}$ containing the basis element B_n. The collection $\mathcal{A}'$ of the sets A_n is countable, since it is indexed with a subset J of the positive integers. Furthermore, it covers X: Given a point $x \in X$, we can choose an element A of $\mathcal{A}$ containing x. Since A is open, there is a basis element B_n such that $x \in B_n \subset A$. Because B_n lies in an element of $\mathcal{A}$, the index n belongs to the set J, so A_n is defined; since A_n contains B_n, it contains x. Thus $\mathcal{A}'$ is a countable subcollection of $\mathcal{A}$ that covers X.

(b) From each nonempty basis element B_n, choose a point x_n. Let D be the set consisting of the points x_n. Then D is dense in X: Given any point x of X, every basis element containing x intersects D, so x belongs to $\bar{D}$. ∎

The two properties listed in Theorem 30.3 are sometimes taken as alternative countability axioms. A space for which every open covering contains a countable subcovering is called a ***Lindelöf space***. A space having a countable dense subset is often said to be ***separable*** (an unfortunate choice of terminology).[†] Weaker in general than the second countability axiom, each of these properties is equivalent to the second countability axiom when the space is metrizable (see Exercise 5). They are less important than the second countability axiom, but you should be aware of their existence, for they are sometimes useful. It is often easier, for instance, to show that a space X has a countable dense subset than it is to show that X has a countable basis. If the space is metrizable (as it usually is in analysis), it follows that X is second-countable as well.

We shall not use these properties to prove any theorems, but one of them—the Lindelöf condition—will be useful in dealing with some examples. They are not as well behaved as one might wish under the operations of taking subspaces and cartesian products, as we shall see in the examples and exercises that follow.

EXAMPLE 3. *The space $\mathbb{R}_\ell$ satisfies all the countability axioms but the second.*

Given $x \in \mathbb{R}_\ell$, the set of all basis elements of the form $[x, x + 1/n)$ is a countable basis at x. And it is easy to see that the rational numbers are dense in $\mathbb{R}_\ell$.

To see that $\mathbb{R}_\ell$ has no countable basis, let $\mathcal{B}$ be a basis for $\mathbb{R}_\ell$. Choose for each x, an element B_x of $\mathcal{B}$ such that $x \in B_x \subset [x, x+1)$. If $x \neq y$, then $B_x \neq B_y$, since $x = \inf B_x$ and $y = \inf B_y$. Therefore, $\mathcal{B}$ must be uncountable.

To show that $\mathbb{R}_\ell$ is Lindelöf requires more work. It will suffice to show that every open covering of $\mathbb{R}_\ell$ by basis elements contains a countable subcollection covering $\mathbb{R}_\ell$. (You can check this.) So let

$$\mathcal{A} = \{[a_\alpha, b_\alpha)\}_{\alpha \in J}$$

be a covering of $\mathbb{R}$ by basis elements for the lower limit topology. We wish to find a countable subcollection that covers $\mathbb{R}$.

Let C be the set

$$C = \bigcup_{\alpha \in J} (a_\alpha, b_\alpha),$$

which is a subset of $\mathbb{R}$. We show the set $\mathbb{R} - C$ is countable.

Let x be a point of $\mathbb{R} - C$. We know that x belongs to no open interval (a_α, b_α); therefore $x = a_\beta$ for some index β. Choose such a β and then choose q_x to be a rational number belonging to the interval (a_β, b_β). Because (a_β, b_β) is contained in C, so is the interval $(a_\beta, q_x) = (x, q_x)$. It follows that if x and y are two points of $\mathbb{R} - C$ with $x < y$, then $q_x < q_y$. (For otherwise, we would have $x < y < q_y \leq q_x$, so that y would lie in the interval (x, q_x) and hence in C.) Therefore the map $x \to q_x$ of $\mathbb{R} - C$ into $\mathbb{Q}$ is injective, so that $\mathbb{R} - C$ is countable.

