Extending Baire–one functions on topological spaces ⋆

O. F. K. Kalenda [1] and J. Spurný [2]

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75, Praha 8, Czech Republic

Abstract

We investigate the possibility of extension of Baire–one functions from subspaces of topological spaces. In particular we prove that any Baire–one function on a Lindelöf hereditarily Baire completely regular space can be extended to a Baire–one function on any completely regular superspace.

Key words: Baire–one function, Lindelöf space, hereditarily Baire space, extension, Michael space, compact convex set
2000 MSC: 54C20, 54C30, 54H05

1 Introduction

Much is known on the possibility of extending continuous functions on topological spaces. The classical Tietze theorem asserts that a topological space is normal if and only if any real–valued continuous function on a closed subset can be continuously extended to the whole space. Further, Čech–Stone compactification is defined via extensions of bounded continuous functions. In this paper we investigate possibility of extending Baire–one functions (i.e., pointwise limits of sequences of continuous functions).

This work was inspired by results of the second author [11]. He studied abstract Dirichlet problem for Baire–one functions (i.e., the possibility of extending a

⋆ Supported by the Research Project MSM 1132 00007 from the Czech Ministry of Education.
Email addresses: kalenda@karlin.mff.cuni.cz (O. F. K. Kalenda), spurny@karlin.mff.cuni.cz (J. Spurný).
[1] Supported by Research grant GA ČR 201/03/0933.
[2] Supported by Research grants GA ČR 201/03/0935 and GA ČR 201/03/D120.

Baire–one function defined on the set of extreme points of a compact convex set to an affine Baire–one function on the whole set). Some problems in this area remained open and it turns out to be worthwhile to better understand the situation in general topological spaces.

It is well–known that a Baire–one function on a G_δ–subset of a metric space can be extended to a Baire–one function defined on the whole space (see [6, § 35, VI]). As a simple example (see Example 18) shows this is not true for general topological spaces. On the other hand, it is easy to prove that this result is true for Lindelöf G_δ–subsets of completely regular spaces (see Theorem 10). However, this result is not satisfactory enough as, within topological spaces, the notion of G_δ–set is much more special than within metric spaces. A natural generalization of G_δ–sets are $(F \vee G)_\delta$–sets, i.e., sets of the form $\bigcap_n (F_n \cup G_n)$ with each F_n closed and G_n open. But another example (Example 21) presents a closed Lindelöf subset of a normal space such that the extension result is not valid.

The precise statement of our main result is the following (see Theorem 13).

Let Y be a Lindelöf hereditarily Baire subset of a completely regular space X and f be a Baire–one function on Y. Then there exists a Baire–one function g on X such that $f = g$ on Y.

If X is a hereditarily Baire space and $Y \subset X$ is a $(F \vee G)_\delta$–set, it is easy to see that Y is hereditarily Baire as well and thus Theorem 13 is applicable in the particular case of Lindelöf $(F \vee G)_\delta$–subsets of a hereditarily Baire space. In fact, the same is true for a more general class of sets, so called H_δ–sets (i.e., countable intersections of H–sets, see e.g. [6, § 12, II]).

Remark that our main theorem gives some new results even in case of separable metric spaces. For example, the Bernstein set is a hereditarily Baire Lindelöf space which is wildly non–measurable, however any Baire–one function on the Bernstein set can be extended to a Baire–one function on any completely regular superspace (in particular on $\mathbb{R}$).

The most important step in the proof of Theorem 13 is a separation result for countable intersections of cozero sets (Coz_δ–sets, see definitions below). Once we have this separation result, we are able to extend mappings of the first Borel class which have values in separable complete metric spaces. This is the content of Section 5.

The last section of the paper is devoted to an application in convex analysis (which motivated our research). Let X be a compact convex subset of a locally convex space and let $\operatorname{ext} X$ stand for the set of extreme points of X. We prove in Theorem 30 that any bounded Baire–one function defined on $\operatorname{ext} X$ can be extended to a bounded Baire–one function on X provided the set $\operatorname{ext} X$ is a

Lindelöf space. This fact yields a partial answer to a question left unsolved in [11] as we explain later.

2 Preliminaries

All topological spaces will be considered as Hausdorff. A subset A of a topological space X is called a *zero set* if $A = f^{-1}(\{0\})$ for a continuous real–valued function f on X. It is clear that such a function f can be chosen with values in $[0, 1]$. A *cozero set* is the complement of a zero set. It is easy to check that zero sets are preserved by finite unions and countable intersections. Hence cozero sets are preserved by finite intersections and countable unions. Countable unions of zero sets will be denoted by Zer_σ, countable intersections of cozero sets by Coz_δ. Note that any zero set is Coz_δ and any cozero set is Zer_σ.

Any zero set is closed and G_δ, any cozero set is open and F_σ. If X is normal, the converse implications hold as well. Completely regular spaces are exactly those in which cozero sets form a basis of the topology.

A real–valued function f on a space X is a *Baire–one function* (or a *function of the first Baire class*) if f is a pointwise limit of a sequence of continuous functions on X. As it is well known, the family $\mathcal{B}^1(X)$ of all Baire–one functions on X forms a vector space which contains the space of all continuous functions $\mathcal{C}(X)$ and which is closed with respect to the uniform convergence. Moreover, $f \cdot g$ and $\max(f, g)$ are Baire–one functions whenever $f, g \in \mathcal{B}^1(X)$.

If $\mathcal{A}$ is a family of sets in X, a mapping $f : X \to P$ from X to a space P is *$\mathcal{A}$–measurable* if $f^{-1}(U) \in \mathcal{A}$ for every open $U \subset P$. If $\mathcal{A}$ is the family of all F_σ–sets in X, the mapping f is said to be *of the first Borel class*.

We recall that a topological space X is a *Baire space* if the intersection of any sequence of open dense subsets of X is dense in X. If every closed subset of X is also a Baire space, X is said to be *hereditarily Baire*. A set $A \subset X$ is *of the first category in* X if A can be covered by countably many nowhere dense subsets of X. The complement of a set of the first category in X is a *residual* set in X.

We will denote by $\mathbb{N}^{<\mathbb{N}}$ the set of all finite sequences of positive integers, $\emptyset$ denotes the empty sequence, $|s|$ the length of the sequence s and $s^\wedge n$ the sequence made from s by adding the element n at the end as the last element. For a sequence $\sigma \in \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$ we write $\sigma \restriction n$ for the sequence $(\sigma_1, \ldots, \sigma_n)$. If σ, τ are sequences in $\mathbb{N}^\mathbb{N}$, we write $\sigma \leq \tau$ if $\sigma_n \leq \tau_n$ for every $n \in \mathbb{N}$. If $\mathcal{A}$ is a family of sets in a space X, a set $A \subset X$ is said to be the *result of the*

Souslin operation applied on sets from $\mathcal{A}$ if there exists a family

$$\{F_s : s \in \mathbb{N}^{<\mathbb{N}}\} \subset \mathcal{A}$$

such that

$$A = \bigcup_{\sigma \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} F_{\sigma \restriction n} \, .$$

If X is a topological space and every set F_s, $s \in \mathbb{N}^{<\mathbb{N}}$, is closed, we simply say that A is a *Souslin set*.

A topological space X is said to be $\mathcal{K}$*–countably determined* if X is the image of a set $S \subset \mathbb{N}^{\mathbb{N}}$ under an upper semicontinuous compact–valued mapping. According to [10, Section 5.1], a completely regular space is $\mathcal{K}$–countably determined if and only if there exists $S \subset \mathbb{N}^{\mathbb{N}}$ and closed sets $\{F_s : s \in \mathbb{N}^{<\mathbb{N}}\}$ in some compactification of X such that

$$X = \bigcup_{\sigma \in S} \bigcap_{n=1}^{\infty} F_{\sigma \restriction n} \, .$$

We remark that any separable metric space or a Souslin subset of a compact space is a $\mathcal{K}$–countably determined space.

If f is a real–valued function on a set X and $a \in \mathbb{R}$, we write $[f \geq a]$ for the set $\{x \in X : f(x) \geq a\}$. Similarly we use $[f \leq a]$, $[f < a]$, $[f > a]$ and $[f = a]$.

