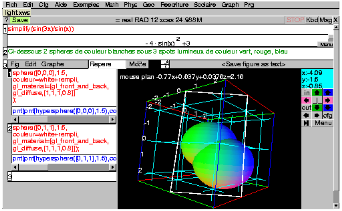
Introduction to Computer Algebra

Carlos D'Andrea

Oslo, December 1st 2016



What is Computer Algebra?

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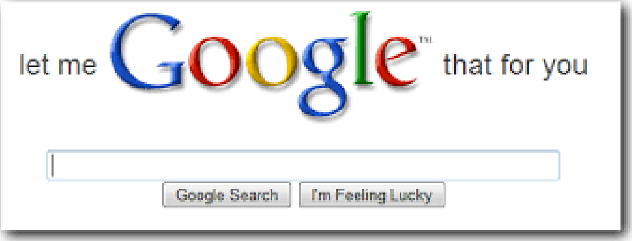

Computer Algebra

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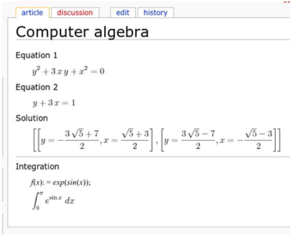

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Computer algebra

Equation 1

$y^2 + 3\,x\,y + x^2 = 0$

Equation 2

$y + 3\,x = 1$

Solution

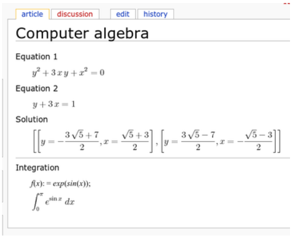
$\left[\left[y = -\frac{3\sqrt{5}+7}{2}, x = \frac{\sqrt{5}+3}{2}\right], \left[y = \frac{3\sqrt{5}-7}{2}, x = -\frac{\sqrt{5}-3}{2}\right]\right]$

Integration

$f(x) := exp(sin(x));$

$\int_0^{\pi} e^{\sin x}\, dx$

Computer Algebra



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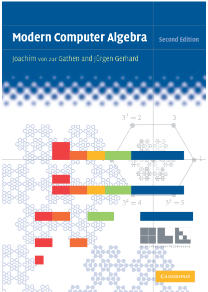

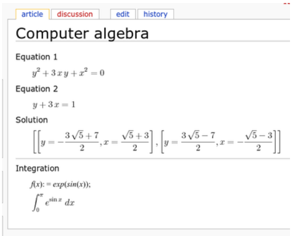

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f(x): = exp(sin(x));
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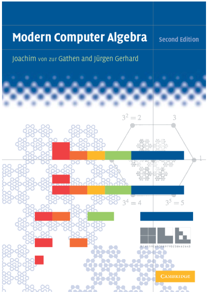
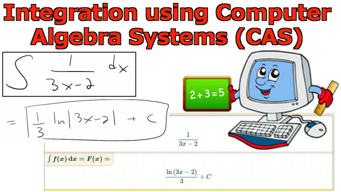
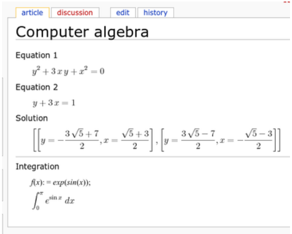


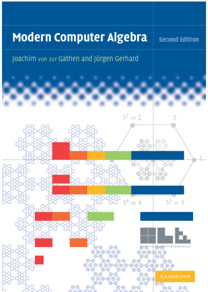
Modern Computer Algebra
Second Edition
Joachim von zur Gathen and Jürgen Gerhard
CAMBRIDGE

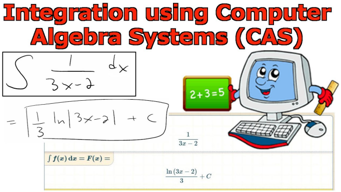


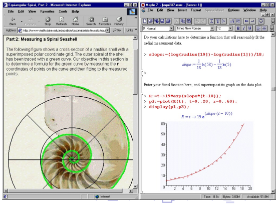
Integration using Computer Algebra Systems (CAS)
$\int \frac{1}{3x-2}\,dx$
$= \frac{1}{3}\ln|3x-2| + C$
2+3=5
$\frac{1}{3x-2}$
$\int f(x)\,dx = F(x) =$
$\frac{\ln(3x-2)}{3}+C$

Computer Algebra

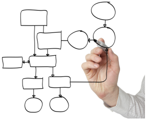








From Wikipedia (Symbolic Computation)



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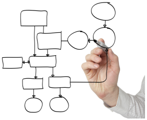

"...is a scientific area that refers to the study and development of

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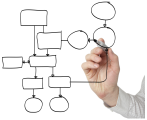

"...is a scientific area that refers to the study and development of **algorithms** and **software**

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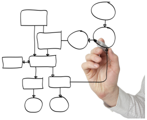

"...is a scientific area that refers to the study and development of **algorithms** and **software** for manipulating mathematical **expressions** and other mathematical **objects**..."

