Finite Element Method for Sturm-Liouville Problems

Consider a Sturm-Liouville boundary value problem with Dirichlet boundary conditions on some interval.

$$-\frac{d}{dx}\left(P(x)\,\frac{du(x)}{dx}\right) + R(x)\,u(x) = \lambda\,w(x)\,u(x)$$

$$u(a) = u(b) = 0$$

We can apply the finite element method to this problem in the usual way by first constructing a weak form for the equation.

$$\int_a^b \left(-\frac{d}{dx}\left(P(x)\,\frac{du(x)}{dx}\right) + R(x)\,u(x)\right) v(x)\,dx = \int_a^b \lambda\,w(x)\,u(x)\,v(x)\,dx$$

By splitting the integral on the left into two distinct integrals and then applying integration by parts to the first of the two integrals we obtain

$$\left(-P(x)\,\frac{du(x)}{dx}\,v(x)\right)\Big|_a^b + \int_a^b P(x)\,\frac{du(x)}{dx}\,\frac{dv(x)}{dx}\,dx + \int_a^b R(x)\,u(x)\,v(x)\,dx = \int_a^b \lambda\,w(x)\,u(x)\,v(x)\,dx$$

Since the test functions satisfy Dirichlet conditions, the first term on the left will vanish leaving us with

$$\int_a^b P(x)\,\frac{du(x)}{dx}\,\frac{dv(x)}{dx} + R(x)\,u(x)\,v(x)\,dx = \int_a^b \lambda\,w(x)\,u(x)\,v(x)\,dx$$

We now proceed as usual by introducing a family of spike functions $\{\phi_j(x)\}$ defined on $[a,b]$. We assume that an approximate solution can be written

$$v_n(x) = \sum_{j=1}^{n} u_j\,\phi_j(x)$$

If we substitute this into the weak form and use test functions of the form $v(x) = \phi_i(x)$ we obtain

$$\sum_{j=1}^{n} u_j \left(\int_a^b P(x)\,\frac{d\phi_j(x)}{dx}\,\frac{d\phi_i(x)}{dx} + R(x)\,\phi_j(x)\,\phi_i(x)\,dx\right) = \lambda \sum_{j=1}^{n} u_j \int_a^b w(x)\,\phi_j(x)\,\phi_i(x)\,dx$$

If we introduce matrices

$$A_{i,j} = \int_a^b P(x)\,\frac{d\phi_j(x)}{dx}\,\frac{d\phi_i(x)}{dx} + R(x)\,\phi_j(x)\,\phi_i(x)\,dx$$

$$M_{i,j} = \int_a^b w(x)\,\phi_j(x)\,\phi_i(x)\,dx$$

these equations for $i = 1$ to n can be written as a matrix equation

$$A\,\mathbf{u} = \lambda\, M\,\mathbf{u}$$

To find the desired approximate eigenfunctions and eigenvalues, we simply have to find the eigenvalues and eigenvectors of the matrix equation

$$M^{-1}A\,\mathbf{u} = \lambda\,\mathbf{u}$$

The accompanying Mathematica notebook will show a couple of examples of this process.