Now we show that some countable subcollection of $\mathcal{A}$ covers $\mathbb{R}$. To begin, choose for each element of $\mathbb{R} - C$ an element of $\mathcal{A}$ containing it; one obtains a countable subcollection $\mathcal{A}'$ of $\mathcal{A}$ that covers $\mathbb{R} - C$. Now take the set C and topologize it as a subspace of $\mathbb{R}$; in this topology, C satisfies the second countability axiom. Now C is covered by the sets (a_α, b_α), which are open in $\mathbb{R}$ and hence open in C. Then some countable subcollection

[†]This is a good example of how a word can be overused. We have already defined what we mean by a *separation* of a space; and we shall discuss the *separation* axioms shortly.

covers C. Suppose this subcollection consists of the elements (a_α, b_α) for $\alpha = \alpha_1, \alpha_2, \ldots$. Then the collection

$$\mathcal{A}'' = \{[a_\alpha, b_\alpha) \mid \alpha = \alpha_1, \alpha_2, \ldots\}$$

is a countable subcollection of $\mathcal{A}$ that covers the set C, and $\mathcal{A}' \cup \mathcal{A}''$ is a countable subcollection of $\mathcal{A}$ that covers $\mathbb{R}_\ell$.

EXAMPLE 4. *The product of two Lindelöf spaces need not be Lindelöf.* Although the space $\mathbb{R}_\ell$ is Lindelöf, we shall show that the product space $\mathbb{R}_\ell \times \mathbb{R}_\ell = \mathbb{R}_\ell^2$ is not. The space $\mathbb{R}_\ell^2$ is an extremely useful example in topology called the ***Sorgenfrey plane***.

The space $\mathbb{R}_\ell^2$ has as basis all sets of the form $[a, b) \times [c, d)$. To show it is not Lindelöf, consider the subspace

$$L = \{x \times (-x) \mid x \in \mathbb{R}_\ell\}.$$

It is easy to check that L is closed in $\mathbb{R}_\ell^2$. Let us cover $\mathbb{R}_\ell^2$ by the open set $\mathbb{R}_\ell^2 - L$ and by all basis elements of the form

$$[a, b) \times [-a, d).$$

Each of these open sets intersects L in at most one point. Since L is uncountable, no countable subcollection covers $\mathbb{R}_\ell^2$. See Figure 30.1.