Proposition 1. *Let f be a real–valued function on a topological space X. Then f is of the first Baire class if and only if f is* Zer_σ*–measurable.*

If X is moreover normal, then f is of the first Baire class if and only if f is F_σ–measurable.

Proof. See [7, Exercise 3.A.1]. □

Proposition 2. *If A is a* Coz_δ*–subset of a space X, then there exists a Baire–one function f with values in* $[0, 1]$ *such that* $A = [f = 0]$.

If A and B is a pair of disjoint Coz_δ*–subsets of X, then there exists a Baire–one function f on X with values in* $[0, 1]$ *such that* $A = [f = 0]$ *and* $B = [f = 1]$.

Proof. First observe that the characteristic function χ_U of a set U is of the first Baire class whenever U is a cozero subset of X. Indeed, let $h : X \to [0, 1]$ be continuous with $U = [h > 0]$. Then $h_n = \sqrt[n]{h}$, $n \in \mathbb{N}$, is a sequence of continuous functions pointwise converging to χ_U.

Further, let $A = \bigcap_n U_n$ where each U_n is a cozero subset of X. Then, by the previous paragraph, the characteristic function χ_{U_n} is a Baire–one function for every $n \in \mathbb{N}$. Then

$$f := 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{U_n}$$

satisfies $A = [f = 0]$. Moreover, f is Baire–one as it is a uniform limit of Baire–one functions.

Concerning the second assertion, given a couple A and B of disjoint Coz_δ–subsets of X, let f_1 and f_2 be Baire–one functions on X with values in $[0,1]$ such that

$$A = [f_1 = 0] \quad \text{and} \quad B = [f_2 = 0] \, .$$

Then the function

$$f := \frac{f_1}{f_1 + f_2}$$

has the required properties. (If $f_1 = \lim_n g_n^1$ and $f_2 = \lim_n g_n^2$ with g_n^i continuous, then $f = \lim_n \frac{g_n^1}{n^{-1} + \max(g_n^1 + g_n^2, 0)}$ and thus f is Baire–one.) □

Proposition 3. *Let $f : X \to P$ be a mapping of a normal space X to a metric space P. Then f is F_σ–measurable if and only if f is Zer_σ–measurable.*

Proof. Obviously, any Zer_σ–measurable mapping is F_σ–measurable. For the proof of the converse implication, assume that f is an F_σ–measurable mapping. Given an open set $U \subset P$, we consider a continuous function $\varphi : P \to \mathbb{R}$ defined as

$$\varphi(p) := \mathrm{dist}(p, P \setminus U) \, , \quad p \in P \, .$$

Then $\varphi \circ f$ is an F_σ–measurable function from X to $\mathbb{R}$ and thus it is of the first Baire class due to Proposition 1. Hence $\varphi \circ f$ is Zer_σ–measurable. Thus

$$\begin{aligned} f^{-1}(U) &= \{x \in X : \mathrm{dist}(f(x), P \setminus U) > 0\} = \{x \in X : \varphi(f(x)) \in (0, \infty)\} \\ &= [\varphi \circ f > 0] \end{aligned}$$

is a Zer_σ–set. □

Proposition 4. *Let X be a completely regular space.*

(a) If $A \subset B \subset X$, A is Lindelöf and B is a G_δ–set, then there exists a Coz_δ–set C so that $A \subset C \subset B$.
(b) Any Lindelöf G_δ–set A is Coz_δ.
(c) If $Y \subset X$ is Lindelöf and A is a Coz_δ–subset of Y, then there is a Coz_δ–subset $\widehat{A}$ of X with $Y \cap \widehat{A} = A$.

Proof. For the proof of (a), given a Lindelöf set A and an open set G with $A \subset G$, using the Lindelöf property we can find a cozero set U such that $A \subset U \subset G$. From this observation the assertion (a) easily follows.

Since (b) is an immediate consequence of (a), we proceed to the proof of (c). Obviously it is enough to show that any cozero subset of Y is a trace of some cozero subset of X. But this easily follows from the fact that cozero subsets of Y are relatively open and Lindelöf. $\square$

Later on we will need an information whether a Coz_δ–subset of a Lindelöf space is also Lindelöf. Since the so–called Michael space shows that this is not true in general (see Example 22), we have collected below a few conditions ensuring this property.

Proposition 5. *Let X be a topological space such that $X \times \mathbb{N}^\mathbb{N}$ is Lindelöf. Then any Coz_δ–subset of X is Lindelöf.*

In particular, any Coz_δ–subset of X is Lindelöf if X is hereditarily Lindelöf (i.e., every open subset of X is Lindelöf) or X is $\mathcal{K}$–countably determined.

Proof. Assume that A is a Coz_δ–subset of X and $X \times \mathbb{N}^\mathbb{N}$ is Lindelöf. Since A is a Souslin subset of X, there exists a closed set H in $X \times \mathbb{N}^\mathbb{N}$ such that $A = \pi_X(H)$ where π_X denotes the projection onto the first coordinate (see e.g. the proof of [4, Theorem 5.2]). As $X \times \mathbb{N}^\mathbb{N}$ is Lindelöf, H is Lindelöf as well. Thus A, as the continuous image of a Lindelöf space, is Lindelöf.

Let X be a hereditarily Lindelöf space. We will check that $X \times \mathbb{N}^\mathbb{N}$ is Lindelöf. To this end, let $\mathcal{U}$ be an open cover of $X \times \mathbb{N}^\mathbb{N}$. Fix a countable basis $\mathcal{B}$ of $\mathbb{N}^\mathbb{N}$. Obviously, we may suppose that $\mathcal{U}$ consists of sets of the form $U \times B$, where U is open in X and B is an element of $\mathcal{B}$. For every $B \in \mathcal{B}$ set

$$\mathcal{U}_B := \{U \subset X : U \times B \in \mathcal{U}\} .$$

Using the assumption we select a countable subfamily $\mathcal{C}_B$ from $\mathcal{U}_B$ so that

$$\bigcup \mathcal{C}_B = \bigcup \mathcal{U}_B .$$

Then

$$\{U \times B : U \in \mathcal{C}_B \ , \ B \in \mathcal{B}\}$$

is a countable subfamily of $\mathcal{U}$ covering $X \times \mathbb{N}^\mathbb{N}$.

If X is a $\mathcal{K}$–countably determined space, $X \times \mathbb{N}^\mathbb{N}$, as the product of $\mathcal{K}$–countably determined spaces, is $\mathcal{K}$–countably determined as well. Since any $\mathcal{K}$–countably determined space is Lindelöf (see [10, Section 2.7]), the proof is completed. $\square$

Remark 6. A regular Lindelöf space, whose product with $\mathbb{N}^\mathbb{N}$ is not Lindelöf, is called a *Michael space* and first was constructed by E. Michael in [8] under the Continuum Hypothesis. In Example 22 we use his construction in order to show that there are Coz_δ–subsets of a regular Lindelöf space which are not

Lindelöf. It seems to be an open question whether it is possible to construct a Michael space in ZFC.

3 Extension of Baire–one functions

In this section we prove our main result on extending Baire–one functions. We begin by the following proposition showing equivalence of the possibility of extending bounded Baire–one functions and the possibility to separate relative Coz_δ–sets.

Proposition 7. *Let X be a topological space and $Y \subset X$. Then the following assertions are equivalent.*

(i) For any bounded Baire–one function f on Y there is a Baire–one function g on X extending f such that $\inf f(Y) = \inf g(X)$ and $\sup f(Y) = \sup g(X)$.

(ii) Any bounded Baire–one function on Y can be extended to a Baire–one function on X.

(iii) For any pair A, B of disjoint Coz_δ–subsets of Y there are disjoint Coz_δ–subsets $\widehat{A}$, $\widehat{B}$ of X such that $A = \widehat{A} \cap Y$ and $B = \widehat{B} \cap Y$.

