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Measure and Integration 2006-Selected Solutions Chapter 4

1. (**Exercise 4.11, p.29**) Let λ be the one-dimensional Lebesgue measure (see def.4.8, p.27)

   (i) Show that for all $x \in \mathbb{R}$ the set $\{x\}$ is a Borel set with $\lambda(\{x\}) = 0$.

   (ii) Prove that $\mathbb{Q}$ is a Borel set and that $\lambda(\mathbb{Q}) = 0$.

   (iii) Show that the uncountable union of null sets need not be a non-set.

   Proof (i) Notice that $\{x\} = \bigcap_{n\in\mathbb{N}}[x - 1/n, x + 1/n)$. Since $[x - 1/n, x + 1/n)$ is a Borel set for all n, and the Borel σ-algebra is closed under countable intersection, it follows that $\{x\}$ is a Borel set. Further, $\lambda(\{x\}) \leq \lambda([x-1/n, x+1/n)) = 2/n$ for all n. Taking limits we see that $\lambda(\{x\}) = 0$.

   Proof (ii) $\mathbb{Q} = \bigcup_{q\in\mathbb{Q}}\{q\}$ is a countable union of Borel sets (see part (i)). Since the Borel σ-algebra is closed under countable unions, it follows that $\mathbb{Q}$ is Borel measurable. Now, $\lambda(\mathbb{Q}) \leq \sum_{q\in\mathbb{Q}} \lambda(\{q\}) = 0$.

   Proof (iii) Consider $[0, 1] = \cup_{r\in\mathbb{R}}\{r\}$ which is an uncountable union of null sets (since $\lambda(\{r\}) = 0$), however $\lambda([0, 1]) = 1$.

2. (**Exercise 4.13, p.21**). Let $(X, \mathcal{A}, \mu)$ be a measure space.

   (i) Show that $\mathcal{A}^* = \{A\cup N : A \in \mathcal{A}, N$ is a subset of some $\mathcal{A}-$measurable null set$\}$ is a σ-algebra containing $\mathcal{A}$.

   (ii) Define $\bar{\mu}$ on $\mathcal{A}^*$ by $\bar{\mu}(A^*) = \bar{\mu}(A \cup N) = \mu(A)$. Show that $\bar{\mu}$ is well-defined.

   (iii) Show that $\bar{\mu}$ is a measure extending μ, i.e. $\bar{\mu}(A) = \mu(A)$ for all $A \in \mathcal{A}$.

   (iv) Show that $(X, \mathcal{A}^*, \bar{\mu})$ is complete.

   (v) Show that $\mathcal{A}^* = \{A^* \subset X : \exists A, B \in \mathcal{A},\ A \subset A^* \subset B,\ \mu(B \setminus A) = 0\}$.

   Proof (i) Clearly $X \in \mathcal{A}^*$ since $X = X \cup \emptyset$, $X, \emptyset \in \mathcal{A}$ and $\mu(\emptyset) = 0$. Now let $(B_n)_n \subset \mathcal{A}^*$. Then there exist $(A_n)_n, (C_n)_n \subset \mathcal{A}$ with $\mu(C_n) = 0$ such that $B_n = A_n \cup N_n$ for some $N_n \subset C_n$. We want to show that $\bigcup_n B_n \in \mathcal{A}^*$. Let $A = \bigcup_n A_n$, $C = \bigcup_n C_n$ and $N = \bigcup_n C_n$. Then, $\bigcup_n B_n = A \cup N$ with $A \in \mathcal{A}$ and $N \subset C$ with $C \in \mathcal{A}$ satisfies $\mu(C) = 0$. Hence, $\bigcup_n B_n \in \mathcal{A}^*$. Now, let $B \in \mathcal{A}^*$.

Then $B = A \cup N$ with $A \in \mathcal{A}$ and $N \subset C$ for some $C \in \mathcal{A}$ with $\mu(C) = 0$. Notice that $A^c \cap C^c \subset A^c \cap N^c = B^c$. Thus, $B^c = (A^c \cap C^c) \cup [(A^c \cap N^c) \setminus (A^c \cap C^c)]$ with $A^c \cap C^c \in \mathcal{A}$, $(A^c \cap N^c) \setminus (A^c \cap C^c) = C \cap A^c \cap N^c \subset C$ and $\mu(C) = 0$. Thus, $B^c \in \mathcal{A}^*$. Therefore, $\mathcal{A}^*$ is a σ-algebra. Finally, for each $A \in \mathcal{A}$, one has $A \cup \emptyset$ hence, $\mathcal{A} \subset \mathcal{A}^*$.