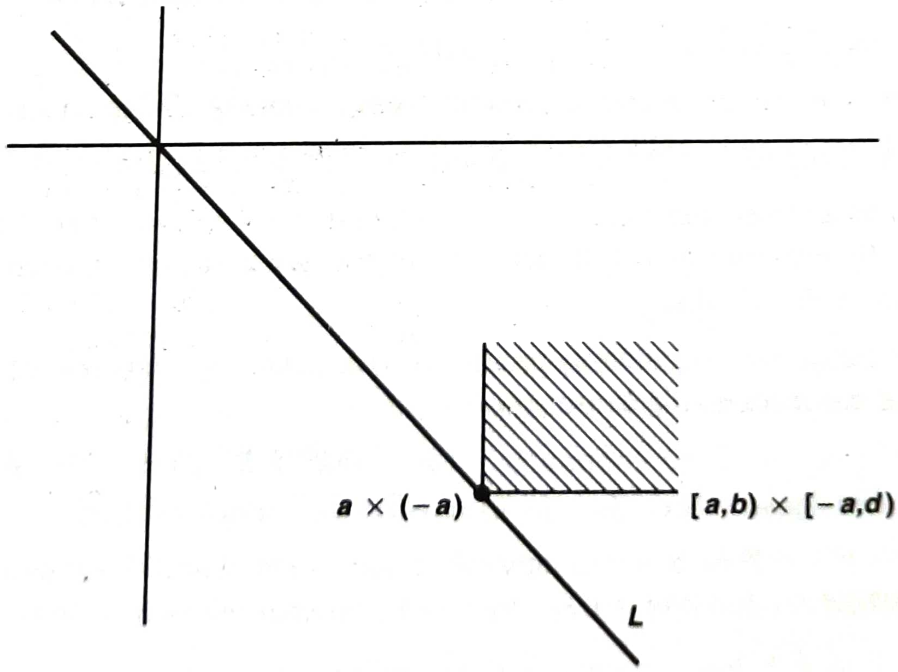

Figure 30.1

EXAMPLE 5. *A subspace of a Lindelöf space need not be Lindelöf.* The ordered square I_o^2 is compact; therefore it is Lindelöf, trivially. However, the subspace $A = I \times (0, 1)$ is not Lindelöf. For A is the union of the disjoint sets $U_x = \{x\} \times (0, 1)$, each of which is open in A. This collection of sets is uncountable, and no proper subcollection covers A.

16. (a) Show that the product space $\mathbb{R}^I$, where $I = [0, 1]$, has a countable dense subset.

(b) Show that if J has cardinality greater than $\mathcal{P}(\mathbb{Z}_+)$, then the product space $\mathbb{R}^J$ does not have a countable dense subset. [*Hint:* If D is dense in $\mathbb{R}^J$, define $f : J \to \mathcal{P}(D)$ by the equation $f(\alpha) = D \cap \pi_\alpha^{-1}((a, b))$, where (a, b) is a fixed interval in $\mathbb{R}$.]

***17.** Give $\mathbb{R}^\omega$ the box topology. Let $\mathbb{Q}^\infty$ denote the subspace consisting of sequences of rationals that end in an infinite string of 0's. Which of our four countability axioms does this space satisfy?

***18.** Let G be a first-countable topological group. Show that if G has a countable dense subset, or is Lindelöf, then G has a countable basis. [*Hint:* Let $\{B_n\}$ be a countable basis at e. If D is a countable dense subset of G, show the sets dB_n, for $d \in D$, form a basis for G. If G is Lindelöf, choose for each n a countable set C_n such that the sets cB_n, for $c \in C_n$, cover G. Show that as n ranges over $\mathbb{Z}_+$, these sets form a basis for G.]

§31 The Separation Axioms

In this section, we introduce three separation axioms and explore some of their properties. One you have already seen—the Hausdorff axiom. The others are similar but stronger. As always when we introduce new concepts, we shall examine the relationship between these axioms and the concepts introduced earlier in the book.

Recall that a space X is said to be *Hausdorff* if for each pair x, y of distinct points of X, there exist disjoint open sets containing x and y, respectively.

Definition. Suppose that one-point sets are closed in X. Then X is said to be ***regular*** if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively. The space X is said to be ***normal*** if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.

It is clear that a regular space is Hausdorff, and that a normal space is regular. (We need to include the condition that one-point sets be closed as part of the definition of regularity and normality in order for this to be the case. A two-point space in the indiscrete topology satisfies the other part of the definitions of regularity and normality, even though it is not Hausdorff.) For examples showing the regularity axiom stronger than the Hausdorff axiom, and normality stronger than regularity, see Examples 1 and 3.

These axioms are called separation axioms for the reason that they involve "separating" certain kinds of sets from one another by disjoint open sets. We have used the word "separation" before, of course, when we studied connected spaces. But in that case, we were trying to find disjoint open sets *whose union was the entire space*.

The present situation is quite different because the open sets need not satisfy this condition.