Proof. The implication (i) $\Longrightarrow$ (ii) is trivial. For the proof of (ii)$\Longrightarrow$ (iii), let A, B be disjoint Coz_δ–subsets of Y. By Proposition 2 there is a Baire–one function $f : Y \to [0, 1]$ satisfying $A = [f = 0]$ and $B = [f = 1]$. Let $g : X \to \mathbb{R}$ be a Baire–one function extending f. Then $\widehat{A} = [g = 0]$ and $\widehat{B} = [g = 1]$ have the required properties.

(iii)$\Longrightarrow$(i). Set

$$t := \begin{cases} f & \text{on } Y\ , \\ \inf f(Y) & \text{on } X \setminus Y\ , \end{cases} \qquad \text{and} \qquad s := \begin{cases} f & \text{on } Y\ , \\ \sup f(Y) & \text{on } X \setminus Y\ . \end{cases}$$

According to [7, Theorem 3.2], there exists a Baire–one function g on X satisfying $s \le g \le t$ if and only if the following condition is satisfied: *given a couple of real numbers $a < b$, there is a Baire–one function φ on X such that*

$$\varphi = 0 \quad \textit{on} \quad A := [s \le a] \qquad \textit{and} \qquad \varphi = 1 \quad \textit{on} \quad B := [t \ge b]\ . \tag{1}$$

So assume that $a < b$ are given. Without loss of generality we may suppose that

$$\inf f(Y) \le a < b \le \sup f(Y)\ .$$

Then A and B is a couple of Coz_δ–subsets of Y. By (iii) we can find disjoint Coz_δ–sets $\widehat{A}$ and $\widehat{B}$ in X such that $A \subset \widehat{A}$ and $B \subset \widehat{B}$. Then Proposition 2 provides a Baire–one function φ with the required property (1).

Thus there is a Baire–one function g on X such that $t \le g \le s$. Obviously, g is the sought extension. $\square$

Next we give a similar characterization of the possibility to extend all Baire–one functions (not necessarily bounded).

Proposition 8. *Let X be a topological space and $Y \subset X$. Then the following assertions are equivalent.*

(i) For any Baire–one function f on Y there is a Baire–one function g on X extending f such that $\inf f(Y) = \inf g(X)$ and $\sup f(Y) = \sup g(X)$.
(ii) Any Baire–one function on Y can be extended to a Baire–one function on X.
(iii) For any Coz_δ–subset A of Y there is a Coz_δ–subset $\widehat{A}$ of X with $A = \widehat{A} \cap Y$, and, moreover, for any Coz_δ–subset G of X disjoint with Y there is a Coz_δ–set $H \subset X$ satisfying $Y \subset H \subset X \setminus G$.

Proof. The implication (i)$\Longrightarrow$(ii) is obvious. In order to prove (ii)$\Longrightarrow$(iii), pick a Coz_δ–subset A of Y. By Proposition 2 there is a Baire–one function f on Y with $A = [f = 0]$. If g is a Baire–one extension of f defined on X, then $\widehat{A} = [g = 0]$ is a Coz_δ–subset of X with $\widehat{A} \cap Y = A$.

Further, let $G \subset X$ be a Coz_δ–set disjoint with Y. Proposition 2 provides a Baire–one function $h : X \to [0, 1]$ with $G = [h = 0]$. By setting

$$\varphi(t) = \frac{t}{1 + |t|}\,, \quad t \in \mathbb{R}\,,$$

we obtain homeomorphism of $\mathbb{R}$ onto $(-1, 1)$. The function

$$f := (\varphi^{-1}) \circ (1 - h) \restriction Y$$

is a Baire–one function on Y. Let g be a Baire–one extension of f defined on X. Then

$$H := \{x \in X : \ \varphi(g(x)) = 1 - h(x)\} = [\varphi \circ g - 1 + h = 0]$$

is a Coz_δ–set containing Y and disjoint with G.

(iii)$\Longrightarrow$ (i). First we claim that the condition (iii) of Proposition 7 holds. Indeed, let A, B be disjoint Coz_δ–subsets of Y. The hypothesis yields the existence of Coz_δ–subsets A_0, B_0, of X such that $A = A_0 \cap Y$ and $B = B_0 \cap Y$. Then $G := A_0 \cap B_0$ is a Coz_δ–set in X which is disjoint with Y. Due to our assumption there is a Coz_δ–set $H \subset X$ such that $Y \subset H \subset X \setminus G$. Then

$\widehat{A} := A_0 \cap H$ and $\widehat{B} := B_0 \cap H$ are disjoint Coz_δ–sets satisfying $A = \widehat{A} \cap Y$ and $B = \widehat{B} \cap Y$. Proposition 7 finishes the proof for bounded Baire–one functions on Y.

Assume now that f is a (possibly unbounded) Baire–one function on Y. Obviously we may assume that f is nonconstant and thus there is a homeomorphism $\varphi : \mathbb{R} \to (-1, 1)$ such that

$$\inf(\varphi \circ f)(Y) < 0 < \sup(\varphi \circ f)(Y) .$$

Then $\varphi \circ f$ is a bounded Baire–one function on Y, and hence we can find a Baire–one function h on X such that $h = \varphi \circ f$ on Y,

$$\sup h(X) = \sup(\varphi \circ f)(Y) \leq 1 \quad \text{and} \quad -1 \leq \inf(\varphi \circ f)(Y) = \inf h(X) .$$

By setting

$$G := h^{-1}(\{-1\} \cup \{1\})$$

we obtain a Coz_δ–subset of X which is disjoint with Y. According to the assumption, there is a Coz_δ–set H in X such that

$$Y \subset H \subset X \setminus G .$$

Proposition 2 yields the existence of a Baire–one function ψ on X with values in $[0, 1]$ such that $\psi = 1$ on H and $\psi = 0$ on G. One can readily verify that

$$g := \varphi^{-1} \circ (h \cdot \psi)$$

is a Baire–one function on X which satisfies our requirements. This concludes the proof. $\square$

Remark 9. Let X be an absolute Souslin metric space (i.e., X is a Souslin set in the completion $\widehat{X}$ of X) and Y be a subset of X such that the complement $X \setminus Y$ is a Souslin set in X. Then any Baire–one function on Y is extensible on X if and only if Y is a G_δ–subset of X.

Indeed, sufficiency of the condition was already mentioned in the introduction (see also Theorem 10(c)). Concerning the necessity, assume that $X \setminus Y$ is not an F_σ–set in X. Since $\widehat{X} \setminus Y$ is a Souslin set in $\widehat{X}$, due to [5, Theorem 2(d)] there exists a compact set $F \subset X$ such that $F \cap Y$ is countable and

$$\overline{F \setminus Y} = \overline{F \cap Y} = F .$$

Let $A \subset F \cap Y$ be a dense subset of $F \cap Y$ such that $(F \cap Y) \setminus A$ is also dense in $F \cap Y$. Then $f := \chi_A$ is a Baire–one function on $F \cap Y$. As $F \cap Y$ is a G_δ–set in Y, the function f can be extended to a Baire–one function g on Y. Nevertheless, g cannot be extended to a Baire–one function on F because it is impossible to find a couple of disjoint G_δ–sets in F containing A and $(F \cap Y) \setminus A$, respectively.

Now we are ready to prove the following theorem on extending Baire–one functions in some easy cases.

Theorem 10. *Let X be a topological space, $Y \subset X$ and*

(a) Y is a cozero subset of X, or
(b) X is completely regular and Y is its Lindelöf G_δ–subset, or
(c) X is a metric space and Y is its G_δ–subset.

Then for any Baire–one function f on Y there is a Baire–one function g on X such that $f = g$ on Y,

$$\inf f(Y) = \inf g(X) \quad \textit{and} \quad \sup f(Y) = \sup g(X)\ .$$

Proof. In all three cases we are going check that the assertion (iii) of Proposition 8 is valid.

(a) In this case the second part is trivial as Y itself is Coz_δ. As for the first part, we verify that any Coz_δ–subset of Y is Coz_δ in X as well. To see this it is enough to observe that a cozero subset of Y is cozero in X. To this end, let A be a cozero subset of X, $g : Y \to [0,1]$ and $h : X \to [0,1]$ continuous functions with $A = [g > 0]$ and $Y = [h > 0]$. Define the function $f : X \to [0,1]$ by

$$f(x) = \begin{cases} g(x) \cdot h(x)\ , & x \in Y\ , \\ 0\ , & x \in X \setminus Y\ . \end{cases}$$

Then $A = [f > 0]$ and f is clearly continuous on X.