From Wikipedia (Symbolic Computation)

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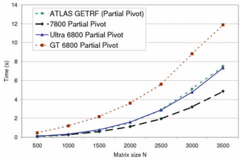
From Wikipedia (Symbolic Computation)

"A large part of the work in the field consists in revisiting classical algebra in order to make it **effective** and to discover **efficient** algorithms to implement this effectiveness"



“Effective” Operations

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- Finite-Dimensional Linear Algebra

What is “an algorithm” for CA?

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Given $A, B \in \mathbb{Z}$,

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Given $A,\ B \in \mathbb{Z}$,compute the product

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$$\begin{array}{r}
\overset{1}{5127} \\
\text{x } 4265 \\
\hline
25635 \\
307620 \\
1025400
\end{array}$$

$$\begin{array}{r}
453 \\
\times \; 1001205 \\
\hline
2265 \\
906\cdot \\
453 \\
453\cdot\cdot \\
\hline
453545865
\end{array}$$

Example: The Multiplication Algorithm

- Input: $A, B \in \mathbb{Z}$

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- Input: $A, B \in \mathbb{Z}$
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- Input: $A, B \in \mathbb{Z}$
- Output: $A \cdot B \in \mathbb{Z}$
- Procedure:

```
   36
 x 24
 ----
  144     (4 x 36 = 144)
+ 720     (2 x 36 = 72,
 ----     with a 0 added in
  864     the ones' position)
```

Effective and Efficient

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Effective: the procedure must finish after a finite number of operations, and give the right answer

Efficient: the procedure must be as **short** as possible and use as **less space** as possible

Short and less Space

- Input: $A, B \in \mathbb{Z}$

Short and less Space

- Input: $A, B \in \mathbb{Z}$ of N digits

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It is effective

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It is effective, but is it efficient?

Efficient Multiplication

Karatsuba's algorithm (1960): $\mathcal{O}(N^{1.585})$

Multiplication of large numbers					
X	9876256719 8712349865				
	0054739234 5643218574				Y
Decomposition					
X	L	9876256719 * 0054739234		L	Y
	R	+ 8712349865 * 5643218574 +		R	
	S	= 18588606584 * 5697957808 =		S	
Partial multiplications					
P	1	540618727585413246	=	X_L * Y_L	
	2	49165694581354392510	=	X_R * Y_R	
	3	105917096025143007872	=	X_S * Y_S	
Result					
P_1 * 10 ^ N + (P_3 - P_2 - P_1) * 10 ^ (N/2) + P_2					
N = Max(Length_X, Length_Y) and if N odd then N = N + 1					
54 061 873 320 649 151 811 197 715 741 354 392 510					

Toom-Cook's algorithm (1966):

$\mathcal{O}(N^{\log(2k-1)/\log k}),\ k \geq 3$

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Schönhage-Strassen's algorithm (1971): $\mathcal{O}(N \log N \log\log N)$

$$W = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & & \omega^{2n-1} \\ 1 & \omega^2 & \omega^4 & & \omega^{4n-2} \\ & & & & \\ 1 & \omega^{2n-1} & \omega^{4n-2} & & \omega^{2n^2-n} \end{pmatrix}$$

Efficient Multiplication

Fürer's algorithm (2007):
$\mathcal{O}(N \log N \, 2^{\mathcal{O}(\log^* N)})$

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Is it the most efficient??

Numerical vs Computer Algebra

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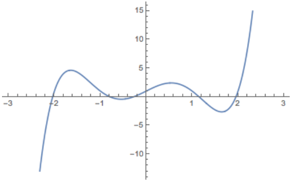
Numerical vs Computer Algebra

"As **numerical** software are highly efficient for approximate numerical computation, it is common, in **computer algebra**, to emphasize on **exact computation** with **exactly represented data**"

Compute the roots of

$$x^5 - 5x^3 + 4x + 1$$



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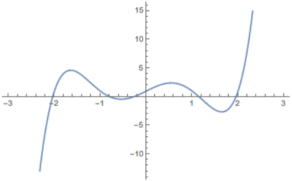

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Numerical vs Computer Algebra