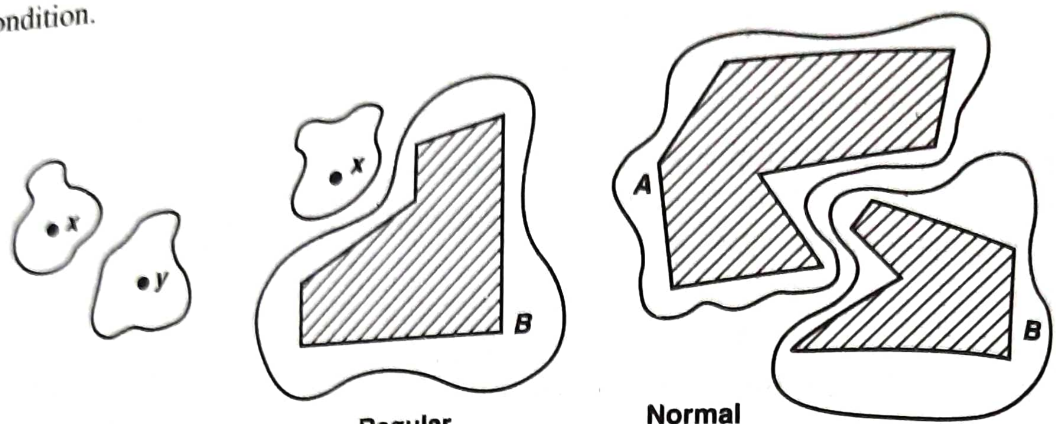

Figure 31.1

The three separation axioms are illustrated in Figure 31.1.

There are other ways to formulate the separation axioms. One formulation that is sometimes useful is given in the following lemma:

Lemma 31.1. *Let X be a topological space. Let one-point sets in X be closed.*

(a) X is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood V of x such that $\bar{V} \subset U$.

(b) X is normal if and only if given a closed set A and an open set U containing A, there is an open set V containing A such that $\bar{V} \subset U$.

Proof. (a) Suppose that X is regular, and suppose that the point x and the neighborhood U of x are given. Let $B = X - U$; then B is a closed set. By hypothesis, there exist disjoint open sets V and W containing x and B, respectively. The set $\bar{V}$ is disjoint from B, since if $y \in B$, the set W is a neighborhood of y disjoint from V. Therefore, $\bar{V} \subset U$, as desired.

To prove the converse, suppose the point x and the closed set B not containing x are given. Let $U = X - B$. By hypothesis, there is a neighborhood V of x such that $\bar{V} \subset U$. The open sets V and $X - \bar{V}$ are disjoint open sets containing x and B, respectively. Thus X is regular.

(b) This proof uses exactly the same argument; one just replaces the point x by the set A throughout. ∎

Now we relate the separation axioms with the concepts previously introduced.

Theorem 31.2. *(a) A subspace of a Hausdorff space is Hausdorff; a product of Hausdorff spaces is Hausdorff.*

(b) A subspace of a regular space is regular; a product of regular spaces is regular.

Proof. (a) This result was an exercise in §17. We provide a proof here. Let X be Hausdorff. Let x and y be two points of the subspace Y of X. If U and V are disjoint neighborhoods in X of x and y, respectively, then $U \cap Y$ and $V \cap Y$ are disjoint neighborhoods of x and y in Y.

Let $\{X_\alpha\}$ be a family of Hausdorff spaces. Let $\mathbf{x} = (x_\alpha)$ and $\mathbf{y} = (y_\alpha)$ be distinct points of the product space $\prod X_\alpha$. Because $\mathbf{x} \neq \mathbf{y}$, there is some index β such that $x_\beta \neq y_\beta$. Choose disjoint open sets U and V in X_β containing x_β and y_β, respectively. Then the sets $\pi_\beta^{-1}(U)$ and $\pi_\beta^{-1}(V)$ are disjoint open sets in $\prod X_\alpha$ containing $\mathbf{x}$ and $\mathbf{y}$, respectively.

(b) Let Y be a subspace of the regular space X. Then one-point sets are closed in Y. Let x be a point of Y and let B be a closed subset of Y disjoint from x. Now $\bar{B} \cap Y = B$, where $\bar{B}$ denotes the closure of B in X. Therefore, $x \notin \bar{B}$, so, using regularity of X, we can choose disjoint open sets U and V of X containing x and $\bar{B}$, respectively. Then $U \cap Y$ and $V \cap Y$ are disjoint open sets in Y containing x and B, respectively.

