Path Integral Approach to Relativistic Quantum Mechanics
—— *Two-Dimentional Dirac Equation* ——

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(Received March 11, 1987)

Countably additive path space measures are constructed, in two space-time dimensions, to give rigorous path integral formulas representing the fundamental solution of the Cauchy problem for the Dirac equation as well as the retarded and advanced propagators for the Dirac particle. The theory also applies to a free particle, a particle in a central electric field and a particle in parallel electric and uniform magnetic fields in four-dimensional space-time.

Contents

§ 1. Introduction

The fundamental physical and mathematical concepts which underlie *path integral* were first developed by Feynman[1),2)] in nonrelativistic quantum mechanics, with a suggestion of Dirac's remarks.[3)] In pure-imaginary-time quantum mechanics Kac[4)] has given a mathematical realization of it in terms of the Wiener measure.[5)] Namely he established a rigorous path integral formula representing the solution of the heat equation for the quantum Hamiltonian of a nonrelativistic spinless particle in a scalar potential, which is called the Feynman-Kac formula.[5)] Its further extension to the Hamiltonian with both scalar and vector potentials is the Feynman-Kac-Itô formula,[5)] which involves the Itô integral.[5)] The Laplace transform of this formula for the

Schrödinger operator in four-dimensional space with respect to a fifth variable as the fictitious time gives rise to a path integral representation of the Euclidean propagator for a Klein-Gordon particle in an external electromagnetic field.

In Refs. 11)~13), we have made a path integral approach to relativistic quantum mechanics and established rigorous path integral formulas for the two-space-time-dimensional Dirac equation. We have constructed countably additive measures, on the space of the continuous paths, which differs from the Wiener measure. In our treatment the time variable is real. It makes a contrast with Kac's approach regarding the heat equation as the Schrödinger equation in pure-imaginary-time quantum mechanics. The aim of this article is to present the full story in a coherent way, recapitulating the basic physical and mathematical ideas. We also include some recent result[34),35)] which improves on the support property of the path space measures constructed in the previous work.

Our construction of these countably additive path space measures follows Nelson's method[10)] of construction of the Wiener measure. The crucial step is to prove continuity of a certain linear functional defined, on the Banach space of the continuous functions on the path space, through the fundamental solution for the free Dirac equation. The problem is connected with the L^∞ well-posedness of the Cauchy problem for a hyperbolic system of the first order with two independent variables. Then the Riesz-type representation theorem assures this linear functional to bring forth a countably additive measure on the path space.

The path space measures constructed turn out to be concentrated on the set of the Lipschitz continuous paths which have differential coefficients of magnitude equal to the light velocity in every finite time interval with the possible exception of finite instants of time. So the trajectory of the particle shuttles back and forth in one-dimensional space with slopes of the light velocity; it is a zigzag of a finite number of straight segments in each finite time interval. At the end points of the segments the particle changes its direction of motion. This property may remind us of the "Zitterbewegung"[7)] of the Dirac particle.

These path space measures are then used to represent by path integral the fundamental solution of the Cauchy problem for the Dirac equation with vector and scalar potentials as well as the retarded and advanced propagators, both in two space-time dimensions. The path integral formulas established show a close analogy with the Feynman-Kac formula and the Feynman-Kac-Itô formula for the heat equation.

The theory can be extended to a certain hyperbolic system of the first order,[11)] but does not apply to the Dirac equation in four space-time dimensions except for *three* special cases. These are the free Dirac equation, the Dirac equation for a central electric field and the Dirac equation for parallel electric and uniform magnetic fields, for they are reduced to equations with two independent variables by use of, respectively, the Radon transform, the spherical coordinates and the Fourier transform in one variable together with an Hermite function expansion in another variable. Nor it applies to the pure-imaginary-time or Euclidean Dirac equation.

Finally we add here brief mention of path integral for a relativistic *spinless* particle in an electromagnetic field. A path integral formula is obtained[36),37)] for the solution of the pure-imaginary-time Schrödinger equation with its Weyl quantized Hamiltonian.

We begin § 2 with a heuristic argument deriving the path integral for the Dirac equation, in order to get an intuitive understanding of the subject of this article. Section 3 is devoted to path integral representations for the fundamental solutions of two Cauchy problems for the Dirac equation and for the retarded and advanced propagators, in two space-time dimensions. The proof is given in § 4. Section 5 is concerned with the path integral for the Dirac equation in four space-time dimensions.

The natural units are used in which both the light velocity c and the constant $\hbar = h/2\pi$ with Planck's constant h equal 1.

$\boldsymbol{C}^d$ is the vector space of complex d-column-vectors and $(\boldsymbol{C}^d)'$ that of complex d-row-vectors. $M_d(\boldsymbol{C})$ is the vector space of complex $d\times d$ matrices. The norm of a $d\times d$ matrix $N=(N_{jk})$ is defined by $|N|=\max_{1\leq j\leq d}\sum_{k=1}^{d}|N_{jk}|$. $\langle\cdot,\cdot\rangle$ is the bilinear inner product and $(\cdot,\cdot)$ the physicist's inner product.

§ 2. Heuristic derivation of path integral for the Dirac equation

In this section we shall heuristically see what should be the path integral for the fundamental solution of the Cauchy problem for the $d+1$-dimensional Dirac equation

$$\partial_t\phi(t,\boldsymbol{x})=[-\boldsymbol{\alpha}(\partial-i\boldsymbol{A}(t,\boldsymbol{x}))-im\beta-ie\boldsymbol{\Phi}(t,\boldsymbol{x})]\phi(t,\boldsymbol{x}),$$
$$t\in\boldsymbol{R},\ x\in\boldsymbol{R}^d \tag{2·1}$$

for a particle of mass m and charge e in an external electromagnetic field. Here $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\cdots,\alpha_d)$, and $\alpha_1,\alpha_2,\cdots,\alpha_d$ and β are the Dirac matrices. $\boldsymbol{\Phi}(t,\boldsymbol{x})$ and $\boldsymbol{A}(t,\boldsymbol{x})$ are, respectively, the scalar and vector potentials of the field. Our strategy is to exploit the method of phase space path integral[8] or Hamiltonian path integral.[9]

We begin with presuming that the action for a Dirac particle in an electromagnetic field is given by

$$S(s,r;\boldsymbol{P},\boldsymbol{X})=\int_r^s[\boldsymbol{P}(t)\dot{\boldsymbol{X}}(t)-\boldsymbol{\alpha}(\boldsymbol{P}(t)-e\boldsymbol{A}(t,\boldsymbol{X}(t)))$$
$$-m\beta-e\Phi(t,\boldsymbol{X}(t))]dt\,. \tag{2·2}$$

Here $\boldsymbol{X}(t)$ and $\boldsymbol{P}(t)$ are the position and momentum. A reasonable ground for this is that (2·2) is what results, by application of Dirac's prescription

$$\pm\sqrt{(\boldsymbol{P}(t)-e\boldsymbol{A}(t,\boldsymbol{X}(t)))^2+m^2}=\boldsymbol{\alpha}(\boldsymbol{P}(t)-e\boldsymbol{A}(t,\boldsymbol{X}(t)))+m\beta\,,$$

from the action[33]

$$\int_r^s\{\boldsymbol{P}(t)\dot{\boldsymbol{X}}(t)-[\pm\sqrt{(\boldsymbol{P}(t)-e\boldsymbol{A}(t,\boldsymbol{X}(t)))^2+m^2}+e\Phi(t,\boldsymbol{X}(t))]\}dt$$

for a relativistic *spinless*, positive-energy(+) [resp., negative-energy(−)] particle of mass m and charge e in an electromagnetic field. We assume $s>r$ for definiteness.

The method of phase space path integral or Hamiltonian path integral assumes that the fundamental solution $K(s, \boldsymbol{x}; r, \boldsymbol{y})$ of the Cauchy problem for the Dirac equation (2·1), which is the probability amplitude that a quantized particle at position $\boldsymbol{y}$ at time r will be at position $\boldsymbol{x}$ at time s, is given by the formal "integral"

$$\int_{\mathcal{P}_{s,x;r,y}} e^{iS(s,r\,;\,\boldsymbol{P},\boldsymbol{X})}\mathcal{D}(\boldsymbol{P})\mathcal{D}(\boldsymbol{X})\,. \tag{2·3}$$

Here $\mathcal{D}(\boldsymbol{P})\mathcal{D}(\boldsymbol{X})$ is a formal "measure" $\prod_{t\in[r,s]}(2\pi)^{-d}d\boldsymbol{P}(t)d\boldsymbol{X}(t)$ on the space $\mathcal{P}_{s,x\,;\,r,y}$ of the phase space paths $(\boldsymbol{P}(t), \boldsymbol{X}(t))$ satisfying $\boldsymbol{X}(r)=\boldsymbol{y}$ and $\boldsymbol{X}(s)=\boldsymbol{x}$ with $\boldsymbol{P}(t)$ unrestricted. In this formal phase space path "integral" we make the change of variables: $\boldsymbol{X}'(t)=\boldsymbol{X}(t)$, $\boldsymbol{P}'(t)=\boldsymbol{P}(t)-e\boldsymbol{A}(t, \boldsymbol{X}(t))$. Then we have, writing $(\boldsymbol{P}(t), \boldsymbol{X}(t))$ again instead of $(\boldsymbol{P}'(t), \boldsymbol{X}'(t))$,

$$K(s, \boldsymbol{x}; r, \boldsymbol{y})=\int_{\mathcal{P}_{s,x;r,y}} T\exp\Big\{i\int_r^s[(\boldsymbol{P}(t)+e\boldsymbol{A}(t, \boldsymbol{X}(t)))\dot{\boldsymbol{X}}(t)-(\boldsymbol{\alpha}\boldsymbol{P}(t)+m\beta)$$
$$-e\Phi(t, \boldsymbol{X}(t))]dt\Big\}\prod_{t\in[r,s]}(2\pi)^{-d}d\boldsymbol{P}(t)d\boldsymbol{X}(t)\,, \tag{2·4}$$

where T stands for the time-ordering symbol. We understand (2·4) to be defined with a time division procedure, i.e.,

$$K(s, \boldsymbol{x}; r, \boldsymbol{y})=\lim\int_{\boldsymbol{R}^d}\overbrace{\int_{\boldsymbol{R}^{2d}}\cdots\int_{\boldsymbol{R}^{2d}}}^{n-1}\prod_{j=1}^{n}$$
$$\times\exp\Big\{i\Big[(\boldsymbol{p}_{j-1}+e\boldsymbol{A}(t_{j-1}, \boldsymbol{x}_{j-1}))\frac{\boldsymbol{x}_j-\boldsymbol{x}_{j-1}}{t_j-t_{j-1}}$$
$$-(\boldsymbol{\alpha}\boldsymbol{p}_{j-1}+m\beta)-e\Phi(t_{j-1}, \boldsymbol{x}_{j-1})\Big](t_j-t_{j-1})\Big\}$$
$$\times(2\pi)^{-d}d\boldsymbol{p}_0(2\pi)^{-d}(d\boldsymbol{p}_1d\boldsymbol{x}_1)\cdots(2\pi)^{-d}(d\boldsymbol{p}_{n-1}d\boldsymbol{x}_{n-1})\,. \tag{2·5}$$

Note that the integrand of the integral on the right-hand side of (2·5) is rewritten as

$$\prod_{j=1}^{n}\{\exp[i\boldsymbol{x}_j\boldsymbol{p}_{j-1}]\exp[-i(\boldsymbol{\alpha}\boldsymbol{p}_{j-1}+m\beta)(t_j-t_{j-1})]\exp[-i\boldsymbol{x}_{j-1}\boldsymbol{p}_{j-1}]\}$$
$$\times\exp\{i\sum_{k=1}^{n}[e\boldsymbol{A}(t_{k-1}, \boldsymbol{x}_{k-1})(\boldsymbol{x}_k-\boldsymbol{x}_{k-1})-e\Phi(t_{k-1}, \boldsymbol{x}_{k-1})(t_k-t_{k-1})]\}\,.$$

Here $r=t_0<t_1<\cdots<t_n=s$ and $\boldsymbol{x}_0=\boldsymbol{y}$, $\boldsymbol{x}_j=\boldsymbol{X}(t_j)$, $j=1, \cdots, n-1$, $\boldsymbol{x}_n=\boldsymbol{x}$, and the product $\prod_{j=1}^{n}$ is time-ordered with time increasing from the right to the left. The limit is taken for $n\to\infty$ and $\max_{1\le j\le n}(t_j-t_{j-1})\to 0$. If $K_0(t, \boldsymbol{x})$ is the fundamental solution of the Cauchy problem for the free Dirac equation, i.e., $K_0(t, \boldsymbol{x})=\int_{\boldsymbol{R}^d}\exp[-it(\boldsymbol{\alpha}\boldsymbol{p}+m\beta)+i\boldsymbol{x}\boldsymbol{p}](2\pi)^{-d}d\boldsymbol{p}$, then the $\boldsymbol{p}_j$ integrations on the right-hand side of (2·5) yield

$$K(s, \boldsymbol{x}; r, \boldsymbol{y}) = \lim \overbrace{\int_{\boldsymbol{R}^d} \cdots \int_{\boldsymbol{R}^d}}^{n-1} K_0(t_n - t_{n-1}, \boldsymbol{x}_n - \boldsymbol{x}_{n-1}) \cdots K_0(t_1 - t_0, \boldsymbol{x}_1 - \boldsymbol{x}_0)$$

$$\times \exp\{i \sum_{j=1}^{n} [e\boldsymbol{A}(t_{j-1}, \boldsymbol{x}_{j-1})(\boldsymbol{x}_j - \boldsymbol{x}_{j-1})$$

$$- e\Phi(t_{j-1}, \boldsymbol{x}_{j-1})(t_j - t_{j-1})]\} d\boldsymbol{x}_1 \cdots d\boldsymbol{x}_{n-1}.$$

If a path space measure $\nu_{s,\boldsymbol{x};r,\boldsymbol{y}}$ should be constructed from the product of K_0's and the Lebesgue measures $d\boldsymbol{x}_j$, we should get

$$K(s, \boldsymbol{x}; r, \boldsymbol{y}) = \int_{\mathscr{X}_{s,\boldsymbol{x};r,\boldsymbol{y}}} d\nu_{s,\boldsymbol{x};r,\boldsymbol{y}}(\boldsymbol{X})$$

$$\times \exp\left[-i \int_r^s e\Phi(t, \boldsymbol{X}(t))dt + i \int_r^s e\boldsymbol{A}(t, \boldsymbol{X}(t))d\boldsymbol{X}(t)\right], \qquad (2\cdot6)$$

where $\mathscr{X}_{s,\boldsymbol{x};r,\boldsymbol{y}}$ is the space of the paths $\boldsymbol{X}(t)$ satisfying $\boldsymbol{X}(r) = \boldsymbol{y}$ and $\boldsymbol{X}(s) = \boldsymbol{x}$.

What we carry out in the following sections is to justify with mathematical rigor the above heuristic argument, in two space-time dimensions or $d=1$, to establish the formula (2·6) and related ones.