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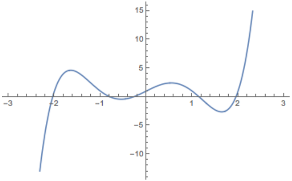

Numerical "solution":

$-2.0385; -0.790734; -0.275834; 1.15098; 1.95408$

Exactly Represented Data

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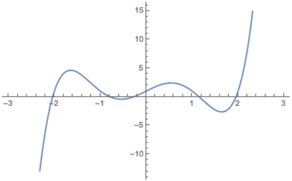

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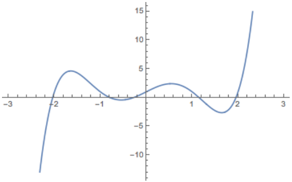

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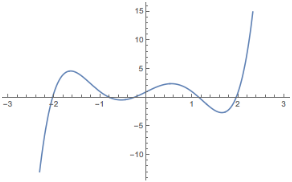
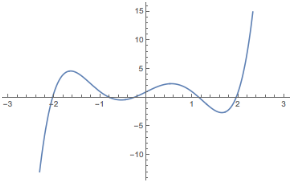


- $x^5 - 5x^3 + 4x + 1$
- $[-2.5, -2]; [-2, -0.5]; [-0.5, 0]; [0, 1.5]; [1.5, 2]$

Exactly Represented Data



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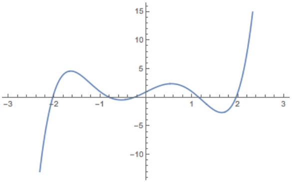
- $[-2.5, -2]; [-2, -0.5]; [-0.5, 0]; [0, 1.5]; [1.5, 2]$
- $\{+, -, +, -, +\}$, $\{-, +, +, -, +\}$, $\{+, +, -, -, +\}$, $\{-, -, +, +, +\}$, $\{+, +, +, +, +\}$

Exactly Represented Data



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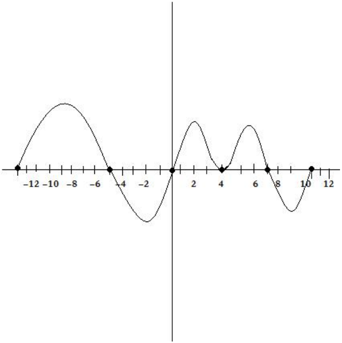
- $\{+, -, +, -, +\}$, $\{-, +, +, -, +\}$, $\{+, +, -, -, +\}$, $\{-, -, +, +, +\}$, $\{+, +, +, +, +\}$
- $-2.5;\ -0.75;\ -0.25;\ 1;\ 2$

Computational Challenge



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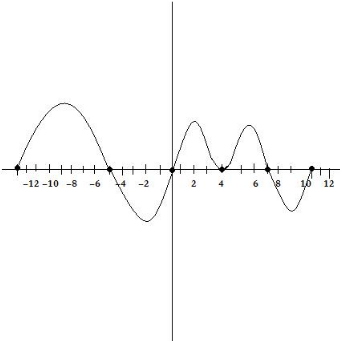
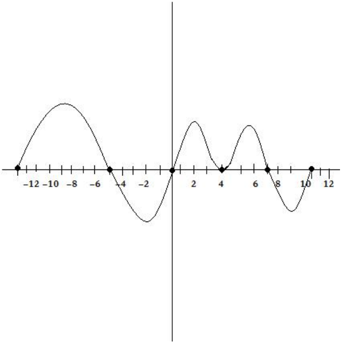


Given $f(x) \in \mathbb{Z}[x]$, compute a set of **small** approximate roots of it

Computational Challenge



Given $f(x) \in \mathbb{Z}[x]$, compute a set of **small** approximate roots of it **fast**

What is a small number?

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The size of an integer N is its number of digits:

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Size of Algebraic Numbers

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Given $\alpha \in \mathbb{C}$ the root of

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The "size" of α is

$$\left(n, \ \max\{\log |a_i|_{1 \leq i \leq n}\}\right)$$

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Given $\alpha \in \mathbb{C}$ the root of

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The "size" of α is

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if $(*)$ is the minimal polynomial of α

With this definition...

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- size of 2016 :

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- size of $2016: \ (1, \log 2016 \approx 11)$

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With this definition...

- size of 2016 : $(1, \log 2016 \approx 11)$
- size of $\sqrt{2}$: $(2, \log 2 = 1)$

With this definition...

- size of $2016: \ (1, \log 2016 \approx 11)$
- size of $\sqrt{2}: (2, \log 2 = 1)$
- size of $-\frac{1}{2} + \frac{\sqrt{3}}{2} i:$

With this definition...