(b) The first part follows from Proposition 4(c). The second part is trivial as Y is Coz_δ by Proposition 4(b).

(c) The both requirements of (iii) in Proposition 8 are obviously fulfilled because any G_δ–subset of a metric space is also a Coz_δ–set. □

We continue by a key result on separating disjoint Lindelöf sets which enables us to prove deeper extension results.

Proposition 11. *Let A and B be a couple of disjoint Lindelöf subsets of a completely regular space X.*

If there is no Coz_δ–set G satisfying $A \subset G \subset X \setminus B$, then there exists a nonempty closed set $H \subset X$ such that $\overline{H \cap A} = \overline{H \cap B} = H$.

Proof. Assume that such a set G does not exist. We set

$$\widehat{B} := \{x \in B : \text{there exist an open set } U_x \text{ containing } x \text{ and a } \mathrm{Zer}_\sigma\text{–set} \\ F_x \text{ such that } F_x \cap A = \emptyset \text{ and } B \cap U_x \subset F_x\} \,. \tag{2}$$

If $\widehat{B} = B$, by the Lindelöf property we may find countably many $x_n \in B$, $n \in \mathbb{N}$, such that $B \subset \bigcup_n U_{x_n}$. Then $F := \bigcup_n F_{x_n}$ is a Zer_σ–set disjoint with A which covers B. Hence $G := X \setminus F$ is a Coz_δ–set separating A from B, a contradiction with our assumption.

Thus $B \setminus \widehat{B}$ is a nonempty set. We set

$$H := \overline{B \setminus \widehat{B}} \,.$$

We claim that H is the desired set. Since $H \cap B$ is obviously dense in H, we have to verify that $\overline{A \cap H} = H$.

Assuming the contrary, we may find a point $b_0 \in B \cap H$ and a cozero set U such that $b_0 \in U$ and $A \cap H \cap U = \emptyset$, in other words, $A \cap U \subset U \setminus H$. For every $a \in A \cap U$ we find a cozero set V_a containing a such that $V_a \cap H = \emptyset$. Since $A \cap U$ is Lindelöf, we can select countably many $a_n \in A \cap U$, $n \in \mathbb{N}$, so that

$$A \cap U \subset \bigcup_{n=1}^{\infty} V_{a_n} \,.$$

Then

$$V := \bigcup_{n=1}^{\infty} V_{a_n}$$

is a cozero set containing $A \cap U$ which is disjoint with H. Since $B \cap V \subset \widehat{B}$, for every $b \in B \cap V$ we use the property (2) and find its open neighbourhood U_b and a Zer_σ–set F_b such that $F_b \cap A = \emptyset$ and $B \cap U_b \subset F_b$. Using the Lindelöf property of $B \cap V$ we choose countably many points $b_n \in B \cap V$, $n \in \mathbb{N}$, so that

$$B \cap V \subset \bigcup_{n=1}^{\infty} U_{b_n} \,.$$

Then

$$F := (U \setminus V) \cup \bigcup_{n=1}^{\infty} F_{b_n}$$

is a Zer_σ–set which does not intersect A and does cover $B \cap U$. Thus b_0 is contained in $\widehat{B}$ which is a contradiction.

Hence $A \cap H$ is dense in H and the proof is finished. □

Proposition 12. *Let A be a Lindelöf hereditarily Baire subspace of a completely regular space X. If B is a Coz_δ–set in X disjoint with A, then there exists a Coz_δ–set $G \subset X$ such that $A \subset G \subset X \setminus B$.*

Proof. We claim that we can consider the space X to be even compact. Indeed, let βX stand for the Čech–Stone compactification of X and $B = \bigcap_n B_n$ where $B_n = f_n^{-1}(\mathbb{R} \setminus \{0\})$, $n \in \mathbb{N}$, for some bounded continuous functions f_n on X. We denote by $\widehat{f}_n$ the continuous extension of f_n on βX. Then

$$\widehat{B} := \bigcap_{n=1}^{\infty} \widehat{f}_n^{-1}(\mathbb{R} \setminus \{0\})$$

is Coz_δ–subset of βX disjoint with A. If we are able to separate A from B by a Coz_δ–subset $\widehat{G}$ of βX, the trace $\widehat{G} \cap X$ is a Coz_δ–subset of X separating A from B. This justifies our additional hypothesis that X is a compact space.

We assume that such a set G is impossible to find. Since B is a Lindelöf subset of X due to Proposition 5, Proposition 11 provides a nonempty closed set $H \subset X$ such that

$$\overline{H \cap A} = \overline{H \cap B} = H \ .$$

Then $H \cap B$ is a dense G_δ–set in H and thus it is a residual set in H. Hence $H \cap A$ is of the first category in H and consequently $H \cap A$ is of the first category in itself. (Note that for any set $F \subset H$ nowhere dense in H the set $F \cap A$ is nowhere dense in A.) But this contradicts the fact that $H \cap A$ is a Baire space. Thus our assumption is false which concludes the proof. □

Theorem 13. *Let Y be a Lindelöf hereditarily Baire subset of a completely regular space X and f be a Baire–one function on Y. Then there exists a Baire–one function g on X such that $f = g$ on Y,*

$$\inf f(Y) = \inf g(X) \quad \textit{and} \quad \sup f(Y) = \sup g(X) \ .$$

Proof. It is enough to check that the assertion (iii) of Proposition 8 is satisfied. The first part follows again from Proposition 4(c). The second part is a consequence of Proposition 12. □

Since any $(F \vee G)_\delta$–subset of a hereditarily Baire space is also hereditarily Baire, we get the following corollary.

Corollary 14. *Let Y be a Lindelöf $(F \vee G)_\delta$–subset of a hereditarily Baire space X. Then any Baire–one function f on Y can be extended to a Baire–one function g on X so that $f = g$ on Y,*

$$\inf f(Y) = \inf g(X) \quad \textit{and} \quad \sup f(Y) = \sup g(X) \ .$$

Remark 15. One is tempted to investigate the question whether every Lindelöf $(F \vee G)_\delta$–set is even a Coz_δ–set. An affirmative answer would yield an easier proof of Corollary 14. But this is not true since the "one–point lindelöfication" $X := \{\omega\} \cup Y$ of an uncountable discrete space Y is of type $F \cup G$

in the Čech–Stone compactification βX of X and X is not a $\mathcal{K}$–countably determined space (cf. e.g. [3]), in particular, X is not a Coz_δ–subset of βX.

The result of Corollary 14 holds also for a more general class of so called H_δ–sets. These are countable intersections of H–sets. Properties of H–sets are described for example in [6, § 12, II]. Let us recall one of the equivalent definitions. A set $A \subset X$ is a H–set if for any nonempty $B \subset X$ there is a nonempty relatively open set $V \subset B$ such that either $V \subset A$ or $V \cap A = \emptyset$. It is easy to check that the family of H–sets is an algebra containing open sets, and hence any $(F \vee G)_\delta$–set is H_δ. H–sets can be described explicitly as scattered unions of sets of the form $F \cap G$ with F closed and G open. One can readily verify that any H_δ–subset of a hereditarily Baire space is again hereditarily Baire, hence $(F \vee G)_\delta$ can be replaced by H_δ in Corollary 14.

We now formulate one more theorem on extending of Baire–one functions.

Theorem 16. *Let X be a Lindelöf completely regular space such that its each Coz_δ–subset is Lindelöf. Let Y be a Lindelöf H–subset of X. Then any Baire–one function f on Y can be extended to a Baire–one function g on X so that $f = g$ on Y,*

$$\inf f(Y) = \inf g(X) \quad \textit{and} \quad \sup f(Y) = \sup g(X) \ .$$

In particular it is true if Y is Lindelöf and belongs to the algebra generated by open sets in X.