§ 3. Path integral representations

In two space-time dimensions, we give path integral formulas for the fundamental solutions of two Cauchy problems with the Dirac equation.[6] One of them is further used to give path integral formulas for the retarded and advanced propagators. By $|r, s|$ is meant the closed interval $r \le t \le s$ or $r \ge t \ge s$ according to $r < s$ or $r > s$. We use the convention of summation over repeated Greek indices.

3.1. *The fundamental solutions of the Cauchy problems*

We consider the Dirac equation (2·1) in two-dimensional space-time,

$$\partial_t \phi(t, x) = [-\alpha(\partial_x - ieA(t, x)) - im\beta - ie\Phi(t, x)]\phi(t, x),$$

$$t \in \boldsymbol{R}, x \in \boldsymbol{R}. \qquad (3\cdot1)$$

Here α and β are 2×2 Hermitian matrices with $\alpha^2 = \beta^2 = \boldsymbol{1}$ and $\alpha\beta + \beta\alpha = 0$. The vector and scalar potential $A(t, x)$ and $\Phi(t, x)$ are real-valued functions in space-time $\boldsymbol{R}\times\boldsymbol{R} = \boldsymbol{R}^2$. We assume for simplicity that they are *continuous* functions on $\boldsymbol{R}^2$. The case for somewhat more general A and Φ is referred to Ref. 11). One Cauchy problem we consider is that for (3·1) with data $\phi(r, x) = g(x)$. Since Eq. (3·1) is a first-order hyperbolic system with two independent variables, this Cauchy problem can be solved[21] along the characteristics and so has a unique solution.

We write $A^0(t, x) = \Phi(t, x)$ and $A^1(t, x) = A(t, x)$. Set $A_\rho(t, x) = g_{\rho\sigma}A^\sigma(t, x)$, $\rho = 0, 1$, with the metric tensor

$$(g_{\rho\sigma})=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

By putting $x^0=t$ and $x^1=x$, (3·1) is rewritten as

$$iH^{\rm II}\phi(\boldsymbol{x})\equiv[(\partial_0+ieA_0(\boldsymbol{x}))+\alpha(\partial_1+ieA_1(\boldsymbol{x}))+im\beta]\phi(\boldsymbol{x})=0\,, \tag{3·2}$$

where $\boldsymbol{x}=(x^0, x^1)\in\boldsymbol{R}^2$ and $\partial_\rho=\partial/\partial x^\rho$, $\rho=0, 1$. $H^{\rm II}$ is essentially self-adjoint on $C_o^\infty(\boldsymbol{R}^2; \boldsymbol{C}^2)$, and defines[14] a self-adjoint operator in $L^2(\boldsymbol{R}^2; \boldsymbol{C}^2)$.

We introduce the fictitious time τ to consider the other Cauchy problem for

$$\partial_\tau\psi(\tau, \boldsymbol{x})=-iH^{\rm II}\psi(\tau, \boldsymbol{x})\,, \qquad \tau\in\boldsymbol{R}\,, \qquad \boldsymbol{x}\in\boldsymbol{R}^2 \tag{3·3}$$

with data $\psi(r, \boldsymbol{x})=g(\boldsymbol{x})$. Let $K^{\rm I}(s, x; r, y)$ and $K^{\rm II}(s, \boldsymbol{x}; r, \boldsymbol{y})$ be the fundamental solutions of the Cauchy problems for (3·1) and (3·2), respectively:

$$\phi(s, x)=\int_{\boldsymbol{R}}K^{\rm I}(s, x; r, y)g(y)dy\,, \tag{3·4}$$

$$\psi(s, \boldsymbol{x})=(e^{-i(s-r)H^{\rm II}}g)(\boldsymbol{x})=\int_{\boldsymbol{R}^2}K^{\rm II}(s, \boldsymbol{x}; r, \boldsymbol{y})g(\boldsymbol{y})d\boldsymbol{y}\,. \tag{3·5}$$

Then they admit the following path integral representations. We say that a measure μ on a space Ω is *concentrated* on a subset E of Ω if μ vanishes on $\Omega\setminus E$. $M_2(\boldsymbol{C})$ is the space of complex 2×2 matrices.

THEOREM 3.1. There exists a unique $\mathcal{S}'(\boldsymbol{R}\times\boldsymbol{R}; M_2(\boldsymbol{C}))$-valued countably additive measure $\nu^{\rm I}_{s;r}$ on the Banach space $C(|r, s|; \boldsymbol{R})$ of the one-dimensional continuous path $X: |r, s|\to\boldsymbol{R}$ such that for every continuous $A(t, x)$ and $\Phi(t, x)$,

$$\begin{aligned}(f, \phi(s, \cdot))&=\iint_{\boldsymbol{R}\times\boldsymbol{R}}\overline{f(x)}K^{\rm I}(s, x; r, y)g(y)dxdy\\ &=\int(f, d\nu^{\rm I}_{s;r}(X)g)\exp\Big[-i\int_r^s e\Phi(t, X(t))dt+i\int_r^s eA(t, X(t))dX(t)\Big]\end{aligned} \tag{3·6}$$

with (f, g) in $\mathcal{S}(\boldsymbol{R}; (\boldsymbol{C}^2)')\times\mathcal{S}(\boldsymbol{R}; \boldsymbol{C}^2)$. The measure $\nu^{\rm I}_{s;r}$ is concentrated on the set of those Lipschitz continuous paths $X: |r, s|\to R$ which satisfy

for some finite partition: $r=t_0\leqq t_1\leqq\cdots\leqq t_k=s$ of $|r, s|$, depending on X,

$$X(t)-X(r)=\sum_{i=1}^{j-1}(-1)^i(t_i-t_{i-1})+(-1)^j(t-t_{j-1}) \qquad \text{or}$$

$$X(t)-X(r)=\sum_{i=1}^{j-1}(-1)^{i-1}(t_i-t_{i-1})+(-1)^{j-1}(t-t_{j-1})$$

for $t\in|t_{j-1}, t_j|\,, \qquad 1\leq j\leq k$

$$[|X(t)-X(r)|=|t-r| \text{ for } t\in|r, s|, \text{ in case } m=0]\,. \tag{3·7}$$

The set function $\nu^{\rm I}_{s,f;r,g}$ defined by

$$\nu^{\rm I}_{s,f;r,g}(\cdot)=\langle\bar{f}\otimes g, \nu^{\rm I}_{s;r}(\cdot)\rangle=(f, \nu^{\rm I}_{s;r}(\cdot)g) \tag{3·8}$$

is a complex-valued countably additive measure on the Banach space $C(|r, s|; \boldsymbol{R})$ which is concentrated on the set of the Lipschitz continuous paths X satisfying (3·7) and $X(r)\in \mathrm{supp}\,g$, $X(s)\in \mathrm{supp}\,f$.

THEOREM 3.2. There exists a unique $\mathcal{S}'(\boldsymbol{R}^2\times\boldsymbol{R}^2; M_2(\boldsymbol{C}))$-valued countably additive measure $\nu^{\mathrm{II}}_{s;r}$ on the Banach space $C(|r, s|; \boldsymbol{R}^2)$ of the two-dimensional continuous paths $\boldsymbol{X}: |r, s|\to\boldsymbol{R}^2$, $\boldsymbol{X}(\tau)=(X^0(\tau), X^1(\tau))$, such that for every continuous $\boldsymbol{A}(\boldsymbol{x})=(A_0(\boldsymbol{x}), A_1(\boldsymbol{x}))$,

$$(f, e^{-i(s-r)H^{\mathrm{II}}}g)=\iint_{\boldsymbol{R}^2\times\boldsymbol{R}^2}\overline{f(\boldsymbol{x})}K^{\mathrm{II}}(s, \boldsymbol{x}; r, \boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y}$$

$$=\int(f, d\nu^{\mathrm{II}}_{s;r}(\boldsymbol{X})g)\exp\left[-i\int_r^s eA_\rho(\boldsymbol{X}(\tau))dX^\rho(\tau)\right] \qquad (3\cdot 9)$$

with (f, g) in $\mathcal{S}(\boldsymbol{R}^2; (\boldsymbol{C}^2)')\times\mathcal{S}(\boldsymbol{R}^2; \boldsymbol{C}^2)$. The measure $\nu^{\mathrm{II}}_{s,r}$ is concentrated on the set of those Lipschitz continuous paths $\boldsymbol{X}: |r, s|\to\boldsymbol{R}^2$ which satisfy

$$X^0(t)-X^0(r)=t-r \qquad \text{for} \qquad t\in|r, s|,$$

and, for some partition : $r=t_0\leqq t_1\leqq\cdots\leqq t_k=s$ of $|r, s|$, depending on X,

$$X^1(t)-X^1(r)=\sum_{i=1}^{j-1}(-1)^i(t_i-t_{i-1})+(-1)^j(t-t_{j-1}) \quad \text{or}$$

$$X^1(t)-X^1(r)=\sum_{i=1}^{j-1}(-1)^{i-1}(t_i-t_{i-1})+(-1)^{j-1}(t-t_{j-1}),$$

for $t\in|t_{j-1}, t_j|$, $1\leq j\leq k$

$$[|X^1(t)-X^1(r)|=|t-r| \quad \text{for } t\in|r, s|, \text{ in case } m=0]. \qquad (3\cdot 10)$$

The set function $\nu^{\mathrm{II}}_{s,f;r,g}$ defined by

$$\nu^{\mathrm{II}}_{s,f;r,g}(\cdot)=\langle\bar{f}\otimes g, \nu^{\mathrm{II}}_{s;r}(\cdot)\rangle=(f, \nu^{\mathrm{II}}_{s;r}(\cdot)g) \qquad (3\cdot 11)$$

is a complex-valued countably additive measure on the Banach space $C(|r, s|; \boldsymbol{R}^2)$ which is concentrated on the set of the Lipschitz continuous paths $\boldsymbol{X}$ satisfying (3·10) and $\boldsymbol{X}(r)\in\mathrm{supp}\,g$, $\boldsymbol{X}(s)\in\mathrm{supp}\,f$.

The proofs of Theorems 3.1 and 3.2 are given in § 4.

These theorems were first shown in Refs. 11) and 13), with a somewhat weaker statement on the support property of the path space measures, to the effect that $\nu^{\mathrm{I}}_{s;r}$ is, when $m>0$, concentrated on the set of the Lipschitz continuous paths $X:|r, s|\to\boldsymbol{R}$ satisfying $|X(b)-X(a)|\leq|b-a|$ for every a, b with $r\leqq a\leqq b\leqq s$, and similarly for $\nu^{\mathrm{II}}_{s;r}$. Next it was improved in Ref. 34) as follows: $\nu^{\mathrm{I}}_{s;r}$ is, when $m>0$, concentrated on the set of the Lipschitz continuous paths $X: |r, s|\to\boldsymbol{R}$ which are differentiable, outside a closed subset E_X of Lebesgue measure zero depending on X, with differential coefficients $(d/dt)X(t)$ equal to 1 or -1 on each (relatively) open interval in $|r, s|\backslash E_X$, and similarly for $\nu^{\mathrm{II}}_{s;r}$. Finally the support property of the ultimate form, (3·7) and (3·10), has been shown in our recent note.[35)]

Remark 1. $\nu^{\mathrm{I}}_{s;r}$ and $\nu^{\mathrm{II}}_{s;r}$ may be regarded as *conditional* path space measures (cf.

conditional Wiener measure[5]). In fact, introduce formal *conditional* path space measures $\nu^{\mathrm{I}}_{s,x;\,r,y}$ and $\nu^{\mathrm{II}}_{s,\boldsymbol{x};\,r,\boldsymbol{y}}$ as

$$\nu^{\mathrm{I}}_{s,f;\,r,g}(\cdot)=\iint_{\boldsymbol{R}\times\boldsymbol{R}}\overline{f(x)}\,\nu^{\mathrm{I}}_{s,x;\,r,y}(\cdot)g(y)\,dxdy$$

and

$$\nu^{\mathrm{II}}_{s,f;\,r,g}(\cdot)=\iint_{\boldsymbol{R}^2\times\boldsymbol{R}^2}\overline{f(\boldsymbol{x})}\,\nu^{\mathrm{II}}_{s,\boldsymbol{x};\,r,\boldsymbol{y}}(\cdot)g(\boldsymbol{y})\,d\boldsymbol{x}d\boldsymbol{y}\,.$$

Then Theorems 3.1 and 3.2 look like

$$K^{\mathrm{I}}(s,x;r,y)=\int d\nu^{\mathrm{I}}_{s,x;\,r,y}(X)\times\exp\Big[-i\int_r^s e\Phi(t,X(t))dt+i\int_r^s eA(t,X(t))dX(t)\Big] \tag{3·6}'$$

and

$$K^{\mathrm{II}}(s,\boldsymbol{x};r,\boldsymbol{y})=\int d\nu^{\mathrm{II}}_{s,\boldsymbol{x};\,r,\boldsymbol{y}}(\boldsymbol{X})\exp\Big[-i\int_r^s eA_\rho(\boldsymbol{X}(\tau))dX^\rho(\tau)\Big]\,. \tag{3·9}'$$

The formal measure $\nu^{\mathrm{I}}_{s,x;\,r,y}[\nu^{\mathrm{II}}_{s,\boldsymbol{x};\,r,\boldsymbol{y}}]$ is concentrated on the set of the Lipschitz continuous paths X satisfying (3·7) [$\boldsymbol{X}$ satisfying (3·10)] and $X(r)=y$, $X(s)=x$ [$\boldsymbol{X}(r)=\boldsymbol{y}$, $\boldsymbol{X}(s)=\boldsymbol{x}$].

Notice that (3·6)′ is coincident with the formula (2·6) in two space-time dimensions.

Remark 2. The support property of $\nu^{\mathrm{I}}_{s;\,r}$ and $\nu^{\mathrm{II}}_{s;\,r}$ in Theorems 3.1 and 3.2 tells us the nature of the "Zitterbewegung"[7] of the Dirac particle. The motion described by a path satisfying (3·7) or (3·10) is such that the velocity is, in magnitude, equal to 1, the light velocity, at every finite time interval except for finite instants of time where the velocity alters the sign. Here the role the mass m plays is not to render the magnitude of the velocity smaller than 1, but to change the direction of motion of the particle time after time.

Remark 3. Feynman and Hibbs[2] give briefly a cryptic description of the fundamental solution of the Cauchy problem for the *free* Dirac equation in two space-time dimensions. For this context we refer also to Riazanov[15] and Rosen.[16] There are some recent contributions to this problem based on Poisson process. See Gaveau et al.,[38] Gaveau,[39] Blanchard et al.,[40] Jacobson[41] and de Angelis et al.[42]

Remark 4. Daletskii[17] dealt with related problems but did construct no countably additive path space measure.

3.2. *The retarded and advanced propagators*

The propagator[18] for a two-space-time-dimensional Dirac particle is a 2×2 matrix-valued function (distribution) which is a solution of Green's function equation

$$[\gamma^\rho(-i\partial_\rho+eA_\rho(\boldsymbol{x}))+m]S(\boldsymbol{x},\boldsymbol{y})=\delta(\boldsymbol{x}-\boldsymbol{y})\mathbf{1}\,,\qquad \boldsymbol{x},\boldsymbol{y}\in\boldsymbol{R}^2\,. \tag{3·12}$$

Here $\gamma^0=\beta$ and $\gamma^1=\beta\alpha$. γ^0 is Hermitian and γ^1 anti-Hermitian with $\gamma^\rho\gamma^\sigma+\gamma^\sigma\gamma^\rho=2\delta^{\rho\sigma}\mathbf{1}$, where $\rho, \sigma=0, 1$. The convention of summation over repeated Greek indices is used. The left-hand side of (3·12) is nothing but $\gamma^0 H^{\mathrm{II}} S(\boldsymbol{x}, \boldsymbol{y})$ with H^{II} in (3·2).