Proof (ii) Suppose that $A^* \in \mathcal{A}^*$ can be written as $A^* = A \cup N = B \cup M$ with $A, B \in \mathcal{A}$ and $N \subset C$, $M \subset D$ with $C, D \in \mathcal{A}$ with $\mu(C) = \mu(D) = 0$. We need to show that $\mu(A) = \mu(B)$. First note that $A \setminus B \subset A^* \setminus B \subset M \subset D$. Hence $\mu(A \setminus B) = 0$. Now $\mu(A) = \mu(A \cap B) + \mu(A \setminus B) = \mu(A \cap B)$. Similarly, $B \setminus A \subset N \subset C$ so that $\mu(B \setminus A) = 0$. Then, $\mu(B) = \mu(B \cap A) + \mu(B \setminus A) = \mu(B \cap A) = \mu(A)$.

Proof (iii) If $A \in \mathcal{A}$, then $A = A \cup \emptyset$ so by part (ii), $\bar{\mu}(A) = \mu(A)$. We show that $\bar{\mu}$ is a measure. Clearly, $\bar{\mu}(\emptyset) = 0$. Now let (A_n^*) be a disjoint sequence in $\mathcal{A}^*$. Then, $A_n^* = A_n \cup N_n$ with $A_n \in \mathcal{A}$ and N_n is a subset of an $\mathcal{A}$-measurable null set. By definition $\bar{\mu}(A_n^*) = \mu(A_n)$. Since $\bigcup_n A_n^* = \bigcup_n A_n \cup \bigcup_n N_n$ with $\bigcup_n A_n \in \mathcal{A}$ and $\bigcup_n N_n$ is a subset of a null set, then

$$\bar{\mu}(\bigcup_n A_n^*) = \mu(\bigcup_n A_n) = \sum_n \mu(A_n) = \sum_n \bar{\mu}(A_n^*).$$

Thus, $\bar{\mu}$ is σ-additive, hence $\bar{\mu}$ is a measure.

Proof (iv) Let $M \in \mathcal{A}^*$ with $\bar{\mu}(M) = 0$. Let $C \subset M$ be any subset of M, we need to show that $C \in \mathcal{A}^*$, from which it will immediately follow that $\bar{\mu}(C) = 0$. Since $M \in \mathcal{A}^*$, then $M = A \cup N$ with $A \in \mathcal{A}$, $N \subset N'$, $N' \in \mathcal{A}$ and $\mu(N') = 0$. Then, $\mu(A) = \bar{\mu}(M) = 0$, and $C \subset M = A \cup N \subset A \cup N' \in \mathcal{A}$. Furthermore, $\mu(A \cup N') \leq \mu(A) + \mu(N') = 0$. Hence, C is a subset of an $\mathcal{A}$-measurable μ-null set. Since $C = \emptyset \cup C$ and $\emptyset \in \mathcal{A}$, it follows that $C \in \mathcal{A}^*$. Thus, $(X, \mathcal{A}^*, \bar{\mu})$ is complete.

Proof (v) Let $\mathcal{B} = \{A^* \subset X : \exists A, B \in \mathcal{A},\ A \subset A^* \subset B,\ \mu(B \setminus A) = 0\}$. We need to show that $\mathcal{A}^* = \mathcal{B}$. Let $A^* \in \mathcal{B}$, then there exist $A, B \in \mathcal{A}$ such that $A \subset A^* \subset B$ and $\mu(B \setminus A) = 0$. Notice that $A^* = A \cup (A^* \setminus A)$ with $A \in \mathcal{A}$, $A^* \setminus A \subset (B \setminus A) \in \mathcal{A}$ and $\mu(B \setminus A) = 0$. This implies that $A^* \in \mathcal{A}^*$, and hence $\mathcal{B} \subset \mathcal{A}^*$. Conversely, suppose $A^* \in \mathcal{A}^*$. Then, $A^* = A \cup N$ with $A \in \mathcal{A}$ and $N \subset N' \in \mathcal{A}$ satisfying $\mu(N') = 0$. Set $B = A \cup N'$. Then $B \in \mathcal{A}$, $A \subset A^* \subset B$ and $\mu(B \setminus A) \leq \mu(N') = 0$. Thus, $A^* \in \mathcal{B}$ and $\mathcal{A}^* \subset \mathcal{B}$. Therefore, $\mathcal{A}^* = \mathcal{B}$.