Let $\{X_\alpha\}$ be a family of regular spaces; let $X = \prod X_\alpha$. By (a), X is Hausdorff, so that one-point sets are closed in X. We use the preceding lemma to prove regularity of X. Let $\mathbf{x} = (x_\alpha)$ be a point of X and let U be a neighborhood of $\mathbf{x}$ in X. Choose a basis element $\prod U_\alpha$ about $\mathbf{x}$ contained in U. Choose, for each α, a neighborhood V_α of x_α in X_α such that $\bar{V}_\alpha \subset U_\alpha$; if it happens that $U_\alpha = X_\alpha$, choose $V_\alpha = X_\alpha$. Then $V = \prod V_\alpha$ is a neighborhood of x in X. Since $\bar{V} = \prod \bar{V}_\alpha$ by Theorem 19.5, it follows at once that $\bar{V} \subset \prod U_\alpha \subset U$, so that X is regular. ∎

There is no analogous theorem for normal spaces, as we shall see shortly, in this section and the next.

EXAMPLE 1. *The space $\mathbb{R}_K$ is Hausdorff but not regular.* Recall that $\mathbb{R}_K$ denotes the reals in the topology having as basis all open intervals (a, b) and all sets of the form $(a, b) - K$, where $K = \{1/n \mid n \in \mathbb{Z}_+\}$. This space is Hausdorff, because any two distinct points have disjoint open intervals containing them.

But it is not regular. The set K is closed in $\mathbb{R}_K$, and it does not contain the point 0. Suppose that there exist disjoint open sets U and V containing 0 and K, respectively. Choose a basis element containing 0 and lying in U. It must be a basis element of the form $(a, b) - K$, since each basis element of the form (a, b) containing 0 intersects K. Choose n large enough that $1/n \in (a, b)$. Then choose a basis element about $1/n$ contained in V; it must be a basis element of the form (c, d). Finally, choose z so that $z < 1/n$ and $z > \max\{c, 1/(n+1)\}$. Then z belongs to both U and V, so they are not disjoint. See Figure 31.2.

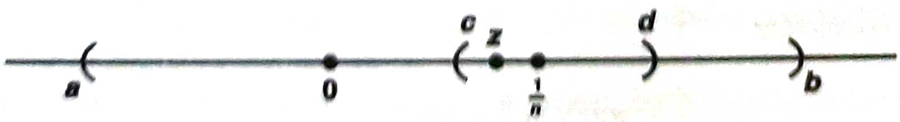

Figure 31.2

*9. Let A be the set of all points of $\mathbb{R}_\ell^2$ of the form $x \times (-x)$, for x rational; let B be the set of all points of this form for x irrational. If V is an open set of $\mathbb{R}_\ell^2$ containing B, show there exists no open set U containing A that is disjoint from V, as follows:
(a) Let K_n consist of all irrational numbers x in $[0, 1]$ such that $[x, x + 1/n) \times [-x, -x + 1/n)$ is contained in V. Show $[0, 1]$ is the union of the sets K_n and countably many one-point sets.
(b) Use Exercise 5 of §27 to show that some set $\bar{K}_n$ contains an open interval (a, b) of $\mathbb{R}$.
(c) Show that V contains the open parallelogram consisting of all points of the form $x \times (-x + \epsilon)$ for which $a < x < b$ and $0 < \epsilon < 1/n$.
(d) Conclude that if q is a rational number with $a < q < b$, then the point $q \times (-q)$ of $\mathbb{R}_\ell^2$ is a limit point of V.

§32 Normal Spaces

Now we turn to a more thorough study of spaces satisfying the normality axiom. In one sense, the term "normal" is something of a misnomer, for normal spaces are not as well-behaved as one might wish. On the other hand, most of the spaces with which we are familiar do satisfy this axiom, as we shall see. Its importance comes from the fact that the results one can prove under the hypothesis of normality are central to much of topology. The Urysohn metrization theorem and the Tietze extension theorem are two such results; we shall deal with them later in this chapter.