Then the retarded and advanced propagators $S_{\mathrm{ret}}(\boldsymbol{x}, \boldsymbol{y})$ and $S_{\mathrm{adv}}(\boldsymbol{x}, \boldsymbol{y})$ have the following path integral representations.

THEOREM 3.3. For (f, g) in $\mathcal{D}(\boldsymbol{R}^2; (\boldsymbol{C}^2)')\times\mathcal{D}(\boldsymbol{R}^2; \boldsymbol{C}^2)$,

$$\iint_{\boldsymbol{R}^2\times\boldsymbol{R}^2}\overline{f(\boldsymbol{x})}S_{\mathrm{ret}}(\boldsymbol{x}, \boldsymbol{y})g(\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y}$$

$$=i\int_0^\infty d\tau\int(f, d\nu^{\mathrm{II}}_{\tau;0}(\boldsymbol{X})\gamma^0 g)\exp\left[-i\int_0^\tau eA_\rho(\boldsymbol{X}(s))dX^\rho(s)\right] \tag{3·13}$$

and

$$\iint_{\boldsymbol{R}^2\times\boldsymbol{R}^2}\overline{f(\boldsymbol{x})}S_{\mathrm{adv}}(\boldsymbol{x}, \boldsymbol{y})g(\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y}$$

$$=-i\int_{-\infty}^0 d\tau\int(f, d\nu^{\mathrm{II}}_{\tau;0}(\boldsymbol{X})\gamma^0 g)\exp\left[-i\int_0^\tau eA_\rho(\boldsymbol{X}(s))dX^\rho(s)\right]. \tag{3·14}$$

Remark 1. With the formal conditional path space measure $\nu^{\mathrm{II}}_{\tau,\boldsymbol{x};0,\boldsymbol{y}}$ introduced in Remark 1 to Theorems 3.1 and 3.2, (3·13) and (3·14) become

$$S_{\mathrm{ret}}(\boldsymbol{x}, \boldsymbol{y})=i\int_0^\infty d\tau\int d\nu^{\mathrm{II}}_{\tau,\boldsymbol{x};0,\boldsymbol{y}}(\boldsymbol{X})\gamma^0\exp\left[-i\int_0^\tau eA_\rho(\boldsymbol{X}(s))dX^\rho(s)\right]$$

and

$$S_{\mathrm{adv}}(\boldsymbol{x}, \boldsymbol{y})=-i\int_{-\infty}^0 d\tau\int d\nu^{\mathrm{II}}_{\tau,\boldsymbol{x};0,\boldsymbol{y}}(\boldsymbol{X})\gamma^0\exp\left[-i\int_0^\tau eA_\rho(\boldsymbol{X}(s))dX^\rho(s)\right]. \tag{3·14$'$}$$

Remark 2. For the Feynman propagator $S_F(\boldsymbol{x}, \boldsymbol{y})$ we have not such a neat formula.

Proof of Theorem 3.3. We simply write H for H^{II}. We prove only (3·13) for the retarded propagator. The proof for the advanced propagator is similar. Let (f, g) be in $\mathcal{D}(\boldsymbol{R}^2; (\boldsymbol{C}^2)')\times\mathcal{D}(\boldsymbol{R}^2; \boldsymbol{C}^2)$. Note that H is a self-adjoint operator in $L^2(\boldsymbol{R}^2; \boldsymbol{C}^2)$ and

$$(\varepsilon+iH)^{-1}=\int_0^\infty d\tau\ e^{-\varepsilon\tau}e^{-i\tau H}\,.$$

Then by Theorem 3.2 we have

$$(f, i(\varepsilon+iH)^{-1}\gamma^0 g)=i\int_0^\infty d\tau\ e^{-\varepsilon\tau}\int(f, d\nu^{\mathrm{II}}_{\tau;0}(\boldsymbol{X})\gamma^0 g)$$

$$\times\exp\left[-i\int_0^\tau eA_\rho(\boldsymbol{X}(s))dX^\rho(s)\right]. \tag{3·15}$$

Since f and g have compact support, the τ-integral on the right-hand side of (3·15) is reduced to that over a finite interval in view of the support property of $\nu^{\mathrm{II}}_{\tau,f;0,g}$. Therefore, it converges to the right-hand side of (3·13) as $\varepsilon\downarrow 0$, by the Lebesgue bounded convergence theorem. On the other hand, by definition of the retarded propagator the left-hand side of (3·15) converges to that of (3·13) as $\varepsilon\downarrow 0$. This proves Theorem 3.3.

§ 4. Proofs of Theorems 3.1 and 3.2

We shall prove only Theorem 3.1. Theorem 3.2 will be shown similarly. (Proof of Theorem 3.2 with a somewhat weaker statement on the support property of $\nu^{\mathrm{II}}_{s;r}$ is given in Ref. 13).) Without loss of generality we may assume $r=0$ and $s>0$ to construct $\nu^{\mathrm{I}}_{s;0}$. The proof consists of three parts. First we construct the path space measure $\nu^{\mathrm{I}}_{s;0}$ on the product space $(\dot{\boldsymbol{R}})^{[0,s]}$ of the uncountably many copies of $\dot{\boldsymbol{R}}$, where $\dot{\boldsymbol{R}}=\boldsymbol{R}\cup\{\infty\}$ is the one-point compactification of $\boldsymbol{R}$. Next its support is determined and finally the path integral formula (3·6) is established. For the first and last parts we follow mainly Refs. 13), 11) and for the second, our recent result in Ref. 35).

4.1. *Construction of the path space measure* $\nu^{I}_{s;0}$

We construct $\nu^{\mathrm{I}}_{s;0}$, using Nelson's method[10),5),19)] of construction of the Wiener measure.

First consider the Cauchy problem for the free equation to (3·1),

$$\partial_t\phi(t,x)=[-\alpha\partial_x-im\beta]\phi(t,x)\,,\qquad t\in\boldsymbol{R},\,x\in\boldsymbol{R} \tag{4·1}$$

with initial data $\phi(0,x)=g(x)$. Let $K_0{}^{\mathrm{I}}(s,x)$ be the fundamental solution:

$$\phi(s,x)=(e^{-s(\alpha\partial_x+im\beta)}g)(x)=\int_{\boldsymbol{R}}K_0{}^{\mathrm{I}}(s,x-y)g(y)dy\,. \tag{4·2}$$

It is given by

$$K_0{}^{\mathrm{I}}(s,x)=2^{-1}[\partial_s-\alpha\partial_x-im\beta](J_0(m(s^2-x^2)^{1/2})\theta(s-|x|))\,, \tag{4·3}$$

where $J_0(t)$ is the Bessel function of order zero, and $\theta(t)$ the Heaviside function $\theta(t)=1$ for $t>0$, $=0$ for $t<0$. $C_\infty(\boldsymbol{R};\boldsymbol{C}^2)$ denotes the Banach space of the $\boldsymbol{C}^2$-valued continuous functions in $\boldsymbol{R}$ which vanish at infinity. Its dual space, denoted by $M(\boldsymbol{R};(\boldsymbol{C}^2)')$, is the Banach space of the $(\boldsymbol{C}^2)'$-valued measures on $\boldsymbol{R}$ with bounded variation.[20)]

Lemma 4.1. (1) $e^{-t(\alpha\partial_x+im\beta)}$ is a continuous linear operator of $C_\infty(\boldsymbol{R};\boldsymbol{C}^2)$ into itself, and satisfies

$$\|Ne^{-t(\alpha\partial_x+im\beta)}g\|\le e^{m|t|}\|Ng\| \tag{4·4}$$

for g in $C_\infty(\boldsymbol{R};\boldsymbol{C}^2)$. Here N is a unitary matrix satisfying

$$N\alpha N^{-1}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

(2) $e^{-t(\alpha\partial_x+im\beta)}$ is a continuous linear operator of $\mathcal{S}(\boldsymbol{R};\boldsymbol{C}^2)$ into itself.

Proof These are derived by straightforward calculation using (4·2) and (4·3).

The content of Lemma 4.1 is that the Cauchy problem for (4·1) is L^∞ well-posed. This holds also for the hyperbolic system of the first order with two independent variables.[21)]

Now we are ready to construct the path space measure $\nu^{\mathrm{I}}_{s;0}$. For each fixed $s>0$ let $\mathcal{X}_{s,0}=\prod_{[0,s]}\dot{\boldsymbol{R}}=(\dot{\boldsymbol{R}})^{[0,s]}$ be the product of the uncountably many copies of $\dot{\boldsymbol{R}}$. By the Tychonoff theorem[22] $\mathcal{X}_{s,0}$ is a compact Haussdorff space in the product topology. It may be regarded as the space of all paths $X:[0, s]\to\dot{\boldsymbol{R}}$, possibly discontinuous and possibly passing through the infinity ∞. Let $C(\mathcal{X}_{s,0})$ be the Banach space of the complex-valued continuous functions on $\mathcal{X}_{s,0}$ and $C_{\mathrm{fin}}(\mathcal{X}_{s,0})$ the subspace of those Ψ in $C(\mathcal{X}_{s,0})$ for which there exist a finite partition:

$$0=s_0<s_1<\cdots<s_n=s \tag{4·5}$$

of the interval $[0, s]$ and a complex-valued bounded continuous function $F(x_0, x_1, \cdots, x_n)$ on $(\dot{\boldsymbol{R}})^{n+1}$ such that

$$\Psi(X)=F(X(s_0), X(s_1), \cdots, X(s_n))\,. \tag{4·6}$$

Define, for each fixed $s>0$, a functional $L_{s,0}(\Phi;\mu, g)$ which is linear in $\Psi\in C_{\mathrm{fin}}(\mathcal{X}_{s,0})$ and sesquilinear in $(\mu, g)\in M(\boldsymbol{R};(\boldsymbol{C}^2)')\times C_\infty(\boldsymbol{R};\boldsymbol{C}^2)$ by

$$\begin{aligned}L_{s,0}(\Psi;\mu, g)=\int_{\boldsymbol{R}}dx_0\cdots\int_{\boldsymbol{R}}dx_{n-1}\int_{\boldsymbol{R}}d\overline{\mu(x_n)}K_0{}^{\mathrm{I}}(s_n-s_{n-1}, x_n-x_{n-1})\\ \times K_0{}^{\mathrm{I}}(s_{n-1}-s_{n-2}, x_{n-1}-x_{n-2})\cdots K_0{}^{\mathrm{I}}(s_1-s_0, x_1-x_0)\\ \times F(x_0, x_1, \cdots, x_n)g(x_0)\,.\end{aligned} \tag{4·7}$$

When stressing the sesquilinearity of $L_{s,0}(\Psi;\mu, g)$ we shall write it also as $(L_{s,0}\Psi)(\mu, g)$.

Crucial is the following Lemma.

Lemma 4.2. (1) For each fixed (μ, g) in $M(\boldsymbol{R};(\boldsymbol{C}^2)')\times C_\infty(\boldsymbol{R};\boldsymbol{C}^2)$, $L_{s,0}(\Psi;\mu, g)$ is well-defined on $C_{\mathrm{fin}}(\mathcal{X}_{s,0})$; it is independent of the choice of F corresponding to Ψ.
(2) The following inequality holds

$$|L_{s,0}(\Psi;\mu, g)|\le Ce^{m|s|}\|\Psi\|\|\mu\|\|g\| \tag{4·8}$$

for every Ψ in $C_{\mathrm{fin}}(\mathcal{X}_{s,0})$ and every pair (μ, g) in $M(\boldsymbol{R};(\boldsymbol{C}^2)')\times C_\infty(\boldsymbol{R};\boldsymbol{C}^2)$ with $C=|N||N^{-1}|\le 2$.

Proof By C_t we denote the operator $Ne^{-t(\alpha\partial x+im\beta)}N^{-1}$ on $C_\infty(\boldsymbol{R};\boldsymbol{C}^2)$. The statement (1) is a consequence of the (semi-)group property of the operator C_t. To prove (2), let Ψ be in $C_{\mathrm{fin}}(\mathcal{X}_{s,0})$ and represented as (4·6) with a continuous function $F(x_0, x_1, \cdots, x_n)$ on $(\dot{\boldsymbol{R}})^{n+1}$. We inductively define a sequence $\{F^{(l)}_{x_{l+1},\cdots,x_n}\}_{l=1}^{n}$ of $\boldsymbol{C}^2$-valued functions $F^{(l)}_{x_{l+1},\cdots,x_n}$ on $\boldsymbol{R}$ with $n-l$ parameters $x_{l+1}, \cdots, x_n\in\boldsymbol{R}$ as follows:

$$F^{(0)}_{x_1,x_2,\cdots,x_n}(x)\equiv Ng(x)F(x, x_1, x_2, \cdots, x_n)\,,$$

$$F^{(l)}_{x_{l+1},\cdots,x_n}(x)\equiv(C_{s_l-s_{l-1}}F^{(l-1)}_{x,x_{l+1},\cdots,x_n})(x)\,,$$

$$F^{(n)}(x)\equiv(C_{s_n-s_{n-1}}F_x{}^{(n-1)})(x)$$

for $l=1, 2, \cdots, n$. By induction and continuity of the operator C_t in Lemma 4.1(1), we see that for each fixed $l=1, \cdots, n$, $F^{(l)}_{x_{l+1},\cdots,x_n}$ is in $C_\infty(\boldsymbol{R};\boldsymbol{C}^2)$ and continuous as a map of the parameter space $\boldsymbol{R}^{n-l}$ into $C_\infty(\boldsymbol{R};\boldsymbol{C}^2)$. Note that (4·7) is rewritten as

$$L_{s,0}(\Psi;\mu,g)=\int_{R}d\overline{\mu(x_n)}N^{-1}F^{(n)}(x_n)\,. \tag{4·9}$$

Making an iterative use of the estimate (4·4), we get

$$\begin{aligned}\|F^{(n)}\|&=\sup_{x,j=1,2}|(C_{s_n-s_{n-1}}F_x{}^{(n-1)})_j(x)|\le\sup_x\|C_{s_n-s_{n-1}}F_x{}^{(n-1)}\|\\&\le e^{m(s_n-s_{n-1})}\sup_{x_n}\|F_{x_n}^{(n-1)}\|\le\cdots\le e^{ms}\sup_{x_1,\cdots,x_n}\|F_{x_1,\cdots,x_n}^{(0)}\|\\&\le e^{ms}\|Ng\|\|F\|\,.\end{aligned} \tag{4·10}$$

Then the estimate (4·8) is derived from (4·9) and (4·10).