Proof. We need to check that the condition (iii) of Proposition 8 is fulfilled. The first part follows from Proposition 4(c). In order to show the second part, pick Coz_δ–set $G \subset X$ disjoint with Y. Then G is Lindelöf by the hypothesis. If there is no Coz_δ–set B such that $Y \subset B \subset X \setminus G$, Proposition 11 provides a nonempty closed set H with $H = \overline{H \cap Y} = \overline{H \cap G}$. But Y is an H–set, and hence there is a nonempty relatively open subset $V \subset H$ with either $V \subset Y$ or $V \cap Y = \emptyset$. If the first case takes place, then $V \cap G = \emptyset$ and hence $H \cap G$ is not dense in H; if the second case takes place, $H \cap Y$ is not dense in H, a contradiction. □

Remark 17. Note that the proof of Theorem 16 for a closed set Y can be carried out in an easier way. Since the first condition of Proposition 8(iii) is satisfied due to Proposition 4(c), we need to verify that, given a Coz_δ–set $G \subset X$ disjoint with Y, there is a Coz_δ–set containing Y and disjoint with G. To this end, for each $x \in G$ we find a cozero set U_x containing x and disjoint with Y. As G is supposed to be Lindelöf, there are countably many points $x_n \in G$, $n \in \mathbb{N}$, such that $G \subset \bigcup_n U_{x_n}$. If we denote the union by U, we get a cozero set such that $G \subset U \subset X \setminus Y$. Hence $X \setminus U$ is the required Coz_δ–set.

4 Counterexamples and questions

In this section we collect several examples showing that the assumptions of our main theorem cannot be weakened in some natural ways. We also collect some questions which are, up to our knowledge, open.

First we show by a trivial example that the Lindelöf property of Y cannot be omitted (even if Y is discrete and hence locally compact and paracompact).

Example 18. Let $X = Y \cup \{\omega\}$ be the Alexandroff compactification of an uncountable discrete space Y. Then there exists a bounded continuous function on Y that cannot be extended to a Baire–one function on X.

Proof. We divide Y into two disjoint uncountable subsets Y_1 and Y_2 and let f be the characteristic function of Y_1. Then there is no Baire–one function on X which coincides with f on Y because every continuous, and consequently every Baire–one function on X satisfies $f(y) = f(\omega)$ for all but countably many points $y \in Y$. □

We continue by another trivial example witnessing that the assumption that Y is hereditarily Baire cannot be omitted (even if Y is countable and hence hereditarily Lindelöf).

Example 19. Let $Y = \mathbb{Q} \cap [0,1]$ and $X = [0,1]$ or $X = \beta Y$. Then there is a bounded Baire–one function on Y which cannot be extended to a Baire–one function on X.

Proof. Let A be a dense subset of Y with $Y \setminus A$ also dense. Then both A and $B := Y \setminus A$ are simultaneously Zer_σ and Coz_δ. If $\widehat{A}$ and $\widehat{B}$ are Coz_δ–subsets of X with $A = \widehat{A} \cap Y$ and $B = \widehat{B} \cap Y$, then $\widehat{A} \cap \widehat{B}$ is dense in X (as Y is dense in X and X is a Baire space). Thus we conclude by Proposition 7. □

The next example shows that the assumption that Y is hereditarily Baire cannot be weakened to the assumption that Y is a Baire space.

Example 20. Let $X = [0,1]^2$ and $Y = [0,1] \times (0,1] \cup ([0,1] \cap \mathbb{Q}) \times \{0\}$. Then there is a bounded Baire–one function on Y which cannot be extended to a Baire–one function on X.

Proof. Set $Y_0 = ([0,1] \cap \mathbb{Q}) \times \{0\}$. Let f_0 be a bounded Baire–one function on Y_0 which cannot be extended to a Baire–one function on $\overline{Y_0}^X$. Since Y_0 is a G_δ–subset of the metric space Y, we can extend f_0 to a bounded Baire–one function f on Y according to Theorem 10(c). Then f clearly cannot be extended to a Baire–one function on X. $\square$

One may further ask for which spaces it is possible to extend Baire–one functions on their closed subsets. As continuous functions can be continuously extended from closed subsets of normal spaces, it is natural to ask whether the same is true for Baire–one functions. The following two examples show that this is not the case.

Example 21. There exists a closed Lindelöf subset F of a Baire paracompact (and hence normal) space X such that it is not possible to extend every bounded Baire–one function on F to a Baire–one function on X.

Proof. Let X be the union of F and G, where

$$F := \mathbb{Q} \times \{0\} \quad \text{and} \quad G := \mathbb{R} \times \{1\} .$$

We let G to be open in X and discrete and neighbourhoods of a point $(p, 0) \in F$ are of the form

$$(\mathbb{Q} \cap (p - \delta, p + \delta)) \times \{0\} \cup ((p - \delta, p + \delta) \setminus K) \times \{1\} ,$$

where K is a finite set and $\delta > 0$.

Then X is clearly regular and F is its closed Lindelöf subset. If $\mathcal{U}$ is an open cover of X, there is a countable $\mathcal{V} \subset \mathcal{U}$ covering F. Put

$$\mathcal{W} = \mathcal{V} \cup \{\{x\} : \ x \in G \setminus \bigcup \mathcal{V}\}.$$

Then $\mathcal{W}$ is a σ–discrete open refinement of $\mathcal{U}$. Thus X is paracompact and hence normal. Moreover, G is a dense Baire set in X and hence X is a Baire space as well.

Let A be a dense subset of F such that $B := F \setminus A$ is also dense. Let U be any open set in X which contains A. Then it is easy to see that $U \cap G$ contains a dense G_δ–set in the Euclidean topology of G. Thus any pair of G_δ–sets containing A and B, respectively, cannot be disjoint. Proposition 7 thus finishes the proof. $\square$

Example 22. Under the Continuum Hypothesis there exists a closed subset F of a regular Lindelöf space X and a bounded Baire–one function on F which has no Baire–one extension on X.

Proof. The unit interval I is viewed as a compactification of $\mathbb{N}^\mathbb{N}$ by adding a countable set Q. Let $\{D_\xi\}_{\xi<\omega_1}$ be a sequence of compact sets in $\mathbb{N}^\mathbb{N}$ such that for each compact $K \subset \mathbb{N}^{\mathbb{N}}$ there exists $\xi < \omega_1$ so that $K \subset D_\xi$. (This sequence can be constructed in the following way: let $\{x_\xi\}_{\xi<\omega_1}$ be an enumeration of $\mathbb{N}^{\mathbb{N}}$. Set

$$D_\xi := \{x \in \mathbb{N}^{\mathbb{N}} : x \leq x_\xi\} , \quad \xi < \omega_1 .$$

Then each D_ξ is compact and every compact set $K \subset \mathbb{N}^{\mathbb{N}}$ is contained in some D_ξ because there is a point $x \in \mathbb{N}^{\mathbb{N}}$ so that $y \leq x$ for every $y \in K$.)

Set

$$X_\xi := I \setminus \bigcup_{\eta<\xi} D_\eta , \quad \xi \leq \omega_1 .$$

Then X_ξ is a dense G_δ–subsets of I for every $\xi < \omega_1$ and $X_{\omega_1} = Q$. Set

$$X := \bigcup_{\xi\leq\omega_1} \{\xi\} \times X_\xi$$

with the topology of the product $[0, \omega_1] \times I$.

We claim that X is Lindelöf. Indeed, let $\mathcal{U}$ be an open cover of X consisting of open rectangles. We find a countable subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that

$$\{\omega_1\} \times X_{\omega_1} \subset \bigcup \mathcal{U}_0$$

and each element of $\mathcal{U}_0$ intersects $\{\omega_1\} \times X_{\omega_1}$. It follows from the definition of the product topology that there exists $\eta < \omega_1$ so that

$$[\eta, \omega_1] \times Q \subset \bigcup \mathcal{U}_0 .$$

Let U be the subset of I defined as the projection of $\bigcup \mathcal{U}_0$ onto I. Then $I \setminus U$ is a compact subset of $\mathbb{N}^{\mathbb{N}}$ and thus there exists an ordinal number $\xi < \omega_1$ such that

$$I \setminus U \subset K_\xi .$$

Then $\{\alpha\} \times X_\alpha$ is covered by $\bigcup \mathcal{U}_0$ for every $\alpha \in [\max(\xi, \eta), \omega_1]$. It is easy to select a countable subfamily $\mathcal{U}_1$ from $\mathcal{U}$ such that $\{\beta\} \times X_\beta$ is covered by $\bigcup \mathcal{U}_1$ for every $\beta \in [0, \max\{\xi, \eta\})$. Thus the family $\mathcal{U}_0 \cup \mathcal{U}_1$ is the sought countable subcover of $\mathcal{U}$ and X is Lindelöf.