We begin by proving three theorems that give three important sets of hypotheses under which normality of a space is assured.

Theorem 32.1. *Every regular space with a countable basis is normal.*

Proof. Let X be a regular space with a countable basis $\mathcal{B}$. Let A and B be disjoint closed subsets of X. Each point x of A has a neighborhood U not intersecting B. Using regularity, choose a neighborhood V of x whose closure lies in U; finally, choose an element of $\mathcal{B}$ containing x and contained in V. By choosing such a basis element for each x in A, we construct a countable covering of A by open sets whose closures do not intersect B. Since this covering of A is countable, we can index it with the positive integers; let us denote it by $\{U_n\}$.

Similarly, choose a countable collection $\{V_n\}$ of open sets covering B, such that each set $\bar{V}_n$ is disjoint from A. The sets $U = \bigcup U_n$ and $V = \bigcup V_n$ are open sets containing A and B, respectively, but they need not be disjoint. We perform the following simple trick to construct two open sets that *are* disjoint. Given n, define

$$U_n' = U_n - \bigcup_{i=1}^{n} \bar{V}_i \quad \text{and} \quad V_n' = V_n - \bigcup_{i=1}^{n} \bar{U}_i.$$

Note that each set U_n' is open, being the difference of an open set U_n and a closed set $\bigcup_{i=1}^n \bar{V}_i$. Similarly, each set V_n' is open. The collection $\{U_n'\}$ covers A, because each x in A belongs to U_n for some n, and x belongs to *none* of the sets $\bar{V}_i$. Similarly, the collection $\{V_n'\}$ covers B. See Figure 32.1.