The consequence of Lemma 4.2 is the following. Since $C_{\rm fin}(\mathcal{X}_{s,0})$ is dense in $C(\mathcal{X}_{s,0})$ by the Stone-Weierstrass theorem,[22] the inequality (4·8) holds also for $\Psi\in C(\mathcal{X}_{s,0})$. By Lemma 4.2, $L_{s,0}\Psi$ is a continuous sesquilinear form on $M(\boldsymbol{R};(\boldsymbol{C}^2)')\times C_\infty(\boldsymbol{R};\boldsymbol{C}^2)$, and so on $\mathcal{S}(\boldsymbol{R};(\boldsymbol{C}^2)')\times\mathcal{S}(\boldsymbol{R};\boldsymbol{C}^2)$, because both linear embeddings of $\mathcal{S}(\boldsymbol{R};(\boldsymbol{C}^2)')$ into $M(\boldsymbol{R};(\boldsymbol{C}^2)')$ and of $\mathcal{S}(\boldsymbol{R};\boldsymbol{C}^2)$ into $C_\infty(\boldsymbol{R};\boldsymbol{C}^2)$ are continuous. Then the kernel theorem[22] enables us to regard $L_{s,0}\Psi$ as an element in the space

$$\mathcal{S}'(\boldsymbol{R}\times\boldsymbol{R};M_2(\boldsymbol{C}))=(\mathcal{S}(\boldsymbol{R};(\boldsymbol{C}^2)')\otimes\mathcal{S}(\boldsymbol{R};\boldsymbol{C}^2))'\,. \tag{4·11}$$

Here note that $M_2(\boldsymbol{C})=\boldsymbol{C}^2\otimes(\boldsymbol{C}^2)'$. Hence $L_{s,0}$ is a continuous linear mapping of $C(\mathcal{X}_{s,0})$ into the space (4·11). Further $L_{s,0}$ is weakly compact[23] because the space is reflexive, and even compact[24] because the space (4·11) is a Montel space. Then the Riesz-type represetation theorem[25] yields the following representation of $L_{s,0}$ in terms of a countably additive measure on $\mathcal{X}_{s,0}$.

THEOREM 4.3. There exists a unique countably additive measure $\nu^{\rm I}_{s;0}$ defined on the Borel sets in $\mathcal{X}_{s,0}$ and having values in $\mathcal{S}'(\boldsymbol{R}\times\boldsymbol{R};M_2(\boldsymbol{C}))$ such that

(a) $\nu^{\rm I}_{s;0}$ is of bounded q-variation for each continuous seminorm q on $\mathcal{S}'(\boldsymbol{R}\times\boldsymbol{R};M_2(\boldsymbol{C}))$, i.e.,

$$q-{\rm Var}\ \nu^{\rm I}_{s;0}\equiv\sup\ q(\sum_j\nu^{\rm I}_{s;0}(E_j)c_j)<\infty\,,$$

where the supremum is taken over all finite partition $\{E_j\}$ of $\mathcal{X}_{s,0}$ into disjoint Borel sets and all collection $\{c_j\}$ of complex numbers with $|c_j|\le1$;

(b) for each (f,g) in $\mathcal{S}(\boldsymbol{R};(\boldsymbol{C}^2)')\times\mathcal{S}(\boldsymbol{R};\boldsymbol{C}^2)$ the set function $\nu^{\rm I}_{s,f;0,g}$ defined by

$$\nu^{\rm I}_{s,f;0,g}(E)=\langle f\times g,\nu^{\rm I}_{s;0}(E)\rangle=(f,\nu^{\rm I}_{s;0}(E)g)$$

is a complex-valued countably additive regular measure on the Borel sets E in $\mathcal{X}_{s,0}$;

(c) $\|L_{s,0}\|_q\equiv\sup\{q(L_{s,0}\Psi);\|\Psi\|\le1,\ \Psi\in C(\mathcal{X}_{s,0})\}=q-{\rm Var}\,\nu^{\rm I}_{s;0}\le C'e^{m|s|}$

with a constant C' depending only on the seminorm q;

(d) for each Ψ in $C(\mathcal{X}_{s,0})$,

$$L_{s,0}\Psi=\int_{\mathcal{X}_{s,0}}d\nu^{\rm I}_{s;0}(X)\Psi(X)\,. \tag{4·12}$$

Remark 1. In terms of the formal path space measure $\nu^{\mathrm{I}}_{s,x;0,y}$ in Remark 1 to Theorems 3.1 and 3.2, the expression (4·12) looks like

$$(L_{s,0}\Psi)(x,y)=\int_{\mathscr{X}_{s,0}} d\nu^{\mathrm{I}}_{s,x;0,y}(X)\Psi(X). \tag{4·12$'$}$$

Remark 2. There is another way to construct a path space measure. First note that we can define $L_{s,0}(\Phi;\mu,g)$ to establish Lemma 4.2 for the pair $M(\dot{\boldsymbol{R}};(\boldsymbol{C}^2)')$, $C(\dot{\boldsymbol{R}};\boldsymbol{C}^2)$ in place of the pair $M(\boldsymbol{R};(\boldsymbol{C}^2)')$, $C_\infty(\boldsymbol{R};\boldsymbol{C}^2)$. Here $C(\dot{\boldsymbol{R}};\boldsymbol{C}^2)$ is the Banach space of the $\boldsymbol{C}^2$-valued continuous functions on $\dot{\boldsymbol{R}}$. Its dual space $M(\dot{\boldsymbol{R}};(\boldsymbol{C}^2)')$ is the Banach space of the $(\boldsymbol{C}^2)'$-valued measures on $\dot{\boldsymbol{R}}$ with bounded variation.[22] Define $L_{s,\mu;0}$ by

$$(L_{s,\mu;0}\Psi)(g)=L_{s,0}(\Psi;\mu,g)$$

with a fixed $\mu\in M(\dot{\boldsymbol{R}};(\boldsymbol{C}^2)')$, for $\Psi\in C(\mathscr{X}_{s,0})$ and $g\in C(\dot{\boldsymbol{R}};\boldsymbol{C}^2)$. By the Riesz representation theorem,[22] $L_{s,\mu;0}\Psi$ is regarded as an element in $M(\dot{\boldsymbol{R}};(\boldsymbol{C}^2)')$, so that $L_{s,\mu;0}$ is a continuous linear operator of $C(\mathscr{X}_{s,0})$ into $M(\dot{\boldsymbol{R}};(\boldsymbol{C}^2)')$. Further it is seen[26] that $L_{s,\mu;0}$ is weakly compact. Then by the Riesz-type representation theorem,[27] there exists a unique countably additive measure $\nu^{\mathrm{I}}_{s,\mu;0}$ defined on the Borel sets in $\mathscr{X}_{s,0}$ and having values in $M(\dot{\boldsymbol{R}};(\boldsymbol{C}^2)')$ such that
(a) $\nu^{\mathrm{I}}_{s,\mu;0}$ is of bounded variation;
(b) for each g in $C(\dot{\boldsymbol{R}};\boldsymbol{C}^2)$ the set function $\nu^{\mathrm{I}}_{s,\mu;0,g}$ defined by

$$\nu^{\mathrm{I}}_{s,\mu;0,g}(E)=\langle\nu^{\mathrm{I}}_{s,\mu;0}(E),g\rangle$$

is a complex-valued countably additive regular measure on the Borel sets E in $\mathscr{X}_{s,0}$;
(c) $\|L_{s,\mu;0}\|=\mathrm{Var}\ \nu^{\mathrm{I}}_{s,\mu;0}\le Ce^{m|s|}\|\mu\|$ with $C=|N||N^{-1}|\le 2$;
(d) for each Ψ in $C(\mathscr{X}_{s,0})$,

$$L_{s,\mu;0}\Psi=\int_{\mathscr{X}_{s,0}} d\nu^{\mathrm{I}}_{s,\mu;0}(X)\Psi(X).$$

Notice we can choose, for μ, $\delta_x{}^1\equiv\begin{bmatrix}\delta_x\\0\end{bmatrix}$ or $\delta_x{}^2\equiv\begin{bmatrix}0\\\delta_x\end{bmatrix}$, where $\delta_x=\delta(\cdot-x)$ is the Dirac measure at x, an element of $M(\dot{\boldsymbol{R}})$. Then each component of the solution $\phi(s,x)=\begin{bmatrix}\phi_1(s,x)\\\phi_2(s,x)\end{bmatrix}$ of (4·1) with data $\phi(0,x)=g(x)$ is represented as

$$\phi_i(s,x)=\int_{\mathscr{X}_{s,0}} d\nu^{\mathrm{I}}_{s,\delta_x{}^i;0}(X)g(X(0)),\qquad i=1,2.$$

4.2. *Support property of the path space measure* $\nu^{I}_{s;0}$

We shall now see the measure $\nu^{\mathrm{I}}_{s;0}$ has the support property described in Theorem 3.1. In order to simplify the notation, we put

$$A=-NaN^{-1}\partial_x \quad\text{and}\quad B=-imN\beta N^{-1},$$

where N is the unitary matrix introduced in Lemma 4.1. We may assume $r=0$ and $s>0$. We have

$$Ne^{-t(\alpha\partial_x+im\beta)}N^{-1}=e^{t(A+B)}. \tag{4·13}$$

Let us make a Taylor expansion of $e^{t(A+B)}$ in m by using iteratively the well-known formula

$$e^{t(A+B)}=e^{tA}+\int_0^t d\tau_1 e^{(t-\tau_1)(A+B)}Be^{\tau_1 A}. \tag{4·14}$$

Set

$$C_t^0\equiv e^{tA}, \qquad C_t^{0\to}\equiv e^{t(A+B)} \tag{4·15a}$$

and

$$C_t^k\equiv\int_0^\infty d\tau_1\cdots\int_0^\infty d\tau_k\theta(t-\sum_{i=1}^k\tau_i)\exp[(t-\sum_{i=1}^k\tau_i)A]Be^{\tau_k A}Be^{\tau_{k-1}A}\cdots Be^{\tau_1 A},$$

$$C_t^{k\to}\equiv\int_0^\infty d\tau_1\cdots\int_0^\infty d\tau_k\theta(t-\sum_{i=1}^k\tau_i)\exp[(t-\sum_{i=1}^k\tau_i)(A+B)]Be^{\tau_k A}\cdots Be^{\tau_1 A} \tag{4·15b}$$

for $k\geq 1$. In the proof of Lemma 4.2, $C_t^{0\to}$ was denoted by C_t.

Then we have

Lemma 4.4. (1) $C_t^{0\to}=\sum_{k=0}^N C_t^k+C_t^{N+1\to}, \qquad N=0, 1, 2, \cdots.$

(2) C_t^k and $C_t^{k\to}$, $k=0, 1, \cdots$, are bounded linear operators of $C_\infty(\boldsymbol{R}; \boldsymbol{C}^2)$ into itself:

$$\|C_t^k\|\leq(k!)^{-1}(mt)^k, \qquad \|C_t^{k\to}\|\leq(k!)^{-1}(mt)^k e^{mt}.$$

Proof By iteration of (4·14), we get (1). The statement (2) is a direct consequence of definition (4·15) and the estimates

$$\|e^{tA}\|\leq 1, \qquad \|B\|=m \quad\text{and}\quad \|e^{t(A+B)}\|\leq e^{mt}.$$

The estimate $\|e^{t(A+B)}\|\leq e^{mt}$ is nothing but (4·4). Since $N\beta N^{-1}$ anti-commutes with $N\alpha N^{-1}$ and $N\alpha N^{-1}$ is diagonal, we have $(N\beta N^{-1})_{11}=(N\beta N^{-1})_{22}=0$, $|(N\beta N^{-1})_{12}|=|(N\beta N^{-1})_{21}|=1$ and hence $\|B\|=m$. Notice that e^{tA} operates on $\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\in C_\infty(\boldsymbol{R}; \boldsymbol{C}^2)$ according to

$$\left(e^{tA}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\right)(x)=\begin{pmatrix}\varphi_1(x-t)\\ \varphi_2(x+t)\end{pmatrix}, \tag{4·16}$$

so that we get $\|e^{tA}\|=1$.

For $\Psi\in C_{\text{fin}}(\mathcal{X}_{s,0})$ represented as (4·6) with a continuous function $F(x_0, x_1, \cdots, x_n)$ on $(\dot{\boldsymbol{R}})^{n+1}$, we introduce a sequence $\{F^{(l)}_{K_1,\cdots,K_l\,;\,x_{l+1},\cdots,x_n}\}_{l=1}^n$ of $\boldsymbol{C}^2$-valued functions on $\boldsymbol{R}$ with parameters, similarly to the proof of Lemma 4.2. Set

$$F^{(0)}_{x_1,\cdots,x_n}(x)\equiv Ng(x)F(x, x_1, \cdots, x_n) \tag{4·17a}$$

and with C_t^k and $C_t^{k\to}$ in (4·15),

$$F^{(l)}_{K_1,\cdots K_l\,;\,x_{l+1},\cdots,x_n}(x)\equiv(C^{K_l}_{s_l-s_{l-1}}F^{(l-1)}_{K_1,\cdots,K_{l-1}\,;\,x,x_{l+1},\cdots,x_n})(x) \tag{4·17b}$$

for $l=1, 2, \cdots, n-1$, and

$$F^{(n)}_{K_1,\cdots,K_n}(x)\equiv(C^{K_n}_{s_n-s_{n-1}}F^{(n-1)}_{K_1,\cdots,K_{n-1};x})(x). \qquad (4\cdot17c)$$

Here K_l is k_l or $k_l\to$ with k_l a nonnegative integer. For $1\le l\le n-1$, $F^{(l-1)}_{K_1,\cdots,K_{l-1};x,x_{l+1},\cdots,x_n}(y)$ in (4·17b) is in $C_\infty(\boldsymbol{R};\boldsymbol{C}^2)$ as a function of y.

For each $k\ge0$, define the functionals $L^k_{s,0}(\Psi;f,g)$ and $L^{k\to}_{s,0}(\Psi;f,g)$ which are linear in $\Psi\in C_{\text{fin}}(\mathcal{X}_{s,0})$ and sesquilinear in $(f,g)\in\mathcal{S}(\boldsymbol{R};(\boldsymbol{C}^2)')\times\mathcal{S}(\boldsymbol{R};\boldsymbol{C}^2)$ by

$$L^0_{s,0}(\Psi;f,g)\equiv\int_{\boldsymbol{R}}\overline{f(x)}N^{-1}F^{(n)}_{\underbrace{0,\cdots,0}_{n}}(x)dx,$$

$$L^{0\to}_{s,0}(\Psi;f,g)\equiv\int_{\boldsymbol{R}}\overline{f(x)}N^{-1}F^{(n)}_{\underbrace{0\to,\cdots,0\to}_{n}}(x)dx \qquad (4\cdot18a)$$

and

$$L^k_{s,0}(\Psi;f,g)\equiv\sum_{\Sigma_{i=1}^{n}k_i=k;\,k_1,\cdots,k_n\ge0}\int_{\boldsymbol{R}}\overline{f(x)}N^{-1}F^{(n)}_{k_1,\cdots,k_n}(x)dx,$$

$$L^{k\to}_{s,0}(\Psi;f,g)\equiv\sum_{p=1}^{n}\sum_{\substack{\Sigma_{i=1}^{p}k_i=k\\ k_1,\cdots,k_{p-1}\ge0;\,k_p\ge1}}$$

$$\times\int_{\boldsymbol{R}}\overline{f(x)}N^{-1}F^{(n)}_{k_1,\cdots,k_{p-1},k_p\to,\underbrace{0\to,\cdots,0\to}_{n-p}}(x)dx \qquad (4\cdot18b)$$

for $k\ge1$. Note that $L^{0\to}_{s,0}(\Psi;f,g)$ in (4·18a) is nothing but $L_{s,0}(\Psi;f,g)$ in (4·7).