Now we are going to find a bounded Baire–one function f on a closed set

$$F := \{\omega_1\} \times Q ,$$

which is not extensible to a Baire–one function on X. To this end, let D be a dense subset of Q such that its complement is dense as well. We claim that the characteristic function of $\{\omega_1\} \times D$ is not extensible to a Baire–one function on X. Let G be an open set in X satisfying

$$\{\omega_1\} \times D \subset G .$$

We find countably many open rectangles $(\xi_n, \omega_1] \times U_n$, U_n open in I, so that

$$\{\omega_1\} \times D \subset \bigcup_{n=1}^{\infty} (\xi_n, \omega_1] \times U_n \subset U \ .$$

Set $\xi := \sup_n \xi_n$ and note that

$$V := (\xi, \omega_1] \times \bigcup_{n=1}^{\infty} U_n \subset U \ .$$

Thus for every $\eta \in [\xi, \omega_1)$, the set

$$V \cap (\{\eta\} \times X_\xi)$$

is open in $\{\eta\} \times X_\xi$ and, moreover, it is a dense subset of $\{\eta\} \times X_\xi$ because D is a dense subset of I.

It follows from the previous considerations that for any G_δ–set $G \subset X$ containing $\{\omega_1\} \times D$ there exists $\xi < \omega_1$ so that $G \cap (\{\eta\} \times X_\eta)$ is a dense G_δ–set in $\{\eta\} \times X_\eta$ for every $\eta \in [\xi, \omega_1)$. Since X_η is a Baire space for every $\eta < \omega_1$, it is impossible to find a pair of disjoint G_δ–sets containing $\{\omega_1\} \times D$ and $\{\omega_1\} \times (Q \setminus D)$, respectively. This concludes the proof using Proposition 7. □

Remark 23. Note that the set

$$A := \bigcup_{0 \leq \xi \leq \omega_1} \{\xi\} \times (K_\xi \setminus Q)$$

is a Coz_δ–subset of the space X from the previous example which is not Lindelöf. Indeed, open sets

$$U_\xi := \bigcup_{0 \leq \alpha \leq \xi} \{\alpha\} \times X_\alpha \ , \quad \xi < \omega_1 \ ,$$

form an open cover of A which has no countable subcover.

Remark 24. The statement of Example 22 remains valid under a weaker set–theoretical assumption $\mathfrak{d} = \mathrm{cov}(M)$. We can just use the space constructed in the proof of Theorem 1.2 of [9]. In fact, it follows from this result of [9] that there exists a Baire–one function on a closed subset of a regular Lindelöf space which is impossible to extend to a Baire–one function on the whole space provided there is a Michael space X such that the smallest cardinality of an open cover of $X \times \mathbb{N}^{\mathbb{N}}$ without a countable subcover is regular. It seems not to be clear whether the existence of such an example can be deduced just from the existence of a Michael space.

The following example shows that it is not possible to extend a Baire–one function from a hereditarily Baire zero set if the space X is not normal.

Example 25. There is a completely regular space X, a hereditarily Baire zero set $Y \subset X$ and a bounded continuous function on Y which cannot be extended to a Baire–one function on X.

Proof. Take X to be the Niemytzki plane (see [2, Example 1.2.4]), i.e., $X = \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$ with the following topology. The neighbourhoods of (x,y) with $y > 0$ are the Eucledian ones, the neighbourhoods of $(x,0)$ are of the form $\{(x,0)\} \cup B((x,r),r)$ for $r > 0$ where $B((x,r),r)$ denotes the open Euclidean ball. Put $Y = \{(x,y) \in X : y = 0\}$. Then Y is a zero set (as the function $(x,y) \mapsto y$ is continuous), it is discrete and hence hereditarily Baire. Further, any function on Y is continuous, so there are 2^{2^ω} different bounded continuous functions on Y. On the other hand, there are at most 2^ω Baire–one functions on X because X, as a separable space, has at most 2^ω continuous functions. This completes the proof. □

We finish this section by asking some questions which seem to be natural and open.

Question 1. Let X be a hereditarily Baire completely regular space and f a Baire–one function on X. Can f be extended to a Baire–one function on βX?

Question 2. Let X be a normal space, Y a closed hereditarily Baire subset of X and f a Baire–one function on Y. Can f be extended to a Baire–one function on X?

Question 3. Let X be a normal space, $Y \subset X$ a Coz_δ–set and f a Baire–one function on Y. Can f be extended to a Baire–one function on X?

Question 4. Let X be a completely regular Lindelöf space such that every Coz_δ–subset of X is Lindelöf, $Y \subset X$ a Lindelöf $(F \vee G)_\delta$–subset and f a Baire–one function on Y. Can f be extended to a Baire–one function on X?

5 Extension of mappings of the first Borel class

The aim of this section is to show that once it is possible to extend Baire–one functions from a subspace, the extension theorems of Section 3 can be obtained even for F_σ–measurable mappings with values in Polish spaces.

We start with the following easy result known as the reduction principle (see the proof of [6, § 26, II, Theorem 1]).

Proposition 26. *Let $\mathcal{A}$ be an algebra of subsets of a set X and $\{F_n : n \in \mathbb{N}\}$ be a cover of X consisting of sets from $\mathcal{A}_\sigma$ (this is the family of all countable* *unions of elements of $\mathcal{A}$). Then there exists a partition $\{H_n : n \in \mathbb{N}\}$ of X such that $H_n \in \mathcal{A}_\sigma$ and $H_n \subset F_n$ for each $n \in \mathbb{N}$.*

We will need the following concrete form of this proposition.

Proposition 27. *Let $\{F_n : n \in \mathbb{N}\}$ be a cover of a space X consisting of* Zer_σ*–sets. Then there exists a partition $\{H_n : n \in \mathbb{N}\}$ of X consisting of* Zer_σ*–sets such that $H_n \subset F_n$ for every $n \in \mathbb{N}$.*

Proof. We apply Proposition 26, where the role of the algebra $\mathcal{A}$ is played by the family of sets which are both Zer_σ and Coz_δ. (Note that this family is an algebra and that any zero set belongs to this family.) □

We continue by the following general result on connection of extensions of Baire–one functions with extensions of Zer_σ–measurable mappings.

Proposition 28. *Let X be a topological space and $Y \subset X$ such that any Baire–one function on Y can be extended to a Baire–one function on X. Then for any* Zer_σ*–measurable mapping $f : Y \to P$ to a Polish space P there exists a* Zer_σ*–measurable mapping $g : X \to P$ such that $f = g$ on Y and $g(X) \subset \overline{f(Y)}$.*

Proof. As $\overline{f(Y)}$ is again Polish we may suppose that $P = \overline{f(Y)}$. We fix on P a compatible complete metric ρ such that the diameter P with respect to ρ is smaller than 1.

Let $\{B(p_s, r_s) : s \in \mathbb{N}^{<\mathbb{N}}\}$ be a family of open balls in P with centers p_s and diameters r_s such that

(a) $B(p_\emptyset, r_\emptyset) = P$,
(b) $\bigcup_{n\in\mathbb{N}} B(p_{s^\wedge n}, r_{s^\wedge n}) = B(p_s, r_s)$ for each $s \in \mathbb{N}^{<\mathbb{N}}$, and
(c) $r_s < \frac{1}{2^{|s|}}$ for each $s \in \mathbb{N}^{<\mathbb{N}}$.

Such a family is easy to construct in any separable metric space with the diameter less than one.