- size of $2016 : \ (1, \log 2016 \approx 11)$
- size of $\sqrt{2} : (2, \log 2 = 1)$
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With this definition...

- size of 2016 : $(1, \log 2016 \approx 11)$
- size of $\sqrt{2}$: $(2, \log 2 = 1)$
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- size of $\sqrt[3]{2}$:

With this definition...

- size of 2016 : $(1, \log 2016 \approx 11)$
- size of $\sqrt{2}$: $(2, \log 2 = 1)$
- size of $-\frac{1}{2} + \frac{\sqrt{3}}{2} i$: $(2, \log 1 = 0)$
- size of $\sqrt[3]{2}$: $(3, \log 2 = 1)$

How does the size grow?

Over the integers: size of
N

How does the size grow?

Over the integers: size of $N = (1, \log N)$

How does the size grow?

Over the integers: size of

$$N = (1, \log N)$$

Size of

- $N_1 + N_2$:

How does the size grow?

Over the integers: size of

$$N = (1, \log N)$$

Size of

- $N_1 + N_2 : \quad (1, \max\{\log N_1, \log N_2\})$

How does the size grow?

Over the integers: size of

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Size of

- $N_1 + N_2 : \quad (1, \max\{\log N_1, \log N_2\})$
- $N_1 \cdot N_2 :$

How does the size grow?

Over the integers: size of

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Size of

- $N_1 + N_2: \quad (1, \max\{\log N_1, \log N_2\})$
- $N_1 \cdot N_2: \quad (1, \log N_1 + \log N_2)$

Size of algebraic numbers

Size of:

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Size of algebraic numbers

Size of:

- $\sqrt{2} \cdot \sqrt[3]{2} : (6, \log 32 = 5)$
  $(2^{\frac{1}{2}+\frac{1}{3}} = 2^{\frac{5}{6}}$

Size of algebraic numbers

Size of:

- $\sqrt{2}\cdot\sqrt[3]{2}$: $(6, \log 32 = 5)$ $(2^{\frac{1}{2}+\frac{1}{3}} = 2^{\frac{5}{6}} \leftrightarrow x^6 - 2^5 = 0)$
- $\sqrt{2}+\sqrt[3]{2}$:???

Computer Algebra helps!

$$\begin{cases} x^2 - 2 = 0 \\ (y - x)^3 - 2 = 0 \end{cases}$$

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$$\begin{cases} x^2 - 2 = 0 \\ (y-x)^3 - 2 = 0 \end{cases}$$

$$\downarrow$$

$$\begin{cases} x^2 - 2 = 0 \\ \mathbf{y^6 - 6y^4 - 4y^3 + 12y^2 - 24y = 4} \end{cases}$$

Computer Algebra helps!

$$\begin{cases} x^2 - 2 = 0 \\ (y-x)^3 - 2 = 0 \end{cases}$$

$$\downarrow$$

$$\begin{cases} x^2 - 2 = 0 \\ \mathbf{y^6 - 6y^4 - 4y^3 + 12y^2 - 24y = 4} \end{cases}$$

$$\boxed{\text{size of } \sqrt{2} + \sqrt[3]{2} = (6, \log 24 \approx 4.5)}$$

In general...

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if the sizes of α_1, α_2 are (d_1, L_1), (d_2, L_2),

In general...

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Triangulating systems of equations

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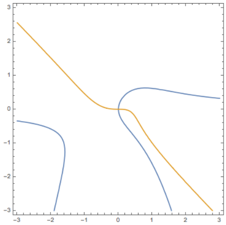

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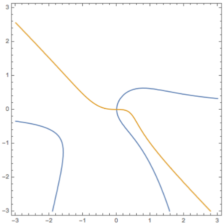

$$\begin{cases} x^3 + y^3 - xy + y = 0 \\ \mathbf{y^9 + 2y^7 + y^5 - 4y^4 + y = 0} \end{cases}$$

What can you get from here?

What can you get from here?

- Number of solutions of a system of equations

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- "Location" of the solutions

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- Factorization of polynomials, matrices, differential operators,...