Then the following lemma holds.

Lemma 4.5. (1) For each fixed $(f,g)\in\mathcal{S}(\boldsymbol{R};(\boldsymbol{C}^2)')\times\mathcal{S}(\boldsymbol{R};\boldsymbol{C}^2)$ and each $k\ge0$, $L^k_{s,0}(\Psi;f,g)$ and $L^{k\to}_{s,0}(\Psi;f,g)$ are well-defined on $C_{\text{fin}}(\mathcal{X}_{s,0})$; they are independent of the choice of F corresponding to Ψ.
(2) The following inequalities hold:

$$|L^k_{s,0}(\Psi;f,g)|\le C(k!)^{-1}(ms)^k\|\Psi\|\|f\|_1\|g\|_\infty,$$

$$|L^{k\to}_{s,0}(\Psi;f,g)|\le C(k!)^{-1}(ms)^k e^{ms}\|\Psi\|\|f\|_1\|g\|_\infty$$

with a constant $C\le2$. The L^1 and L^∞ norms are denoted by $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively.

(3) $$L^{0\to}_{s,0}(\Psi;f,g)=\sum_{l=0}^{k}L^l_{s,0}(\Psi;f,g)+L^{k+1\to}_{s,0}(\Psi;f,g),$$

$$L^{0\to}_{s,0}(\Psi;f,g)=\sum_{l=0}^{\infty}L^l_{s,0}(\Psi;f,g).$$

Proof The statement (1) follows from the identities

$$\sum_{l=0}^{k} C_{\tau_1}^{l} C_{\tau_2}^{k-l} = C_{\tau_1+\tau_2}^{k}$$

and

$$C_{\tau_1}^{0\to} C_{\tau_2}^{k\to} + \sum_{l=1}^{k} C_{\tau_1}^{l\to} C_{\tau_2}^{k-l} = C_{\tau_1+\tau_2}^{k\to}, \qquad \tau_1, \tau_2 > 0,$$

which can be derived from the definition (4·15). To prove (2), we make a multiple use of Lemma 4.4(2) as in the proof of Lemma 4.2(2). The first equality in (3) in a direct consequence of Lemma 4.4(1). Note that $|L_{s,0}^{k+1\to}(\Psi; f, g)| \to 0$ as $k \to \infty$ by (2), then the second equality in (3) holds.

The consequence of Lemma 4.5 is the following theorem.

THEOREM 4.6. For each $k \geq 0$, there exist unique complex-valued countably additive regular measures $\nu_{s,f;0,g}^{k}$ and $\nu_{s,f;0,g}^{k\to}$ on the Borel sets in $\mathcal{X}_{s,0}$ such that for each Ψ in $C(\mathcal{X}_{s,0})$,

$$L_{s,0}^{k}(\Psi; f, g) = \int_{\mathcal{X}_{s,0}} d\nu_{s,f;0,g}^{k}(X)\Psi(X),$$

$$L_{s,0}^{k\to}(\Psi; f, g) = \int_{\mathcal{X}_{s,0}} d\nu_{s,f;0,g}^{k\to}(X)\Psi(X).$$

Moreover the following equality holds for every Borel set E in $\mathcal{X}_{s,0}$:

$$\nu_{s,f;0,g}^{0\to}(E) = \nu_{s,f;0,g}^{\mathrm{I}}(E) = \sum_{k=0}^{\infty} \nu_{s,f;0,g}^{k}(E),$$

where the series in the last member is absolutely convergent. Therefore if, for each $k \geq 0$, the measure $\nu_{s,f;0,g}^{k}$ is concentrated on a Borel subset E_k of $\mathcal{X}_{s,0}$, then the measure $\nu_{s,f;0,g}^{\mathrm{I}}$ is concentrated on the Borel subset $\bigcup_{k=0}^{\infty} E_k$.

Proof The first half of the theorem follows from Lemma 4.5(1), (2) and the Riesz representation theorem. Next this and Lemma 4.5(3) yield

$$\nu_{s,f;0,g}^{\mathrm{I}} = \sum_{k=0}^{N} \nu_{s,f;0,g}^{k} + \nu_{s,f;0,g}^{N+1\to}.$$

Further, by Lemma 4.5(2), we get

$$|\nu_{s,f;0,g}^{N+1\to}(E)| \leq C((N+1)!)^{-1}(ms)^{N+1} e^{ms} \|f\|_1 \|g\|_\infty,$$

which converges to zero as $N \to \infty$. This proves the second half of the theorem.

Our next task is to see the support property of $\nu_{s;0}^{\mathrm{I}}$. We shall show in Theorem 4.8 below that, for each $k \geq 0$, the measure $\nu_{s,f;0,g}^{k}$ is concentrated on the set of the Lipschitz continuous paths $X:[0, s] \to \boldsymbol{R}$ satisfying, for some k-partition: $0 = t_0 < t_1 < \cdots < t_k = s$ of the interval $[0, s]$, depending on X,

$$X(t) - X(0) = \sum_{i=1}^{j-1} (-1)^i (t_i - t_{i-1}) + (-1)^j (t - t_{j-1})$$

or

$$X(t)-X(0)=\sum_{i=1}^{j-1}(-1)^{i-1}(t_i-t_{i-1})+(-1)^{j-1}(t-t_{j-1})$$

for $t_{j-1}\leq t\leq t_j$, $1\leq j\leq k$.

For each $k\geq 1$, let Δ^k be the open k-simplex

$$\Delta^k=\{(\tau_1,\cdots,\tau_k)\in\boldsymbol{R}^k \mid \sum_{i=1}^{k}\tau_i<s \text{ and } \tau_1,\cdots\tau_k>0\},$$

and $\varphi_1{}^k$ and $\varphi_2{}^k$ the maps from $\boldsymbol{R}\times\Delta^k$ into $\mathcal{X}_{s,0}$ defined by

$$\varphi_j{}^k(x,\tau_1,\cdots,\tau_k)(t)=x+(-1)^j[\sum_{l=1}^{N_t}(-1)^l\tau_l+(-1)^{N_t+1}(t-\sum_{l=1}^{N_t}\tau_l)] \tag{4·19}$$

for $x\in\boldsymbol{R}$, $(\tau_1,\cdots,\tau_k)\in\Delta^k$, $t\in[0,s]$ and $j=1,2$, where N_t is the t-dependent integer satisfying

$$\sum_{l=1}^{N_t}\tau_l\leq t<\sum_{l=1}^{N_t+1}\tau_l.$$

In (4·19), the value $\varphi_j{}^k(x,\tau_1,\cdots,\tau_k)$ is a function: $[0,s]\to\boldsymbol{R}$ and so belongs to $\mathcal{X}_{s,0}$.

For $k=0$, we understand Δ^0 to be the set of one point and identify $\boldsymbol{R}\times\Delta^0$ with $\boldsymbol{R}$. Let $\varphi_1{}^0$ and $\varphi_2{}^0$ be the maps from $\boldsymbol{R}=\boldsymbol{R}\times\Delta^0$ to $\mathcal{X}_{s,0}$ defined by

$$\varphi_j{}^0(x)(t)=x+(-1)^j t, \qquad j=1,2.$$

Then the maps $\varphi_j{}^k$, $k\geq 0$, $j=1,2$, have the following properties.

Lemma 4.7. (1) For each $k\geq 0$ and $j=1,2$, $\varphi_j{}^k$ is continuous and Borel-measurable; (2) $\varphi_j{}^k(\boldsymbol{R}\times\Delta^k)$ is an F_σ-set.

Proof The statement (1) is obvious. (2) $\boldsymbol{R}\times\Delta^k$ is expressed as a countable union $\bigcup_{n=1}^\infty K_n$ of compact sets K_n. By the continuity of $\varphi_j{}^k$, each $\varphi_j{}^k(K_n)$ is compact in $\mathcal{X}_{s,0}$ and hence closed, so $\varphi_j{}^k(\boldsymbol{R}\times\Delta^k)$ is an F_σ-set.

For each $k\geq 0$ and $j=1,2$, define the complex-valued regular Borel measure $\mu_{s,f;0,g}^{k,j}$ on $\boldsymbol{R}\times\Delta^k$ by

$$\mu_{s,f;0,g}^{k,j}(E)$$
$$=\int_E\overline{(f(x+(-1)^j[\sum_{l=1}^{k}(-1)^l\tau_l+(-1)^{k+1}(s-\sum_{l=1}^{k}\tau_l)])N^{-1}B^k)_j}$$
$$\times(Ng(x))_j dx d\tau_1\cdots d\tau_k, \tag{4·20}$$

where E is a Borel set in $\boldsymbol{R}\times\Delta^k$.

THEOREM 4.8. For each $(f,g)\in\mathcal{S}(\boldsymbol{R};(\boldsymbol{C}^2)')\times\mathcal{S}(\boldsymbol{R};\boldsymbol{C}^2)$, $k\geq 0$ and $s>0$,

$$\nu_{s,f;0,g}^k=\sum_{j=1}^{2}\mu_{s,f;0,g}^{k,j}(\varphi_j{}^k)^{-1}.$$

Here $\mu_{s,f;0,g}^{k,j}(\varphi_j{}^k)^{-1}$ is the image measure on $\mathcal{X}_{s,0}$ induced[28] from the measure $\mu_{s,f;0,g}^{k,j}$ on $\boldsymbol{R}\times\Delta^k$ by the map $\varphi_j{}^k$.

Before showing Theorem 4.8, we see first what is a consequence of the theorem.

The measure $\mu^{k,j}_{s,f;0,g}(\varphi_j{}^k)^{-1}$ is concentrated on the set $\varphi_j{}^k(\boldsymbol{R}\times\Delta^k)$ and so the measure $\nu^k_{s,f;0,g}$ on the set $\bigcup_{j=1}^2\varphi_j{}^k(\boldsymbol{R}\times\Delta^k)$. It follows by Theorem 4.6 that the measure $\nu^{\mathrm{I}}_{s,f;0,g}$ is concentrated on the set $\bigcup_{k=0}^\infty\bigcup_{j=1}^2\varphi_j{}^k(\boldsymbol{R}\times\Delta^k)$, which is a Borel, in fact, F_{σ}-, set by Lemma 4.7. Then the $\mathcal{S}'(\boldsymbol{R}\times\boldsymbol{R};M_2(\boldsymbol{C}))$-valued measure $\nu^{\mathrm{I}}_{s;0}$ is also concentrated on the set $\bigcup_{k=0}^\infty\bigcup_{j=1}^2\varphi_j{}^k(\boldsymbol{R}\times\Delta^k)$, for this set is independent of (f,g) in $\mathcal{S}(\boldsymbol{R};(\boldsymbol{C}^2)')\times\mathcal{S}(\boldsymbol{R};\boldsymbol{C}^2)$. It is obvious that every X in $\bigcup_{k=0}^\infty\bigcup_{j=1}^2\varphi_j{}^k(\boldsymbol{R}\times\Delta^k)$ satisfies the condition (3·7) in Theorem 3.1. In case $m=0$, we have $\nu^{\mathrm{I}}_{s,f;0,g}=\nu^0_{s,f;0,g}$. By Theorem 4.8 it is concentrated on the set of the straight segments $X:[0,s]\to\boldsymbol{R}$ with $X(t)=X(0)+t$ or $X(t)=X(0)-t$, $0\le t\le s$, and so is $\nu^{\mathrm{I}}_{s;0}$, similarly. Thus we have seen Theorem 4.8 yields the desired support property of $\nu^{\mathrm{I}}_{s;0}$.

Proof of Theorem 4.8. It is enough to show that the equality

$$\int_{\mathcal{X}_{s,0}}d\nu^k_{s,f;0,g}(X)\Psi(X)$$
$$=\sum_{j=1}^2\int_{\boldsymbol{R}\times\Delta^k}d\mu^{k,j}_{s,f;0,g}(\zeta)\Psi(\varphi_j{}^k(\zeta)),\qquad \zeta=(x,\tau_1,\cdots,\tau_k),$$

holds for every $\Psi\in C_{\mathrm{fin}}(\mathcal{X}_{s,0})$. Here we only prove this equality for $k=2$. The proof will be still complicated for general Ψ in $C_{\mathrm{fin}}(\mathcal{X}_{s,0})$. So we only see it when Ψ is represented as $\Psi(X)=F(X(0),X(s_1),X(s))$ with a partition $0=s_0<s_1<s_2=s$ of the interval $[0,s]$ and a bounded continuous function F on $(\dot{\boldsymbol{R}})^3$, i.e., (4·5) and (4·6) with $n=2$. We can similarly prove the general case.

Recall that

$$e^{\tau A}=\begin{pmatrix}e^{-\tau\partial_x} & 0\\ 0 & e^{-\tau\partial_x}\end{pmatrix}\quad\text{and}\quad B=\begin{pmatrix}0 & B_{12}\\ B_{21} & 0\end{pmatrix}$$

with

$$|B_{12}|=|B_{21}|=m\,.$$

By Theorem 4.6 and the definition (4·18), we have

$$\int_{\mathcal{X}_{s,0}}d\nu^2_{s,f;0,g}(X)\Psi(X)=L^2_{s,0}(\Psi;f,g)$$
$$=\int_{\boldsymbol{R}}\overline{f(x)}N^{-1}\{F^{(2)}_{0,2}(x)+F^{(2)}_{1,1}(x)+F^{(2)}_{2,0}(x)\}dx\,.\qquad(4\cdot21)$$

Using (4·17), (4·15) and anti-commutativity of A and B, we get

$$F^{(2)}_{1,1}(x)=(C^1_{s-s_1}F^{(1)}_{1;x})(x)$$
$$=B\int_0^\infty d\tau_2\,\theta(s-s_1-\tau_2)(e^{(-s+s_1+2\tau_2)A}F^{(1)}_{1;x})(x)$$
$$=B\int_0^\infty d\tau_2\,\theta(s-s_1-\tau_2)\begin{pmatrix}(F^{(1)}_{1;x})_1(x+s-s_1-2\tau_2)\\ (F^{(1)}_{1;x})_2(x-s+s_1+2\tau_2)\end{pmatrix}.$$