We will construct by induction Zer_σ–subsets $\{H_s : s \in \mathbb{N}^{<\mathbb{N}}\}$ of X such that

(d) $H_\emptyset = X$,
(e) $H_s \cap Y \subset f^{-1}(B(p_s, r_s))$ for each $s \in \mathbb{N}^{<\mathbb{N}}$, and
(f) $\{H_{s^\wedge k} : k \in \mathbb{N}\}$ is a partition of H_s for each $s \in \mathbb{N}^{<\mathbb{N}}$.

To start the construction set $H_\emptyset := X$. Fix $n \geq 0$ and assume that the sets H_s have been constructed for every $s \in \mathbb{N}^{<\mathbb{N}}$ with $|s| \leq n$. Let

$$\widehat{F}_{s^\wedge k} := f^{-1}(B(p_{s^\wedge k}, r_{s^\wedge k})) \cap H_s\,, \quad s \in \mathbb{N}^{<\mathbb{N}}\,,\ |s| = n\,,\ k \in \mathbb{N}\,.$$

As H_s is Zer_σ and f is Zer_σ–measurable, these sets are Zer_σ–subsets of Y. Moreover, it easily follows from conditions (b) and (e) that

$$\{\widehat{F}_{s^\wedge k} : s \in \mathbb{N}^{<\mathbb{N}} \ , \ |s| = n \ , \ k \in \mathbb{N}\}$$

is a covering of Y. Applying the reduction principle of Proposition 27 we obtain a partition

$$\{F_{s^\wedge k} : s \in \mathbb{N}^{<\mathbb{N}} \ , \ |s| = n \ , \ k \in \mathbb{N}\}$$

of Y consisting of Zer_σ–subsets of Y such that

$$F_{s^\wedge k} \subset \widehat{F}_{s^\wedge k} \ , \quad s \in \mathbb{N}^{<\mathbb{N}} \ , \ |s| = n \ , \ k \in \mathbb{N} \ .$$

Fix a sequence $s \in \mathbb{N}^{<\mathbb{N}}$ of length n. For every $k \in \mathbb{N}$, let $U_{s^\wedge k}$ be a Coz_δ–subset of X such that

$$Y \cap (H_s \setminus F_{s^\wedge k}) = Y \cap U_{s^\wedge k} \ .$$

(We remind that $Y \cap (H_s \setminus F_{s^\wedge k})$ is a Coz_δ–subset of Y and that we use Proposition 8.) Then

$$U_s := \bigcap_{k=1}^{\infty} U_{s^\wedge k}$$

is a Coz_δ–set in X such that $U_s \cap Y = \emptyset$. According to Proposition 8, there exists a Coz_δ–set $G_s \subset X$ so that

$$Y \subset G_s \subset X \setminus U_s \ .$$

Thus

$$V_{s^\wedge k} := U_{s^\wedge k} \cap G_s \ , \quad k \in \mathbb{N} \ ,$$

are Coz_δ–sets in X which satisfy

$$\bigcap_{k=1}^{\infty} V_{s^\wedge k} = \emptyset \quad \text{and} \quad Y \cap (H_s \setminus F_{s^\wedge k}) = Y \cap V_{s^\wedge k} \ , \quad k \in \mathbb{N} \ .$$

Set

$$\widehat{H}_{s^\wedge k} := H_s \setminus V_{s^\wedge k} \ , \quad k \in \mathbb{N} \ .$$

It is easy to verify that

$$\{\widehat{H}_{s^\wedge k} : k \in \mathbb{N}\}$$

is a covering of H_s consisting of sets which are Zer_σ in X. Applying the reduction principle of Proposition 27 to the covering $\{\widehat{H}_{s^\wedge k} : k \in \mathbb{N}\}$ we obtain a partition $\{H_{s^\wedge k} : k \in \mathbb{N}\}$ of H_s consisting of Zer_σ–sets in X such that

$$H_{s^\wedge k} \subset \widehat{H}_{s^\wedge k} \ , \quad k \in \mathbb{N} \ .$$

It easily follows that

$$F_{s^\wedge k} = Y \cap H_{s^\wedge k}$$

for every $k \in \mathbb{N}$. Then the family

$$\{H_{s^\wedge k} : s \in \mathbb{N}^{<\mathbb{N}} \ , \ |s| = n \ , \ k \in \mathbb{N}\}$$

is the required partition of X. This completes the construction.

Now, for any $n \in \mathbb{N} \cup \{0\}$ define $g_n : X \to P$ by the formula

$$g_n(x) := p_s \ , \quad x \in H_s \ , \quad s \in \mathbb{N}^{<\mathbb{N}} \ , \ |s| = n \ .$$

Then each g_n is clearly a Zer_σ–measurable mapping. Obviously the mappings g_n, $n = 0, 1, \ldots$, form a uniformly Cauchy sequence. As (P, ρ) is complete, this sequence converges uniformly to a mapping $g : X \to P$. As a uniform limit of Zer_σ–measurable mappings it is Zer_σ–measurable (see the proof of [6, 2, § 31, VIII, Theorem 2]). Finally, $g = f$ on Y by conditions (e) and (c). □

As a corollary we obtain the following theorem.

Theorem 29. *Let Y be a Lindelöf subset of a completely regular space X. Assume that*

(a) Y is hereditarily Baire, or
(b) every Coz_δ–set in X is Lindelöf and Y is an H–set, or
(c) Y is G_δ–set in X.

Then for any mapping $f : Y \to P$ of the first Borel class to a Polish space P there exists a mapping $g : X \to P$ of the first Borel class such that $f = g$ on Y and $g(X) \subset \overline{f(Y)}$.

Proof. This follows from Proposition 28 and the respective theorems of Section 3 using, moreover, Proposition 3 together with the well–known fact that any regular Lindelöf space is normal. □

6 Extension of Baire–one functions on compact convex sets

The aim of this section is to prove an analogue of the results of Section 3 in a particular case of extending Baire–one functions from the set $\operatorname{ext} X$ of all extreme points of a compact convex set X.

Theorem 30. *Let X be a compact set in a locally convex space such that $\operatorname{ext} X$ is Lindelöf. Let f be a Baire–one function on $\operatorname{ext} X$. Then there exists a Baire–one function g on X such that $f = g$ on X,*

$$\inf f(\operatorname{ext} X) = \inf g(X) \quad \textit{and} \quad \sup f(\operatorname{ext} X) = \sup g(X) \ .$$

In the proof of this theorem we will need the following standard lemma.

Lemma 31. *Let X be a compact set in a locally convex space and $F \subset X$ be a closed set. Then $\overline{\text{co}}\,(F) \cap \text{ext}\, X = F \cap \text{ext}\, X$.*

Proof. Let x be a point in $\overline{\text{co}}\,(F) \cap \text{ext}\, X$ and U be its open neighbourhood. As x is extreme, Proposition 25.13 of [1] provides an affine continuous function on X such that $f(x) > 0 > \max f(X \setminus U)$. Since $x \in \overline{\text{co}}\, F$, the open slice $[f > 0]$ intersects F. Hence $U \cap F \neq \emptyset$ for every open neighbourhood U of x. Thus $x \in \overline{F} = F$. Since the converse inclusion is obvious, the proof is finished. □

Proof of Theorem 30. Denote by E the locally convex space X is embedded in. Again we need to check the validity of condition (iii) in Proposition 8. The first part follows from Proposition 4(c). Thus we have to check the second part. To this end, let C be a Coz_δ–subset of X disjoint with $\text{ext}\, X$. It suffices to find a Coz_δ–set B with $\text{ext}\, X \subset B \subset X \setminus C$.

Suppose that such a set B does not exist. As C is Lindelöf (by Proposition 5), Proposition 11 provides a nonempty closed set $H \subset X$ so that

$$\overline{\text{ext}\, X \cap H} = \overline{C \cap H} = H \ . \tag{3}$$

As C is Coz_δ, it is in particular a G_δ–set. Write $C = \bigcap_n G_n$ where $\{G_n\}$ is a decreasing sequence of open subsets of X. Without loss of generality we may assume that $X \setminus G_1 \neq \emptyset$. Thanks to (3), each $G_n \cap H$ is a dense relatively open subset of H, and hence $G_n \cap H \cap \text{ext}\, X$ is dense in H for every $n \in \mathbb{N}$.