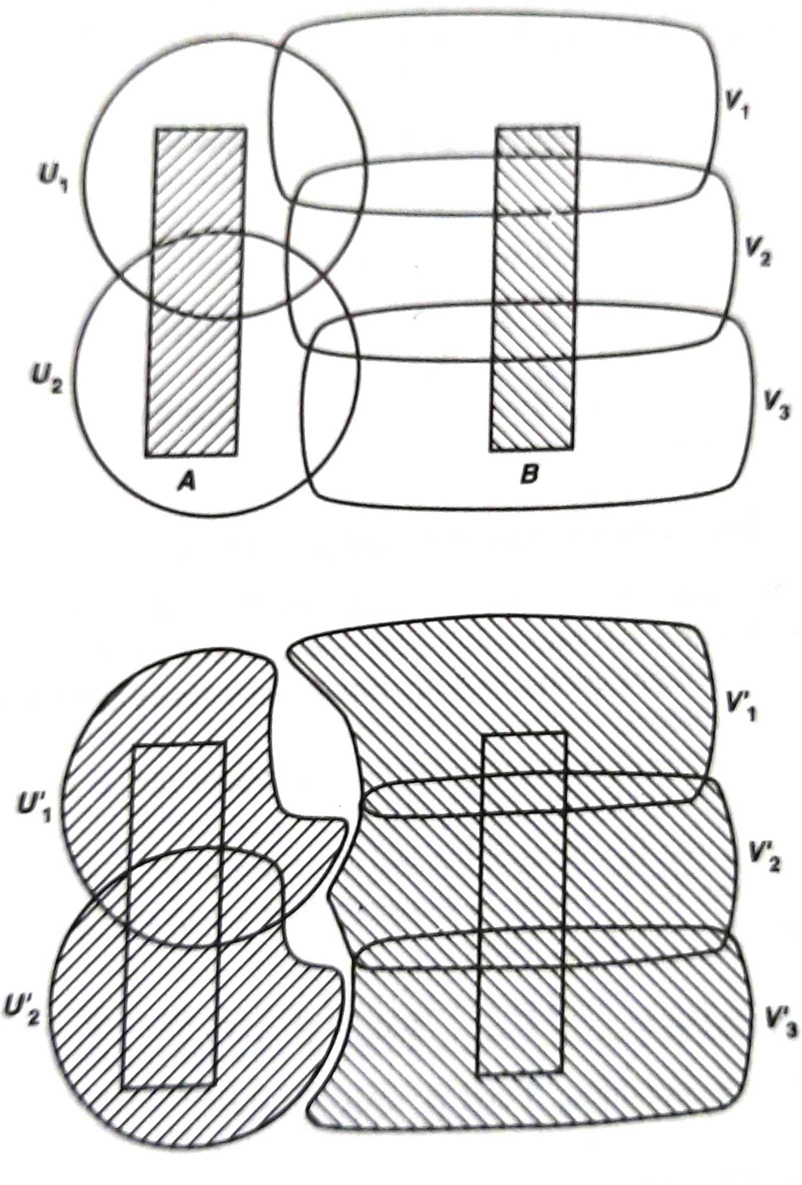

Figure 32.1

Finally, the open sets

$$U' = \bigcup_{n \in \mathbb{Z}_+} U_n' \quad \text{and} \quad V' = \bigcup_{n \in \mathbb{Z}_+} V_n'$$

are disjoint. For if $x \in U' \cap V'$, then $x \in U_j' \cap V_k'$ for some j and k. Suppose that $j \leq k$. It follows from the definition of U_j' that $x \in U_j$; and since $j \leq k$ it follows from the definition of V_k' that $x \notin \bar{U}_j$. A similar contradiction arises if $j \geq k$. ■

Theorem 32.2. *Every metrizable space is normal.*

Proof. Let X be a metrizable space with metric d. Let A and B be disjoint closed subsets of X. For each $a \in A$, choose ϵ_a so that the ball $B(a, \epsilon_a)$ does not intersect B. Similarly, for each b in B, choose ϵ_b so that the ball $B(b, \epsilon_b)$ does not intersect A. Define

$$U = \bigcup_{a \in A} B(a, \epsilon_a/2) \quad \text{and} \quad V = \bigcup_{b \in B} B(b, \epsilon_b/2).$$

Then U and V are open sets containing A and B, respectively; we assert they are disjoint. For if $z \in U \cap V$, then

$$z \in B(a, \epsilon_a/2) \cap B(b, \epsilon_b/2)$$

for some $a \in A$ and some $b \in B$. The triangle inequality applies to show that $d(a, b) < (\epsilon_a + \epsilon_b)/2$. If $\epsilon_a \leq \epsilon_b$, then $d(a, b) < \epsilon_b$, so that the ball $B(b, \epsilon_b)$ contains the point a. If $\epsilon_b \leq \epsilon_a$, then $d(a, b) < \epsilon_a$, so that the ball $B(a, \epsilon_a)$ contains the point b. Neither situation is possible. ∎

Theorem 32.3. *Every compact Hausdorff space is normal.*

Proof. Let X be a compact Hausdorff space. We have already essentially proved that X is regular. For if x is a point of X and B is a closed set in X not containing x, then B is compact, so that Lemma 26.4 applies to show there exist disjoint open sets about x and B, respectively.

Essentially the same argument as given in that lemma can be used to show that X is normal: Given disjoint closed sets A and B in X, choose, for each point a of A, disjoint open sets U_a and V_a containing a and B, respectively. (Here we use regularity of X.) The collection $\{U_a\}$ covers A; because A is compact, A may be covered by finitely many sets $U_{a_1}, \ldots, U_{a_m}$. Then

$$U = U_{a_1} \cup \cdots \cup U_{a_m} \quad \text{and} \quad V = V_{a_1} \cap \cdots \cap V_{a_m}$$

are disjoint open sets containing A and B, respectively. ∎

Here is a further result about normality that we shall find useful in dealing with some examples.

Theorem 32.4. *Every well-ordered set X is normal in the order topology.*

It is, in fact, true that *every* order topology is normal (see Example 39 of [S-S]); but we shall not have occasion to use this stronger result.