Here $F^{(1)}_{1;x}(y)={}^t((F^{(1)}_{1;x})_1(y),(F^{(1)}_{1;x})_2(y))$. For these functions in the integrand of the last member of the above equation, we get

$$
\begin{aligned}
&(F^{(1)}_{1;x})_1(x+s-s_1-2\tau_2)\\
&\quad=(C^1_{s_1}F^{(0)}_{x+s-s_1-2\tau_2,x})_1(x+s-s_1-2\tau_2)\\
&\quad=B_{12}\int_0^\infty d\tau_1\,\theta(s_1-\tau_1)(e^{(-s_1+2\tau_1)A}F^{(0)}_{x+s-s_1-2\tau_2,x})_2(x+s-s_1-2\tau_2)\\
&\quad=B_{12}\int_0^\infty d\tau_1\,\theta(s_1-\tau_1)(Ng)_2(x+s-2s_1-2\tau_2+2\tau_1)\\
&\qquad\qquad\times F(x+s-2s_1-2\tau_2+2\tau_1,\ x+s-s_1-2\tau_2,\ x)
\end{aligned}
$$

and

$$
\begin{aligned}
&(F^{(1)}_{1;x})_2(x-s+s_1+2\tau_2)\\
&\quad=B_{21}\int_0^\infty d\tau_1\,\theta(s_1-\tau_1)(Ng)_1(x-s+2s_1+2\tau_2-2\tau_1)\\
&\qquad\qquad\times F(x-s+2s_1+2\tau_2-2\tau_1,\ x-s+s_1+2\tau_2,\ x)\,.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&F^{(2)}_{1,1}(x)\\
&\quad=B^2\int_0^\infty d\tau_2\int_0^\infty d\tau_1\,\theta(s-s_1-\tau_2)\theta(s_1-\tau_1)\\
&\qquad\times\begin{pmatrix}(Ng)_1(x-s+2s_1+2\tau_2-2\tau_1)F(x-s+2s_1+2\tau_2-2\tau_1,\ x-s+s_1+2\tau_2,\ x)\\(Ng)_2(x+s-2s_1-2\tau_2+2\tau_1)F(x+s-2s_1-2\tau_2+2\tau_1,\ x+s-s_1-2\tau_2,\ x)\end{pmatrix}\\
&\quad=B^2\int_0^\infty d\tau_2\int_0^\infty d\tau_1\,\theta(s-\tau_1-\tau_2)\theta(\tau_1+\tau_2-s_1)\theta(s_1-\tau_1)\\
&\qquad\times\begin{pmatrix}(Ng)_1(x-s+2\tau_2)F(x-s+2\tau_2,\ x-s-s_1+2\tau_2+2\tau_1,\ x)\\(Ng)_2(x+s-2\tau_2)F(x+s-2\tau_2,\ x+s+s_1-2\tau_2-2\tau_1,\ x)\end{pmatrix},
\end{aligned}
\tag{4·22}
$$

where, in the second equality, we have first made the change of variable: $\tau_2'=\tau_2+s_1-\tau_1$ and next written τ_2 again instead of τ_2'. Similarly we have

$$
\begin{aligned}
F^{(2)}_{0,2}(x)=B^2&\int_0^\infty d\tau_2\int_0^\infty d\tau_1\,\theta(s-\tau_1-\tau_2)\theta(\tau_1-s_1)\\
&\times\begin{pmatrix}(Ng)_1(x-s+2\tau_2)F(x-s+2\tau_2,\ x-s+s_1+2\tau_2,\ x)\\(Ng)_2(x+s-2\tau_2)F(x+s-2\tau_2,\ x+s-s_1-2\tau_2,\ x)\end{pmatrix}
\end{aligned}
\tag{4·23}
$$

and

$$
\begin{aligned}
F^{(2)}_{2,0}(x)=B^2&\int_0^\infty d\tau_2\int_0^\infty d\tau_1\,\theta(s_1-\tau_1-\tau_2)\\
&\times\begin{pmatrix}(Ng)_1(x-s+2\tau_2)F(x-s+2\tau_2,\ x-s+s_1,\ x)\\(Ng)_2(x+s-2\tau_2)F(x+s-2\tau_2,\ x+s-s_1,\ x)\end{pmatrix}.
\end{aligned}
\tag{4·24}
$$

Substituting (4·22)~(4·24) into (4·21), we get

$$\int_{\mathscr{X}_{s,0}} d\nu^2_{s,f;0,g}(X)\Psi(X)$$

$$=\int_{\boldsymbol{R}}dx\int_0^\infty d\tau_1\int_0^\infty d\tau_2(\overline{f(x+s-2\tau_2)}N^{-1}B^2)_1\theta(s-\tau_1-\tau_2)$$

$$\times[\theta(\tau_1-s_1)(Ng)_1(x)F(x,x+s_1,x+s-2\tau_2)$$

$$+\theta(\tau_1+\tau_2-s_1)\theta(s_1-\tau_1)(Ng)_1(x)F(x,x-s_1+2\tau_1,x+s-2\tau_2)$$

$$+\theta(s_1-\tau_1-\tau_2)(Ng)_1(x)F(x,x+s_1-2\tau_2,x+s-2\tau_2)]$$

$$+\int_{\boldsymbol{R}}dx\int_0^\infty d\tau_1\int_0^\infty d\tau_2(\overline{f(x-s+2\tau_2)}N^{-1}B^2)_2\theta(s-\tau_1-\tau_2)$$

$$\times[\theta(\tau_1-s_1)(Ng)_2(x)F(x,x-s_1,x-s+2\tau_2)$$

$$+\theta(\tau_1+\tau_2-s_1)\theta(s_1-\tau_1)(Ng)_2(x)F(x,x+s_1-2\tau_1,x-s+2\tau_2)$$

$$+\theta(s_1-\tau_1-\tau_2)(Ng)_2(x)F(x,x-s_1+2\tau_2,x-s+2\tau_2)]\,. \tag{4·25}$$

Here, on the right-hand side, we have first made the change of variables: $x'=x-s+2\tau_2$ in the first term and $x''=x+s-2\tau_2$ in the second term, and next written x instead of x' and x''.

By the definition (4·19) of $\varphi_j{}^2$: $\boldsymbol{R}\times\Delta^2\to\mathscr{X}_{s,0}$, $j=1, 2$, we have

$$\varphi_j{}^2(x,\tau_1,\tau_2)(0)=x\,,$$

$$\varphi_j{}^2(x,\tau_1,\tau_2)(s)=x-(-1)^j(s-2\tau_2)\,,$$

$$\varphi_j{}^2(x,\tau_1,\tau_2)(s_1)=\begin{cases} x-(-1)^j s_1\,, & (s_1<\tau_1) \\ x-(-1)^j(2\tau_1-s_1)\,, & (\tau_1<s_1<\tau_1+\tau_2) \\ x-(-1)^j(s_1-2\tau_2)\,, & (\tau_1+\tau_2<s_1) \end{cases}$$

so that we can get after all

$$\int_{\mathscr{X}_{s,0}} d\nu^2_{s,f;0,g}(X)\Psi(X)$$

$$=\int_{\boldsymbol{R}}dx\int_{\Delta^2}d\tau_1 d\tau_2\sum_{j=1}^2(\overline{f(x-(-1)^j(s-2\tau_2))}N^{-1}B^2)_j$$

$$\times(Ng)_j\Psi(\varphi_j{}^2(x,\tau_1,\tau_2))$$

$$=\sum_{j=1}^2\int_{\boldsymbol{R}\times\Delta^2}d\mu^{2,j}_{s,f;0,g}(x,\tau_1,\tau_2)\Psi(\varphi_j{}^2(x,\tau_1,\tau_2))\,.$$

This proves Theorem 4.8 for $k=2$.

4.3. *Proof of the path integral formula* (3·6)

We prove the formula (3·6) with $r=0$ and $s>0$. We note the proof of the path integral formula (3·9) is analogous. $\Phi(t,x)$ and $A(t,x)$ are continuous on $\boldsymbol{R}^2$. In Ref. 11), it was shown for $\Phi(t,x)$ and $A(t,x)$ more general in x but less general in t, i.e., when both the maps $t\to\Phi(t,\cdot)$ and $t\to A(t,\cdot)$ of $\boldsymbol{R}$ into $L^{\infty}_{\rm loc}(\boldsymbol{R};\boldsymbol{R})$ are continuously differentiable.

We prove (3·6) in three steps; first for t-independent Φ and A which are in $C_0^{\infty}(\boldsymbol{R};\boldsymbol{R})$, next for those which are in $C(\boldsymbol{R};\boldsymbol{R})$ and finally for the general $\Phi(t,x)$ and $A(t,x)$ which are in $C(\boldsymbol{R}^2;\boldsymbol{R})$.

Step 1. So suppose that $\Phi(t,x)=\Phi(x)$ and $A(t,x)=A(x)$ are t-independent functions in $C_0^{\infty}(\boldsymbol{R};\boldsymbol{R})$. Define the operator

$$(T(t)g)(x)=\int_{\boldsymbol{R}}K_0^{\rm I}(t,x-y)\exp[-ie\Phi(y)t+ieA(y)(x-y)]g(y)dy \tag{4·26}$$

for g in $C_{\infty}^{1}(\boldsymbol{R};\boldsymbol{C}^2)$, the Banach space of the $\boldsymbol{C}^2$-valued continuously differentiable functions in $\boldsymbol{R}$ which together with their first derivatives vanish at infinity.

We need the following lemma.

Lemma 4.9. Assume $\Phi(x)$ and $A(x)$ are in $C_0^{\infty}(\boldsymbol{R};\boldsymbol{R})$. Then $T(t)$ defines a bounded linear operator of $C_{\infty}(\boldsymbol{R};\boldsymbol{C}^2)$ into itself and $\|T(t)\|\le Ce^{m|t|}$ with a constant C. Further, if g is in $C_{\infty}^{1}(\boldsymbol{R};\boldsymbol{C}^2)$, $\partial_t(T(t)g)$ converges to $-iHg$ in the norm of L^{∞} as $t\to 0$, where

$$iH^{\rm I}=[\alpha(\partial_x-ieA(x))+im\beta+ie\Phi(x)]\,. \tag{4·27}$$

Proof We simply write H for $H^{\rm I}$. The first half follows from the L^{∞} well-posedness of the Cauchy problem for (4·1). To show the second half note that, for fixed $(t,x)\in\boldsymbol{R}\times\boldsymbol{R}$, the support of $K_0^{\rm I}(t,x-y)$ is bounded in y, and

$$\partial_t K_0^{\rm I}(t,x-y)=[\alpha\partial_y-im\beta]K_0^{\rm I}(t,x-y)\,.$$

Then we have

$$\partial_t(T(t)g)(x)=-\int_{\boldsymbol{R}}[K_0^{\rm I}(t,x-y)(-\partial_t)+\alpha K_0^{\rm I}(t,x-y)\partial_y+im\beta K_0^{\rm I}(t,x-y)]$$
$$\times(\exp[-ie\Phi(y)t+ieA(y)(x-y)]g(y))dy\,.$$

It follows that

$$\begin{aligned}&\partial_t(T(t)g)(x)+i(Hg)(x)\\&=-\Big\{\int_{\boldsymbol{R}}K_0^{\rm I}(t,x-y)\exp[-ie\Phi(y)t+ieA(y)(x-y)]ie\Phi(y)g(y)dy\\&\qquad-ie\Phi(x)g(x)\Big\}\\&\quad-\alpha\Big\{\int_{\boldsymbol{R}}K_0^{\rm I}(t,x-y)\exp[-ie\Phi(y)+ieA(y)(x-y)][\partial_y-ieA(y)\\&\qquad-ie(\partial_y\Phi(y))t+ie(\partial_yA(y))(x-y)]g(y)dy-(\partial_x-ieA(x))g(x)\Big\}\\&\quad-im\beta\Big\{\int_{\boldsymbol{R}}K_0^{\rm I}(t,x-y)\exp[-ie\Phi(y)t+ieA(y)(x-y)]g(y)dy-g(x)\Big\}\,.\end{aligned}$$

The right-hand side above converges to zero in the norm of L^∞ as $t\to 0$, because $K_0^{\mathrm{I}}(t, x-y)\to\delta(x-y)\mathbf{1}$ as $t\to 0$. Lemma 4.9 is thus proved.

Now let us prove the path integral formula (3·6) with t-independent $\Phi(t, x) = \Phi(x)$ and $A(t, x)=A(x)$ in $C_o^\infty(\boldsymbol{R}; \boldsymbol{R})$. By (4·7) and Theorem 4.3 we have for (f, g) in $\mathcal{S}(\boldsymbol{R}; (\boldsymbol{C}^2)')\times\mathcal{S}(\boldsymbol{R}; \boldsymbol{C}^2)$ with $s_j=js/n$

$$(f,\ T(s/n)^n g)=\int(f,\ d\nu^{\mathrm{I}}_{s;0}(X)g)$$
$$\times\exp\{-ie\sum_{j=1}^{n}[\Phi(X(s_{j-1}))s/n-A(X(s_{j-1}))(X(s_j)-X(s_{j-1}))]\}\,. \tag{4·28}$$

The integrand on the right hand of (4·28), which is a function in $C_{\mathrm{fin}}(\mathcal{X}_{s,0})$, is uniformly bounded and convergent to $\exp[-i\int_0^s e\Phi(X(t))dt+i\int_0^s eA(X(t))dX(t)]$ as $n\to\infty$ for every Lipschitz continuous path X:[0, s]$\to\boldsymbol{R}$, i.e., for almost every path X, because $\nu^{\mathrm{I}}_{s,f;0,g}$ has, as seen in § 4.2, support on the set of the Lipschitz continuous paths. Thus by the Lebesgue bounded convergence theorem, the right-hand side of (4·28) converges to that of (3·6) with $r=0$ and with t-independent Φ and A as $n\to\infty$.

As for the left-hand side of (4·28), we show in fact that $T(s/n)^n g$ converges to $e^{-isH}g$ in the norm of L^∞ as $n\to\infty$. By Lemma 4.9 we can see that for every g in $C_\infty^1(\boldsymbol{R}; \boldsymbol{C}^2)$

$$\|\partial_t(T(t)-e^{-itH})g\|\to 0$$

as $t\to 0$. Since

$$\|n(T(s/n)-e^{-i(s/n)H})g\|=n\left\|\int_0^{s/n}\partial_t(T(t)-e^{-itH})g\,dt\right\|$$
$$\leq s\sup_{t\in[0,s/n]}\|\partial_t(T(t)-e^{-itH})g\|\,,$$

we have

$$\|n(T(s/n)-e^{-i(s/n)H})g\|\to 0 \text{ as } n\to\infty\,. \tag{4·29}$$

Further $\{n(T(s/n)-e^{-i(s/n)H})\}_{n=1}^\infty$ is a family of bounded operators of $C_\infty^1(\boldsymbol{R}; \boldsymbol{C}^2)$ into $C_\infty(\boldsymbol{R}; \boldsymbol{C}^2)$ and for each fixed g in $C_\infty^1(\boldsymbol{R}; \boldsymbol{C}^2)$,

$$\|n(T(s/n)-e^{-i(s/n)H})g\| \tag{4·30}$$

is uniformly bounded for n. Therefore, by the uniform boundedness principle,[22] (4·30) is uniformly bounded for both n and g in the unit ball of $C_\infty^1(\boldsymbol{R}; \boldsymbol{C}^2)$. It follows that the convergence in (4·29) is uniform on compact subsets of $C_\infty^1(\boldsymbol{R}; \boldsymbol{C}^2)$. Since

$$\|[T(s/n)^n-e^{-isH}]g\|=\|\sum_{j=1}^{n}T(s/n)^{j-1}(T(s/n)-e^{-i(s/n)H})e^{-i((n-j)s/n)H}g\|$$
$$\leq Cne^{ms}\sup_{t\in[0,s]}\|(T(s/n)-e^{-i(s/n)H})e^{-itH}g\|$$

with $C=|N||N^{-1}|$ by Lemma 4.1, and the set $\{e^{-itH}g;\ t\in[0, s]\}$ is a compact subset of $C_\infty^1(\boldsymbol{R}; \boldsymbol{C}^2)$, the theorem is proved for t-independent Φ and A in $C_o^\infty(\boldsymbol{R}; \boldsymbol{R})$.