“Triangulation” $=$ Elimination of variables

"Triangulation" = Elimination of variables

Find "the condition" on

$a_{10}, a_{11}, a_{20}, a_{21}$ so that the system

$$\begin{cases} a_{10}x_0 + a_{11}x_1 = 0 \\ a_{20}x_0 + a_{21}x_1 = 0 \end{cases}$$

has a solution different from $(0,0)$

The General System with Parameters

The General System with Parameters

For $\mathbf{a} = (a_1, \ldots, a_N)$, k, $n \in \mathbb{N}$ let $f_1(\mathbf{a}, x_1, \ldots, x_n), \ldots, f_k(\mathbf{a}, x_1, \ldots, x_n) \in \mathbb{K}[\mathbf{a}, x_1, \ldots, x_n]$. Find conditions on $\mathbf{a}$ such that

$$\left\{\begin{array}{rcl} f_1(\mathbf{a}, x_1, \ldots, x_n) & = & 0 \\ f_2(\mathbf{a}, x_1, \ldots, x_n) & = & 0 \\ \vdots & \vdots & \vdots \\ f_k(\mathbf{a}, x_1, \ldots, x_n) & = & 0 \end{array}\right.$$

has a solution

Solution?

- Depends on the ground field

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- There is not necessarily a “closed” condition

Solution?

- Depends on the ground field
- There is not necessarily a "closed" condition
- Tools from Geometry are needed!

The "simplest" example

$$k = n = 1,$$

$$a_0 + a_1 x_1 + a_2 {x_1}^2 + \ldots + a_d {x_1}^d = 0$$

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Conditions?

Known and "universal" examples

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$$\left\{\begin{array}{rcl} a_{11}x_1 + \ldots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + \ldots + a_{2n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{k1}x_1 + \ldots + a_{kn}x_n & = & 0 \end{array}\right.$$

with $k \geq n$

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$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + \ldots + a_{2n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{k1}x_1 + \ldots + a_{kn}x_n & = & 0 \end{cases}$$

with $k \geq n$

Conditions: all maximal minors of $\left(a_{ij}\right)_{1\leq i\leq k,\, 1\leq j\leq n}$ equal to zero

Another Classical Example

Another Classical Example

$$\begin{cases} a_{11}v_1 + \ldots + a_{1n}v_n &= \lambda v_1 \\ a_{21}v_1 + \ldots + a_{2n}v_n &= \lambda v_2 \\ \quad\quad\quad \vdots \quad\quad\quad\quad \vdots & \quad \vdots \\ a_{n1}v_1 + \ldots + a_{nn}v_n &= \lambda v_n \end{cases}$$

Another Classical Example

$$\begin{cases} a_{11}v_1 + \ldots + a_{1n}v_n & = & \lambda v_1 \\ a_{21}v_1 + \ldots + a_{2n}v_n & = & \lambda v_2 \\ \\ \vdots \qquad\qquad \vdots & & \vdots \\ a_{n1}v_1 + \ldots + a_{nn}v_n & = & \lambda v_n \end{cases}$$

<u>Condition:</u> $C_A(\lambda) = 0$

Geometry

$$V = \{(\boldsymbol{a}, x_1, \ldots, x_n) : f_1(\boldsymbol{a}, x_1, \ldots, x_n) = 0, \ldots f_k(\boldsymbol{a}, x_1, \ldots, x_n) = 0\}$$

Geometry

$$V = \{(a, x_1, \ldots, x_n) : f_1(a, x_1, \ldots, x_n) = 0, \ldots f_k(a, x_1, \ldots, x_n) = 0\}$$

$$\begin{array}{ccc} V & \subset & \mathbb{K}^N \times \mathbb{K}^n \\ & & \\ \downarrow \pi_1 & & \downarrow \pi_1 \\ & & \\ \pi_1(V) & \subset & \mathbb{K}^N \end{array}$$

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The set of conditions is $\pi_1(\mathbf{V})$, not necessarily described by zeroes of polynomials

Elimination Theorem

$$V = \{(\boldsymbol{a}, x_0, x_1, \ldots, x_n) : f_1(\boldsymbol{a}, x_0, x_1, \ldots, x_n) = 0, \ldots f_k(\boldsymbol{a}, x_0, x_1, \ldots, x_n) = 0\}$$

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$$\pi_1(V) = \{p_1(\mathbf{a}) = 0, \ldots, p_\ell(\mathbf{a}) = 0\}$$

"One" Condition

$$V = \{(\boldsymbol{a}, x_0, x_1, \ldots, x_n) : f_1(\boldsymbol{a}, x_0, x_1, \ldots, x_n) = 0, \ldots, f_{n+1}(\boldsymbol{a}, x_0, x_1, \ldots, x_n) = 0\}$$

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$$\boxed{\pi_1(V) = \{p_1(\mathbf{a}) = 0\}}$$

Example 1

$$\left\{\begin{array}{rcl} a_{00}x_0 + a_{01}x_1 + \ldots + a_{0n}x_n & = & 0 \\ a_{10}x_0 + a_{11}x_1 + \ldots + a_{1n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{n0}x_0 + a_{n1}x_1 + \ldots + a_{nn}x_n & = & 0 \end{array}\right.$$