Proof. Let X be a well-ordered set. We assert that every interval of the form $(x, y]$ is open in X: If X has a largest element and y is that element, $(x, y]$ is just a basis element about y. If y is not the largest element of X, then $(x, y]$ equals the open set (x, y'), where y' is the immediate successor of y.

Now let A and B be disjoint closed sets in X; assume for the moment that neither A nor B contains the smallest element a_0 of X. For each $a \in A$, there exists a basis element about a disjoint from B; it contains some interval of the form $(x, a]$. (Here is where we use the fact that a is not the smallest element of X.) Choose, for each $a \in A$, such an interval $(x_a, a]$ disjoint from B. Similarly, for each $b \in B$, choose an interval $(y_b, b]$ disjoint from A. The sets

$$U = \bigcup_{a \in A} (x_a, a] \quad \text{and} \quad V = \bigcup_{b \in B} (y_b, b]$$

are open sets containing A and B, respectively; we assert they are disjoint. For suppose that $z \in U \cap V$. Then $z \in (x_a, a] \cap (y_b, b]$ for some $a \in A$ and some $b \in B$. Assume that $a < b$. Then if $a \leq y_b$, the two intervals are disjoint, while if $a > y_b$, we have $a \in (y_b, b]$, contrary to the fact that $(y_b, b]$ is disjoint from A. A similar contradiction occurs if $b < a$.

Finally, assume that A and B are disjoint closed sets in X, and A contains the smallest element a_0 of X. The set $\{a_0\}$ is both open and closed in X. By the result of the preceding paragraph, there exist disjoint open sets U and V containing the closed sets $A - \{a_0\}$ and B, respectively. Then $U \cup \{a_0\}$ and V are disjoint open sets containing A and B, respectively. ■

EXAMPLE 1. *If J is uncountable, the product space $\mathbb{R}^J$ is not normal.* The proof is fairly difficult; we leave it as a challenging exercise (see Exercise 9).

This example serves three purposes. It shows that a regular space $\mathbb{R}^J$ need not be normal. It shows that a subspace of a normal space need not be normal, for $\mathbb{R}^J$ is homeomorphic to the subspace $(0, 1)^J$ of $[0, 1]^J$, which (assuming the Tychonoff theorem) is compact Hausdorff and therefore normal. And it shows that an uncountable product of normal spaces need not be normal. It leaves unsettled the question as to whether a finite or a countable product of normal spaces might be normal.

EXAMPLE 2. *The product space $S_\Omega \times \bar{S}_\Omega$ is not normal.*†

Consider the well-ordered set $\bar{S}_\Omega$, in the order topology, and consider the subset S_Ω, in the subspace topology (which is the same as the order topology). Both spaces are normal, by Theorem 32.4. We shall show that the product space $S_\Omega \times \bar{S}_\Omega$ is not normal.

This example serves three purposes. First, it shows that a regular space need not be normal, for $S_\Omega \times \bar{S}_\Omega$ is a product of regular spaces and therefore regular. Second, it shows that a subspace of a normal space need not be normal, for $S_\Omega \times \bar{S}_\Omega$ is a subspace of $\bar{S}_\Omega \times \bar{S}_\Omega$, which is a compact Hausdorff space and therefore normal. Third, it shows that the product of two normal spaces need not be normal.

First, we consider the space $\bar{S}_\Omega \times \bar{S}_\Omega$, and its "diagonal" $\Delta = \{x \times x \mid x \in \bar{S}_\Omega\}$. Because $\bar{S}_\Omega$ is Hausdorff, Δ is closed in $\bar{S}_\Omega \times \bar{S}_\Omega$: If U and V are disjoint neighborhoods of x and y, respectively, then $U \times V$ is a neighborhood of $x \times y$ that does not intersect Δ.

Therefore, in the subspace $S_\Omega \times \bar{S}_\Omega$, the set

$$A = \Delta \cap (S_\Omega \times \bar{S}_\Omega) = \Delta - \{\Omega \times \Omega\}$$

†Kelley [K] attributes this example to J. Dieudonné and A. P. Morse independently.