We will construct by induction continuous affine functions f_n on X and points $x_n \in \text{ext}\, X \cap H$ such that, for every $n \in \mathbb{N}$,

(a) $f_n < \chi_{G_n}$,
(b) $[f_{n+1} > 0] \subset [f_n > 0]$,
(c) $x_n \in G_n \cap \text{ext}\, X \cap H$, and
(d) $f_n(x_n) > 0$.

In the first step of the construction we find a point

$$x_1 \in G_1 \cap \text{ext}\, X \cap H.$$

Set $K_1 := \overline{\text{co}}\,(X \setminus G_1)$. By Lemma 31, $x_1 \notin K_1$ and hence the Hahn–Banach separation theorem provides $\eta_1 \in E^*$ with $\max \eta_1(K_1) < \eta_1(x_1)$. Set

$$f_1 := \frac{1}{\max \eta_1(X) - \min \eta_1(X)} \left(\eta_1 - \frac{\eta_1(x_1) + \max \eta_1(K_1)}{2} \right) .$$

Then f_1 is a continuous affine function fulfilling (a) and (d). This finishes the first step.

Suppose that the construction has been completed up to $n \in \mathbb{N}$. Since $[f_n > 0]$ is a nonempty open set intersecting H (the intersection contains x_n) and the set $G_{n+1} \cap H \cap \operatorname{ext} X$ is dense in H, we can pick a point

$$x_{n+1} \in G_{n+1} \cap [f_n > 0] \cap \operatorname{ext} X \cap H \ .$$

Set

$$K_{n+1} := \overline{\mathrm{co}}\,(X \setminus (G_{n+1} \cup [f_n > 0])) \ .$$

By Lemma 31 we get $x_{n+1} \notin K_{n+1}$ and thus there is $\eta_{n+1} \in E^*$ with

$$\max \eta_{n+1}(K_{n+1}) < \eta_{n+1}(x_{n+1}) \ .$$

Set

$$f_{n+1} := \frac{1}{\max \eta_{n+1}(X) - \min \eta_{n+1}(X)} \left(\eta_{n+1} - \frac{\eta_{n+1}(x_{n+1}) + \max \eta_{n+1}(K_{n+1})}{2} \right) \ .$$

Then f_{n+1} is a continuous affine function on X fulfilling conditions (a), (b) and (d). Thus we have completed the construction.

By setting

$$f := \inf_{n \in \mathbb{N}} f_n$$

we obtain an upper finite upper semicontinuous affine function on X. Therefore f attains its maximum at some point $x_0 \in \operatorname{ext} X$ (see the proof of [1, Theorem 25.9]). Since $\{[f_n \geq 0]\}$ is a centered family of compact sets (by (b) and (d)),

$$[f \geq 0] = \bigcap_{n=1}^{\infty} [f_n \geq 0] \neq \emptyset \ .$$

Therefore $f(x_0) \geq 0$. By (a) we get that $x_0 \in G_n$ for each $n \in \mathbb{N}$. Thus $x_0 \in C$ but this contradicts the assumption that $C \cap \operatorname{ext} X = \emptyset$. This finishes the proof. □

Remark that it is not clear whether this result is a direct consequence of Theorem 13. It is well–known that $\operatorname{ext} X$ is always a Baire space (in fact, an α–favorable space, see [1, Theorem 27.9]). However, in general $\operatorname{ext} X$ need not be hereditarily Baire as the following folklore example shows (see e.g. [12, Corollary 2])

Example 32. Any completely regular space is homeomorphic to a closed subset of $\operatorname{ext} X$ for some convex compact set X.

Proof. Let Y be any completely regular space and K be a compactification of Y. Set

$$L := K \times \{0\} \cup (K \setminus Y) \times \{-1, 1\}$$

and equip this set with the following topology. The set $(K \setminus Y) \times \{-1, 1\}$ is open and discrete, the neighbourhoods of $(k, 0)$ are of the form

$$U \times \{0\} \cup (U \setminus (Y \cup \{k\})) \times \{-1, 1\}$$

where U is a neighbourhood of k in K. Then L is compact. Set

$$A := \{f \in \mathcal{C}(L) : f((k,0)) = \frac{1}{2}\Big(f((k,-1)) + f((k,1))\Big) \text{ for each } k \in K \setminus Y\}$$

and

$$X := \{\xi \in A^* : \xi \geq 0 \text{ and } \xi(1) = 1\}$$

endowed with the weak* topology. Then X is convex compact and $\operatorname{ext} X$ is homeomorphic to $Y \times \{0\} \cup (K \setminus Y) \times \{-1, 1\}$ considered as a subset of L. This completes the proof. □

However, by the method of the previous example we cannot get any example of a compact convex set X with $\operatorname{ext} X$ Lindelöf but not hereditarily Baire. Therefore the following question seems to be natural.

Question 5. Let X be a compact convex set in a locally convex space with $\operatorname{ext} X$ Lindelöf. Is then $\operatorname{ext} X$ hereditarily Baire?

Note that the answer is positive if $\operatorname{ext} X$ is $\mathcal{K}$–countably determined. Indeed, in this case the set $\operatorname{ext} X$ is of type $(F \vee G)_\delta$ as M. Talagrand proved in [13, Théorème 2].

The question of extending bounded Baire–one functions from $\operatorname{ext} X$ to affine Baire–one functions on X was studied in [11]. Due to Theorem 30 we have obtained the following partial improvement of [11, Corollary 1].

Corollary 33. *Let X be a compact convex set such that $\operatorname{ext} X$ is Lindelöf. Then the following conditions are equivalent.*

- *(i) For any bounded Baire–one function f on $\operatorname{ext} X$ there exists an affine Baire–one function h on X with $f = h$ on $\operatorname{ext} X$.*
- *(ii) For any bounded Baire–one function f on X there exists an affine Baire–one function h on X with $f = h$ on $\operatorname{ext} X$.*
- *(iii) X is a Choquet simplex and the function $x \mapsto \delta_x(f)$, $x \in X$, is of the first Baire class for any bounded Baire–one function f on X (here δ_x denotes the unique maximal measure representing a point $x \in X$).*

References

[1] G. Choquet, *Lectures on analysis II.*, W.A. Benjamin, Inc., New York–Amsterdam, 1969.

[2] R. Engelking, *General topology*, Verlag, Berlin, 1989.

[3] D.H.Fremlin, P.Holický and J.Pelant, *Lindelöf modifications and K-analytic spaces*, Mathematatika **40 (1)** (1993), 1–6.

[4] R.W. Hansell, *Descriptive topology*, in: Recent progress in general topology (M. Hušek and J. van Mill, eds.), North–Holland, 1992, 275–315.

[5] J.E. Jayne and C.A. Rogers, *Borel isomorphisms at the first level: Corrigenda et addenda*, Mathematika **27** (1980), 236–260.

[6] K. Kuratowski, *Topology, Vol. I*, Academic Press, New York, 1966.

[7] J. Lukeš, J. Malý and L. Zajíček, *Fine topology methods in real analysis and potential theory*, Lecture Notes in Math. 1189, Springer–Verlag, 1986.

[8] E. Michael, *Paracompactness and the Lindelöf property in finite and countable Cartesian products*, Composito Math. **23** (1971), 199–214.

[9] J. Tatch Moore, *Some of the combinatorics related to Michael's problem*, Proc. Amer. Math. Soc. **127 (8)** (1999), 2459–2467.

[10] C.E. Rogers and J.E. Jayne, *K–analytic sets*, Academic Press, 1980.

[11] J. Spurný, *Affine Baire–one functions on Choquet simplexes*, preprint KMA–MATH–2004/124, available on http://adela.karlin.mff.cuni.cz/~rokyta/preprint/2004–pap/2004–124.ps.

[12] P.J. Stacey, *Choquet simplices with the prescribed extreme and Šilov boundaries*, Quart. J. Math. Oxford **30(2)** (1979), 469–482.

[13] M. Talagrand, *Sur les convexes compacts dont l'ensemble des points extrémaux est K–analytic*, Bull. Soc. Math. France **107** (1979), 49–53.