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$$\boxed{p_1(a) = \det(a_{ij})}$$

Example 2

$$\begin{cases} f_1 = a_{10}{x_0}^{d_1} + a_{11}{x_0}^{d_1-1}x_1 + \ldots + a_{1d_1}{x_1}^{d_1} \\ f_2 = a_{20}{x_0}^{d_2} + a_{21}{x_0}^{d_2-1}x_1 + \ldots + a_{2d_2}{x_1}^{d_2} \end{cases}$$

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$$\boxed{p_1(a) = \det \begin{pmatrix} a_{10} & a_{11} & \ldots & a_{1d_1} & 0 & \ldots & 0 \\ 0 & a_{10} & \ldots & a_{1d_1-1} & a_{1d_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots & \ldots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{10} & \ldots & \ldots & a_{1d_1} \\ a_{20} & a_{21} & \ldots & a_{2d_2} & 0 & \ldots & 0 \\ 0 & a_{20} & \ldots & a_{2d_2-1} & a_{2d_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots & \ldots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{20} & \ldots & \ldots & a_{2d_2} \end{pmatrix}}$$

Example 3

$$\begin{cases} f_1 = \sum_{\alpha_0+\ldots+\alpha_n=d_1} a_{1,\alpha_0,\ldots,\alpha_n} x_0{}^{\alpha_0} \ldots x_n{}^{\alpha_n} \\ f_2 = \sum_{\alpha_0+\ldots+\alpha_n=d_2} a_{2,\alpha_0,\ldots,\alpha_n} x_0{}^{\alpha_0} \ldots x_n{}^{\alpha_n} \\ \vdots \\ f_{n+1} = \sum_{\alpha_0+\ldots+\alpha_n=d_{n+1}} a_{n+1,\alpha_0,\ldots,\alpha_n} x_0{}^{\alpha_0} \ldots x_n{}^{\alpha_n} \end{cases}$$

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$$\boxed{\mathrm{Res}(f_1, f_2, \ldots, f_{n+1})}$$

Effective tools

Linear **Polynomial**

Effective tools

Linear

Determinants

Polynomial

Resultants

Effective tools

Linear

Determinants

Cramer's rule

Polynomial

Resultants

u-resultants

Effective tools

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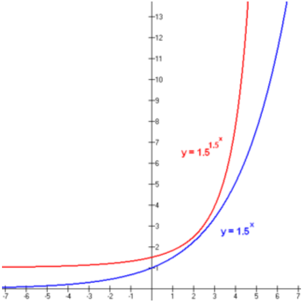
where size of $f_i = (d, L)$

is bounded by and generically equal to

$$\boxed{\left(d^n, nd^{n-1}L\right)}$$

The output is already exponential!!!



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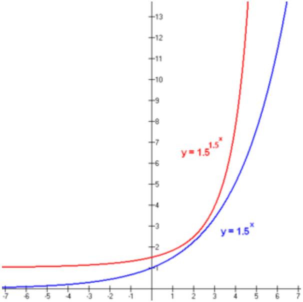
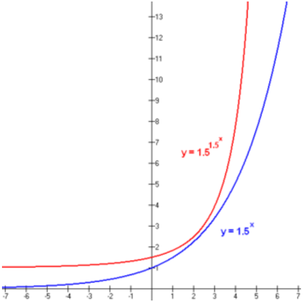


And moreover:

The output is already exponential!!!



And moreover: Complexity of Computing Gröbner bases is doubly exponential

Changing the model

Changing the model

- Probabilistic algorithms

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- Computations "over the Reals"

Changing the model

- Probabilistic algorithms
- Computations "over the Reals"
- Homotopy methods
- ...

Computing over the reals: BSS-machine

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is a **Random Access Machine**

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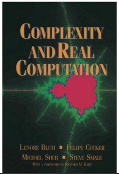
Computing over the reals: BSS-machine

is a **Random Access Machine** with registers that can store **arbitrary real numbers** and that can compute rational functions over reals at unit cost

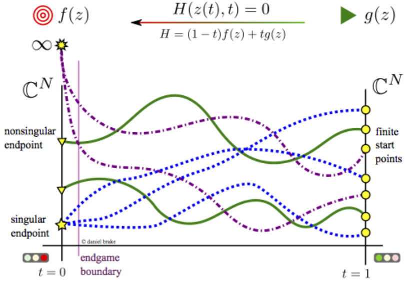


Steve Smale's 17th's problem (1998)

Steve Smale's 17th's problem (1998)

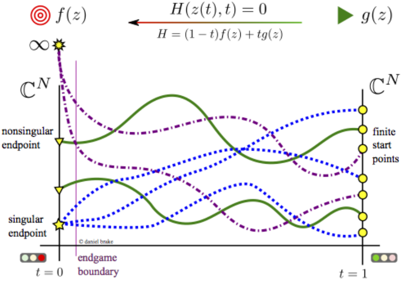
is there an algorithm which computes an **approximate solution** of a system of polynomials in **time polynomial on the average,** in the size of the input ?