Step 2. Next, suppose that $\Phi(t,x)=\Phi(x)$ and $A(t,x)=A(x)$ are t-independent functions in $C(\boldsymbol{R};\boldsymbol{R})$. Let H^{I} be the operator defined by (4·27). Choose sequences $\{\Phi^{(n)}(x)\}_{n=1}^{\infty}$ and $\{A^{(n)}(x)\}_{n=1}^{\infty}$ of functions in $C_0^{\infty}(\boldsymbol{R};\boldsymbol{R})$ such that $\Phi^{(n)}(x)$ and $A^{(n)}(x)$ are uniformly bounded on each compact set in $\boldsymbol{R}$ and $\Phi^{(n)}(x)\to\Phi(x)$, $A^{(n)}(x)\to A(x)$ uniformly on each compact set of $\boldsymbol{R}$ as $n\to\infty$. For each n define the operator $H^{\mathrm{I}(n)}$ by (4·27) with $\Phi^{(n)}(x)$ and $A^{(n)}(x)$ in place of $\Phi(x)$ and $A(x)$, respectively, i.e.,

$$iH^{\mathrm{I}(n)}=[\alpha(\partial_x-ieA^{(n)}(x))+im\beta+ie\Phi^{(n)}(x)]\,. \qquad (4\cdot31)$$

To simplify the notation we write again H, $H^{(n)}$ for H^{I}, $H^{\mathrm{I}(n)}$, respectively. Then for (f,g) in $\mathcal{S}(\boldsymbol{R};(\boldsymbol{C}^2)')\times\mathcal{S}(\boldsymbol{R};\boldsymbol{C}^2)$

$$(f,\exp[-isH^{(n)}]g)=\int d\nu^{\mathrm{I}}_{s,f;0,g}\exp\Big[-i\int_0^s e\Phi^{(n)}(X(t))dt$$
$$+i\int_0^s eA^{(n)}(X(t))dX(t)\Big]\,. \qquad (4\cdot32)$$

Since $\Phi^{(n)}(X(t))$ and $A^{(n)}(X(t))$ converge to $\Phi(X(t))$ and $A(X(t))$ for every Lipschitz continuous path $X:[0,s]\to\boldsymbol{R}$ and so for almost every path X because of the support property of $\nu^{\mathrm{I}}_{s,f;0,g}$, the right-hand side of (4·32) converges to the last member of (3·6) with $r=0$ and with t-independent Φ and A as $n\to\infty$, by the Lebesgue bounded convergence theorem.

On the other hand, $H^{(n)}$ and H are essentially self-adjoint[14] on $C_0^{\infty}(\boldsymbol{R};\boldsymbol{C}^2)$. For g in $C_0^{\infty}(\boldsymbol{R};\boldsymbol{C}^2)$, $H^{(n)}g$ converges to Hg in $L^2(\boldsymbol{R};\boldsymbol{C}^2)$ as $n\to\infty$. It follows[29] that for g in $L^2(\boldsymbol{R};\boldsymbol{C}^2)$, $\exp[-isH^{(n)}]g$ converges to $\exp[-isH]g$ in the norm of L^2 as $n\to\infty$. This proves (3·6) for t-independent Φ and A in $C(\boldsymbol{R};\boldsymbol{R})$.

Step 3. Finally, we show the path integral formula (3·6) in the general case where $\Phi(t,x)$ and $A(t,x)$ are in $C(\boldsymbol{R}^2;\boldsymbol{R})$. Before that, we note the following. Given sequences $\{\Phi^{(n)}\}_{n=1}^{\infty}$ and $\{A^{(n)}\}_{n=1}^{\infty}$ of t-independent functions in $C(\boldsymbol{R};\boldsymbol{R})$, let $\{H^{\mathrm{I}(n)}\}_{n=1}^{\infty}$ be the corresponding operators defined by (4·31). We write again $H^{(n)}$ for $H^{\mathrm{I}(n)}$. For each fixed $n\geq1$, set

$$\tilde{\Phi}^{(n)}(t,x)=\Phi^{(l)}(x),\quad \tilde{A}^{(n)}(t,x)=A^{(l)}(x),$$
$$(l-1)s/n\leq t\leq ls/n,\quad 1\leq l\leq n.$$

Then notice that for each n,

$$e^{-i(s/n)H^{(n)}}e^{-i(s/n)H^{(n-1)}}\cdots\cdots e^{-i(s/n)H^{(1)}}g$$

is the unique solution at $t=s$ of the Cauchy problem for (3·1) with $\tilde{\Phi}^{(n)}$ and $\tilde{A}^{(n)}$ in place of Φ and A, respectively, with initial data g at $t=0$. By similar arguments used in the first step above we get for (f,g) in $\mathcal{S}(\boldsymbol{R};(\boldsymbol{C}^2)')\times\mathcal{S}(\boldsymbol{R};\boldsymbol{C}^2)$

$$(f,e^{-i(s/n)H^{(n)}}e^{-i(s/n)H^{(n-1)}}\cdots e^{-i(s/n)H^{(1)}}g)$$
$$=\int(f,d\nu^{\mathrm{I}}_{s;0}(X)g)\exp\Big[-i\int_0^s e\tilde{\Phi}(t,X(t))dt+i\int_0^s e\tilde{A}(t,X(t))dX(t)\Big]\,. \qquad (4\cdot33)$$

We are now in a position to prove the path integral formula (3·6) with continuous functions Φ and A on $\boldsymbol{R}^2$. Let $\phi(t, x)$ be the solution of the Cauchy problem for (3·1) with initial data $\phi(0, x)=g(x)$ in $\mathcal{S}(\boldsymbol{R}; \boldsymbol{C}^2)$. Define sequences $\{\tilde{\Phi}^{(n)}\}_{n=1}^{\infty}$ and $\{\tilde{A}^{(n)}\}_{n=1}^{\infty}$ by

$$\tilde{\Phi}^{(n)}(t, x)=\Phi(ls/n, x)\,, \quad \tilde{A}^{(n)}(t, x)=A(ls/n, x)\,,$$
$$(l-1)s/n\le t\le ls/n\,, \quad 1\le l\le n\,.$$

Here note that $\tilde{\Phi}^{(n)}\to\Phi$ and $\tilde{A}^{(n)}\to A$ locally uniformly in $\boldsymbol{R}^2$ as $n\to\infty$, because Φ and A are uniformly continuous on each compact set in $\boldsymbol{R}^2$. For each $n\ge 1$, let $\phi^{(n)}(t, x)$ be the solution of the Cauchy problem for (3·1) with $\tilde{\Phi}^{(n)}$ and $\tilde{A}^{(n)}$ in place of Φ and A, respectively, with the same initial data $\phi^{(n)}(0, x)=g(x)$. Then by (4·33) we have

$$(f, \phi^{(n)}(s, \cdot))$$
$$=\int(f, d\nu_{s;0}^{\mathrm{I}}(X)g)\exp\left[-i\int_0^s e\tilde{\Phi}^{(n)}(t, X(t))dt+i\int_0^s e\tilde{A}^{(n)}(t, X(t))dX(t)\right] \tag{4·34}$$

for (f, g) in $\mathcal{S}(\boldsymbol{R}; (\boldsymbol{C}^2)')\times\mathcal{S}(\boldsymbol{R}; \boldsymbol{C}^2)$. The right-hand side of (4·34) converges to the last member of (3·6) with $r=0$ by the Lebesgue bounded convergence theorem. To see the convergence of the left-hand side of (4·34), suppose first that g is in $C_o^{\infty}(\boldsymbol{R}; \boldsymbol{C}^2)$. Then in view of (3·1) we get, denoting the L^2 norm by $\|\cdot\|_2$,

$$\|\phi(s, \cdot)-\phi^{(n)}(s, \cdot)\|_2{}^2=\int_0^s dt\frac{d}{dt}\|\phi(t, \cdot)-\phi^{(n)}(t, \cdot)\|_2{}^2$$
$$=\int_0^s dt\{(\phi^{(n)}(t, \cdot), [ie(\Phi(t, \cdot)-\tilde{\Phi}^{(n)}(t, \cdot))$$
$$+iea(A(t, \cdot)-\tilde{A}^{(n)}(t, \cdot))]\phi(t, \cdot))$$
$$-(\phi(t, \cdot), [ie(\Phi(t, \cdot)-\tilde{\Phi}^{(n)}(t, \cdot))$$
$$+iea(A(t, \cdot)-\tilde{A}^{(n)}(t, \cdot))]\phi^{(n)}(t, \cdot))\}\,.$$

We note here that g has compact support and so do $\phi(t, \cdot)$ and $\phi^{(n)}(t, \cdot)$, by the finite propagation property of the solution for the Cauchy problem for the Dirac equation (3·1). The integrand of the third member of the above equation converges to zero uniformly in t on $[0, s]$ as $n\to\infty$ because $\tilde{\Phi}^{(n)}$ and $\tilde{A}^{(n)}$ converge to Φ and A, respectively, locally uniformly on $[0, s]\times\boldsymbol{R}$. Thus we have $\|\phi(s, \cdot)-\phi^{(n)}(s, \cdot)\|_2\to 0$ and hence the left-hand side of (4·34) converges to the first member of (3·6) with $r=0$ as $n\to\infty$. This proves (3·6) when g is in $C_o^{\infty}(\boldsymbol{R}; \boldsymbol{C}^2)$. Now suppose that g is in $\mathcal{S}(\boldsymbol{R}; \boldsymbol{C}^2)$. For every $\varepsilon>0$ there is a g' in $C_o^{\infty}(\boldsymbol{R}; \boldsymbol{C}^2)$ with $\|g-g'\|_2<\varepsilon$. Let $\phi(t, x)$ and $\phi'(t, x)$ [resp., $\phi^{(n)}(t, x)$ and $\phi'^{(n)}(t, x)$] be the solutions of the Cauchy problem for (3·1) [resp., with $\tilde{\Phi}^{(n)}$ and $\tilde{A}^{(n)}$ in place of Φ and A] with initial data $\phi(0, x)=g(x)$ and $\phi'(0, x)=g'(x)$ [resp., $\phi^{(n)}(0, x)=g(x)$ and $\phi'^{(n)}(0, x)=g'(x)$]. Then we have by unitarity

$$\|\phi(t, \cdot)-\phi'(t, \cdot)\|_2=\|\phi^{(n)}(t, \cdot)-\phi'^{(n)}(t, \cdot)\|_2=\|g-g'\|_2<\varepsilon$$

and so

$$\begin{aligned}
&\|\phi^{(n)}(t,\cdot)-\phi(t,\cdot)\|_2\\
&\le\|\phi^{(n)}(t,\cdot)-\phi'^{(n)}(t,\cdot)\|_2+\|\phi'^{(n)}(t,\cdot)-\phi'(t,\cdot)\|_2+\|\phi'(t,\cdot)-\phi(t,\cdot)\|_2\\
&\le 2\|g-g'\|_2+\|\phi'^{(n)}(t,\cdot)-\phi'(t,\cdot)\|_2\\
&\le 2\varepsilon+\|\phi'^{(n)}(t,\cdot)-\phi'(t,\cdot)\|_2\,.
\end{aligned}$$

The second term in the last member, as already seen above, converges to zero as $n\to\infty$, because g' is in $C_0^\infty(\boldsymbol{R};\boldsymbol{C}^2)$. Since ε is arbitrary, we conclude that as $n\to\infty$, $\phi^{(n)}(t,\cdot)\to\phi(t,\cdot)$ in L^2 and so the left-hand side of (4·34) converges to the first member of (3·6) with $r=0$, when g is in $\mathcal{S}(\boldsymbol{R};\boldsymbol{C}^2)$. This prove (3·6) in the general case, completing the proof of Theorem 3.1.

§ 5. The Dirac equation in four space-time dimensions

We consider the problem of the path integral for the four-space-time-dimensional Dirac equation (2·1) with $d=3$. The Dirac matrices α_j and β are 4×4 Hermitian matrices satisfying $\alpha_j{}^2=\beta^2=1$, $\alpha_j\beta+\beta\alpha_j=0$, $1\le j\le3$, and $\alpha_j\alpha_k+\alpha_k\alpha_j=0$, $j\ne k$.

Our method does not seem to establish the L^∞ well-posedness to get Lemma 4.2 for the Cauchy problem for the free equation to (2·1) with $d=3$,

$$\partial_t\phi(t,\boldsymbol{x})=[-\sum_{j=1}^{3}\alpha_j\partial_j-im\beta]\phi(t,\boldsymbol{x})\,. \tag{5·1}$$

However, we can deal with three special cases, the path integral for the free Dirac equation, that for the Dirac equation with a central electric field and that for the Dirac equation with parallel electric and unifom magnetic fields, which are reduced to the problems for equations with two independent variables as considered in § 3.

5.1. *The free Dirac equation*

We use the Radon transform[30] to reduce the problem with four independent variables to that with two independent variables.[31]

The Radon transform $\hat{g}$ of a function g defined in $\boldsymbol{R}^3$ is by definition

$$\hat{g}(\xi,\omega)=\int_{\boldsymbol{R}^3}g(\boldsymbol{x})\delta(\xi-\boldsymbol{x}\boldsymbol{\omega})d\boldsymbol{x}\,,$$

where $\xi\in\boldsymbol{R}$ and $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3)$ is a unit vector in $\boldsymbol{R}^3$. The following Plancherel theorem holds:

$$\int_{\boldsymbol{R}^3}\overline{f(\boldsymbol{x})}g(\boldsymbol{x})d\boldsymbol{x}=2^{-1}(2\pi)^{-2}\int_{|\boldsymbol{\omega}|=1}d\boldsymbol{\omega}\int_{\boldsymbol{R}}d\xi\,\overline{\hat{f}_\xi(\xi,\boldsymbol{\omega})}\hat{g}_\xi(\xi,\boldsymbol{\omega})\,, \tag{5·2}$$

where $\hat{f}_\xi=\partial_\xi\hat{f}$ and $\hat{g}_\xi=\partial_\xi\hat{g}$.

Then the fundamental solution $K_0(t,\boldsymbol{x}-\boldsymbol{y})$ of the Cauchy problem for the free Dirac equation (5·1) admits the following path integral representation. Note that there is a unitary matrix $N(\boldsymbol{\omega})$ such that $N(\boldsymbol{\omega})(\sum_{j=1}^3\alpha_j\omega_j)N(\boldsymbol{\omega})^{-1}=\alpha_1$ and $N(\boldsymbol{\omega})\beta N(\boldsymbol{\omega})^{-1}=\beta$.