Homotopies



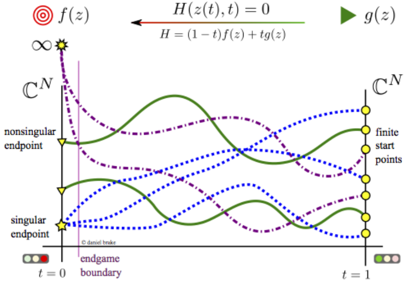
$f(z)$
$H(z(t),t) = 0$
$H = (1-t)f(z) + tg(z)$
$g(z)$
∞
$\mathbb{C}^N$
$\mathbb{C}^N$
nonsingular endpoint
finite start points
singular endpoint
© daniel brake
endgame boundary
$t = 0$
$t = 1$

Homotopies



- Start with an "easy" system

Homotopies



- Start with an "easy" system
- "Chase" the roots with an homotopy + Newton's method

Numerical Algebraic Geometry

By using homotopies, one can compute

Numerical Algebraic Geometry

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- Irreducible components

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Popular software in Computer Algebra

- Maple
- Mathematica
- Bertini
- CoCoA
- Macaulay2
- SageMath
- Singular

Maple

Maple

- Developed by MapleSoft

Maple

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- Core Team: Waterloo (Canada)

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- http://www.maplesoft.com/

Mathematica

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- Core Team: Champaign, IL (USA)

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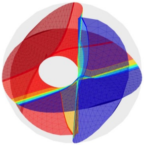
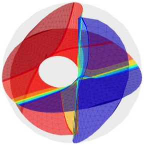
Bertini

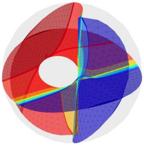


Bertini



- Free software

Bertini



- Free software
- Core Team: University of Notre Dame (USA)

Bertini

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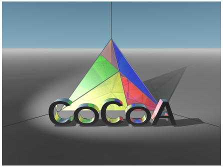
- https://bertini.nd.edu/

CoCoA



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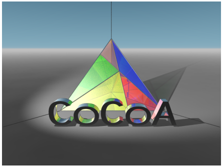

- Free software
- Core Team: University of Genoa (Italy)

CoCoA

- Free software
- Core Team: University of Genoa (Italy)
- http://cocoa.dima.unige.it/

Macaulay2

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Macaulay2

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http://www.math.uiuc.edu/Macaulay2/

SageMath

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- Free open-source

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- Core Team: Worlwide

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Singular

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- https://www.singular.uni-kl.de/

To Learn More about this Area...

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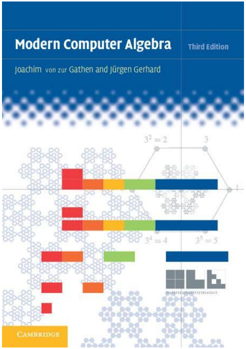

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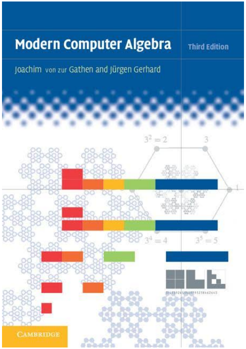

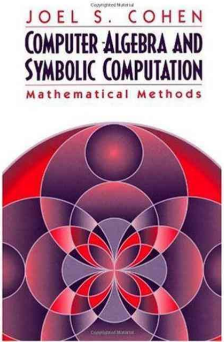

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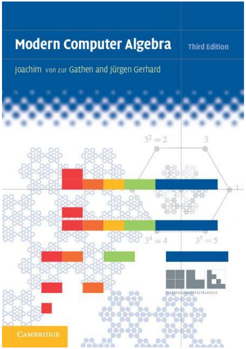

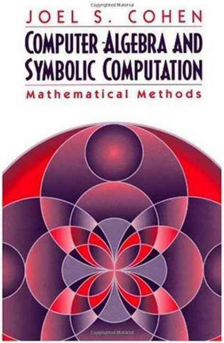

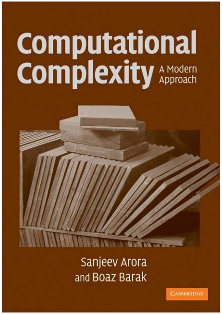

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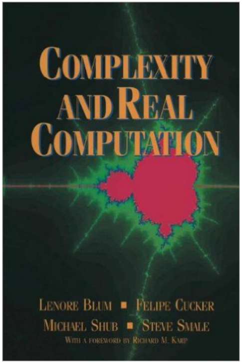

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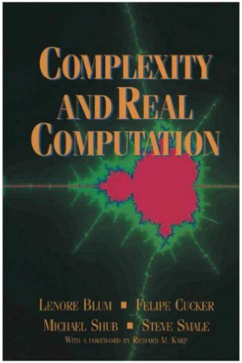

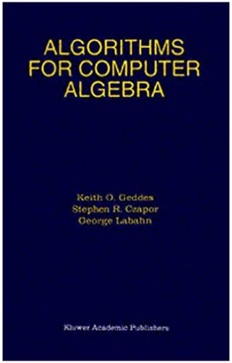

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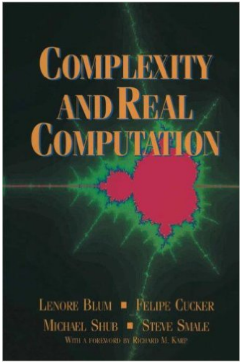

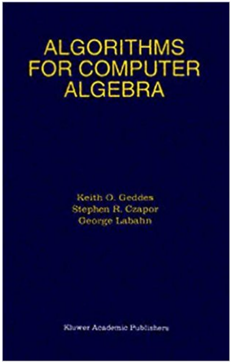

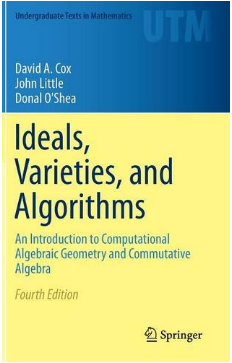

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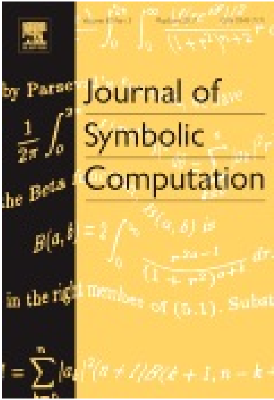

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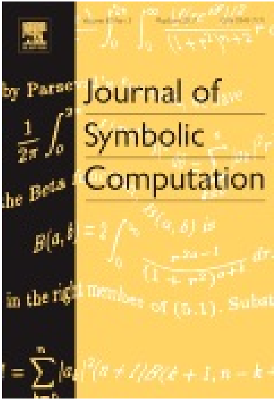

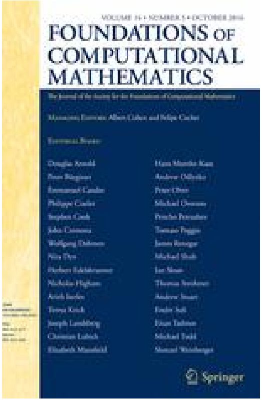

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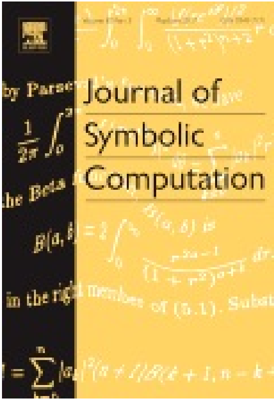

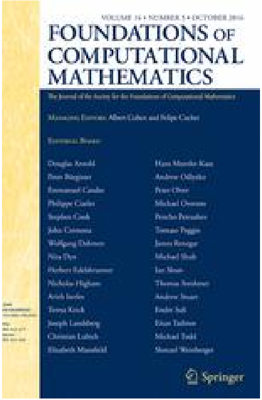

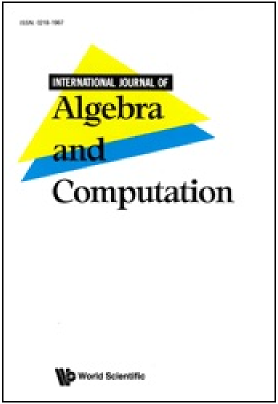

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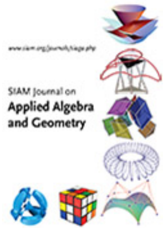

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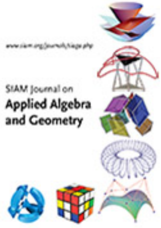

Thanks!