THEOREM 5.1. There exists a unique $\mathcal{S}'(\boldsymbol{R}\times\boldsymbol{R};M_4(\boldsymbol{C}))$-valued countably additive measure $\nu^{\mathrm{I}}_{t;0}$ on the Banach space $C(|0,t|;\boldsymbol{R})$ of the one-dimensional continuous paths $\Xi:|0,t|\to\boldsymbol{R}$ such that for (f,g) in $\mathcal{S}(\boldsymbol{R}^3;(\boldsymbol{C}^4)')\times\mathcal{S}(\boldsymbol{R}^3;\boldsymbol{C}^4)$

$$(f,\phi(t,\cdot))=\iint_{R^3\times R^3}\overline{f(\boldsymbol{x})}K_0(t,\boldsymbol{x}-\boldsymbol{y})g(\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y}$$

$$=2^{-1}(2\pi)^{-2}\int_{|\omega|=1}d\boldsymbol{\omega}\int(\hat{f}_\xi(\cdot,\boldsymbol{\omega}),N(\boldsymbol{\omega})^{-1}d\nu^{\mathrm{I}}_{t;0}(\Xi)N(\boldsymbol{\omega})\hat{g}_\xi(\cdot,\omega)).\quad(5\cdot3)$$

The measure $\nu^{\mathrm{I}}_{t;0}$ is concentrated on the set of those Lipschitz continuous paths $\Xi:|0,t|\to\boldsymbol{R}$ which satisfy

for some finite partition: $0=t_0\gtrless t_1\gtrless\cdots\gtrless t_k=t$

of $|0,t|$, depending on Ξ,

$$\Xi(\tau)-\Xi(0)=\sum_{i=1}^{j-1}(-1)^i(t_i-t_{i-1})+(-1)^j(\tau-t_{j-1})$$

or

$$\Xi(\tau)-\Xi(0)=\sum_{i=1}^{j-1}(-1)^{i-1}(t_i-t_{i-1})+(-1)^{j-1}(\tau-t_{j-1}),$$

for $\tau\in|t_{j-1},t_j|$, $\quad 1\le j\le k$

$[|\Xi(\tau)-\Xi(0)|=|\tau|$ for $\tau\in|0,t|$, in case $m=0]$. (5·4)

The set function

$$\nu^{\mathrm{I}}_{t,\hat{f}_\xi(\cdot,\boldsymbol{\omega});0,\hat{g}_\xi(\cdot,\boldsymbol{\omega})}(\cdot)=(\hat{f}_\xi(\cdot,\boldsymbol{\omega}),N(\boldsymbol{\omega})^{-1}\nu^{\mathrm{I}}_{t;0}(\cdot)N(\boldsymbol{\omega})\hat{g}_\xi(\cdot,\boldsymbol{\omega}))$$

is a complex-valued countably additive measure on the Banach space $C(|0,t|;\boldsymbol{R})$ which is concentrated on the set of the Lipschitz continuous paths Ξ satisfying (5·4) and $\Xi(0)\in\operatorname{supp}\hat{g}_\xi(\cdot,\boldsymbol{\omega})$, $\Xi(t)\in\operatorname{supp}\hat{f}_\xi(\cdot,\boldsymbol{\omega})$.

Proof The Radon transform of (5·1) yields

$$\partial_t\hat{\phi}(t,\xi,\boldsymbol{\omega})=[-(\sum_{j=1}^{3}\alpha_j\omega_j)\partial_\xi-im\beta]\hat{\phi}(t,\xi,\boldsymbol{\omega}).\quad(5\cdot5)$$

Multiply (5·5) by $N(\boldsymbol{\omega})$ from the left. Then we have

$$\partial_t\eta(t,\xi,\boldsymbol{\omega})=[-\alpha_1\partial_\xi-im\beta]\eta(t,\xi,\boldsymbol{\omega})\quad(5\cdot5)'$$

with $\eta(t,\xi,\boldsymbol{\omega})=N(\boldsymbol{\omega})\hat{\phi}(t,\xi,\boldsymbol{\omega})$. For $\boldsymbol{\omega}$ fixed, (5·5)′ is a first-order hyperbolic system with two independent variables t and ξ. In the same way as in the proof of Theorems 3.1 and 3.2 we can construct the path space measure $\nu^{\mathrm{I}}_{t;0}$ with the property mentioned in Theorem 5.1.

To get (5·3), differentiate by ξ both sides of (5·5). Then if $\hat{\phi}_\xi(t,\xi,\boldsymbol{\omega})\equiv\partial_\xi\hat{\phi}(t,\xi,\boldsymbol{\omega})$ is the solution of the Cauchy problem for (5·5) with initial data $\hat{\phi}_\xi(0,\xi,\boldsymbol{\omega})=\hat{g}_\xi(\xi,\boldsymbol{\omega})\equiv\partial_\xi\hat{g}(\xi,\boldsymbol{\omega})$ it has the following path integral representation:

$$(\hat{f}_\xi(\cdot, \boldsymbol{\omega}), \hat{\phi}_\xi(t, \cdot, \boldsymbol{\omega}))$$

$$= \int (\hat{f}_\xi(\cdot, \boldsymbol{\omega}), N(\boldsymbol{\omega})^{-1} d\nu^{\mathrm{I}}_{t;0}(\Xi) N(\boldsymbol{\omega}) \hat{g}_\xi(\cdot, \boldsymbol{\omega})) .$$

The formula (5·3) follows from this with the aid of the Plancherel formula (5·2). This proves Theorem 5.1.

Remark Formal substitution of $\delta_{\boldsymbol{x}} = \delta(\cdot - \boldsymbol{x})$ and $\delta_{\boldsymbol{y}} = \delta(\cdot - \boldsymbol{y})$ into f and g yields the following intuitive expression of (5·3):

$$K_0(t, \boldsymbol{x} - \boldsymbol{y}) = 2^{-1}(2\pi)^{-2} \int_{|\boldsymbol{\omega}|=1} d\boldsymbol{\omega} \int \delta'(\Xi(t) - \boldsymbol{x}\boldsymbol{\omega})$$

$$\times N(\boldsymbol{\omega})^{-1} d\nu^{\mathrm{I}}_{t;0}(\Xi) N(\boldsymbol{\omega}) \delta'(\Xi(0) - \boldsymbol{y}\boldsymbol{\omega}) , \quad (5\cdot3)'$$

where $\delta'(s) = (d/ds)\delta(s)$.

5.2. *The Dirac equation for a central electric field*

The Dirac equation for a central electric field can be separated in spherical coordinates.[6] The radial Dirac equation is

$$\partial_t \chi(t, r) = -iH^\kappa \chi(t, r) , \qquad t \in \boldsymbol{R},\ r \in (0, \infty) ,$$

$$H^\kappa = H_0^\kappa + V(r) ,$$

$$H_0^\kappa = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \partial_r + \begin{pmatrix} m & \kappa/r \\ \kappa/r & -m \end{pmatrix} \qquad (5\cdot6)$$

with $V = e\Phi$, where κ is a positive and negative integer. We assume that $V(r)$ is a real-valued continuous function in $(0, \infty)$ such that H^κ is self-adjoint[32] in $L^2((0, \infty), dr; \boldsymbol{C}^2)$. The following theorem is, though of a rather restrictive character, concerned with a path integral representation for the solution $\kappa(t, r)$ for the Cauchy problem for (5·6) with initial data $\chi(0, r) = g(r)$.

THEOREM 5.2. Let f and g be in $\mathcal{S}(\boldsymbol{R}^+; (\boldsymbol{C}^2)')$ and $\mathcal{S}(\boldsymbol{R}^+; \boldsymbol{C}^2)$, the restrictions of $\mathcal{S}(\boldsymbol{R}; (\boldsymbol{C}^2)')$ and $\mathcal{S}(\boldsymbol{R}; \boldsymbol{C}^2)$ to $\boldsymbol{R}^+ = [0, \infty)$, respectively. If, for each s with $0 \le s \le t$ when $t > 0$ or with $0 \ge s \ge t$ when $t < 0$, the intersection

$$\{r \in \boldsymbol{R}^+; |x - r| \le |t - s|,\ x \in \mathrm{supp} f\} \cap \{r \in \boldsymbol{R}^+; |r - y| \le |s|,\ y \in \mathrm{supp} g\} \qquad (5\cdot7)$$

does not contain 0, there exists a unique complex-valued countably additive measure $\nu^+_{\kappa;t,f;0,g}$ on the set of the one-dimensional continuous paths $R: |0, t| \to \boldsymbol{R}^+$ such that for every continuous $V(r)$,

$$(f, e^{-itH^\kappa} g) = \int_{\boldsymbol{R}^+} \overline{f(r)} \chi(t, r) dr$$

$$= \int d\nu^+_{\kappa;t,f;0,g}(R) \exp\left[-i \int_0^t V(R(s)) ds\right] . \qquad (5\cdot8)$$

The support of $\nu^+_{\kappa;t,f;0,g}$ is on the set of the Lipschitz continuous paths $R: |0, t| \to \boldsymbol{R}^+$ satisfying for each a, b with $0\leq a<b\leq t$ when $t>0$

or $0\geq a>b\geq t$ when $t<0$,

$$|R(b)-R(a)|\leq|b-a| \tag{5·9}$$

and $R(0)\in\operatorname{supp}g$, $R(t)\in\operatorname{supp}f$.

Proof We give only an outline of the proof, for it proceeds with a similar argument used in the proof of Theorem 3.1.

Let t, f and g be as in Theorem 5.2. We consider only the case $t>0$. The free equation to (5·6) is

$$\partial_s\chi(s, r)=-iH_0{}^{\kappa}\chi(s, r)\,, \qquad 0<s\leq t\,, \qquad r\in(0, \infty)\,. \tag{5·10}$$

Then for the Cauchy problem for (5·10) we have the following lemma (cf. Lemma 4.1). Let $r_0>0$ be the minimum of the set (5·7), so that $|\kappa|/r_0$ is an upper bound of $|\kappa|/r$ with r in the set (5·7).

Lemma 5.3. If r is in $\operatorname{supp}f$ with $r\geq\min(\operatorname{supp}g)-s$, then

$$|N\chi(s, r)|\leq e^{Ms}\max\{|N\chi(0, u)|;\, u\in\operatorname{supp}g\,, \quad r-s\leq u\leq r+s\}\,,$$

where $M=m+|\kappa|/r_0$ and $N=2^{-1/2}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}$.

Proof We only note the following. Multiplying Eq. (5·10) by N from the left, we have with $\eta(t, r)=N\chi(t, r)$,

$$\partial_t\eta(t, r)=\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\partial_r+\begin{pmatrix}0 & -m-i\kappa/r\\ m-i\kappa/r & 0\end{pmatrix}\right]\eta(t, r)\,. \tag{5·10$'$}$$

Notice that Lemma 5.3 yields the support property of the fundamental solution $K_0{}^{+}(s, r)$ of the Cauchy problem for (5·10).

To construct the path space measure let $\mathfrak{R}_{t,0}=\prod_{[0,t]}\dot{\boldsymbol{R}}^{+}$ be the product of the uncountably many copies of $\dot{\boldsymbol{R}}^{+}=\boldsymbol{R}^{+}\cup\{\infty\}$, the one-point compactification of $\boldsymbol{R}^{+}$. Let $C(\mathfrak{R}_{t,0})$ be the Banach space of the continuous functions on the compact Hausdorff space $\mathfrak{R}_{t,0}$, and $C_{\text{fin}}(\mathfrak{R}_{t,0})$ its subspace of those Ψ for which there exist a finite partition $0=t_0<t_1<\cdots<t_n=t$ of the interval $[0, t]$ and a complex-valued bounded continuous function $F(r_0, r_1, \cdots, r_n)$ on $(\dot{\boldsymbol{R}}^{+})^{n+1}$ such that $\Psi(R)=F(R(t_0), R(t_1), \cdots, R(t_n))$.

Define a linear functional $L_{\kappa;\,t,f;\,0,g}$ on $C_{\text{fin}}(\mathfrak{R}_{t,0})$ by

$$L_{\kappa;\,t,f;\,0,g}\Psi=\overbrace{\int_{\boldsymbol{R}^{+}}\cdots\int_{\boldsymbol{R}^{+}}}^{n+1}\overline{f(r_n)}K_0{}^{+}(t_n-t_{n-1}, r_n-r_{n-1})\cdots$$

$$\times K_0{}^{+}(t_1-t_0, r_1-r_0)F(r_0, r_1, \cdots, r_n)g(r_0)dr_0dr_1\cdots dr_n\,.$$

Then the following lemma will be shown with use of Lemma 5.3 (cf. Lemma 4.2).

Lemma 5.4. $L_{\kappa;t,f;0,g}\Psi$ is well-defined on $C_{\text{fin}}(\mathcal{R}_{t,0})$ and

$$|L_{\kappa;t,f;0,g}\Psi|\le 2e^{Mt}\|\Psi\|\|f\|_{L^1((0,\infty),dr)}\|g\|_{L^\infty((0,\infty),dr)}$$

for every Ψ in $C_{\text{fin}}(\mathcal{R}_{t,0})$, where $M=m+|\kappa|/r_0$.

Since $C_{\text{fin}}(\mathcal{R}_{t,0})$ is dense in $C(\mathcal{R}_{t,0})$, it follows from Lemma 5.4 with the Riesz-type representation theorem[27] that there exists a unique complex-valued countably additive measure $\nu^+_{\kappa;t,f;0,g}$ defined on the Borel sets in $\mathcal{R}_{t,0}$ such that for every $\Psi \in C(\mathcal{R}_{t,0})$,

$$L_{\kappa;t,f;0,g}\Psi=\int_{\mathcal{R}_{t,0}}d\nu^+_{\kappa;t,f;0,g}(R)\Psi(R)\,.$$

The support property of the measure will be seen from that of $K_0{}^+(s,r)$. For the proof we refer to the argument used in Ref. 13), Section III B. Once the path space measure $\nu^+_{\kappa;t,f;0,g}$ is constructed the proof of the formula (5·8) will be accomplished as in § 4.3.

Remark 1. The restriction for t and the supports of f and g in Theorem 5.2 means that the information which starts from g at time 0 to reach f at time t has never passed through the center of the potential, i.e., $r=0$. We need it, for the free Dirac equation (5·10) contains the $1/r$ singularity in $H_0{}^\kappa$, which invalidates Lemma 5.3. As for the Cauchy problem for (5·6), the theorem gives only a short-time representation of its solution $\chi(t,r)$ with initial data $\chi(0,r)=g(r)$, a function in $\mathcal{S}(\boldsymbol{R}^+;\boldsymbol{C}^2)$ with support not containing $r=0$.

Remark 2. Even when $m=0$, it cannot be asserted that the support of $\nu^+_{\kappa;t,f;0,g}$ is on the set of the paths $R:|0,t|\to\boldsymbol{R}^+$ satisfying $|R(b)-R(a)|=|b-a|$ instead of (5·9) for the same a, b. The presence of the $i\kappa/r$ term in $H_0{}^\kappa$ might warp the paths.

These facts may suggest that this case cannot be effectively reduced to a two-space-time-dimensional problem.

5.3. *The Dirac equation for parallel electric and uniform magnetic fields*

We consider, in 4-dimensional space-time $\boldsymbol{R}\times\boldsymbol{R}^3=\boldsymbol{R}^4$, a uniform magnetic field in the 3-direction and a parallel electric field, which are given by the potentials

$$\Phi(t,x_3),\quad A_1=A_3=0\quad\text{and}\quad A_2=Bx_1\,. \tag{5·11}$$

The coordinates are so chosen that $eB>0$. Then the Dirac equation becomes

$$\partial_t\phi(t,x_1,x_2,x_3)=[-\alpha_1\partial_1-\alpha_2(\partial_2-ieBx_1)-\alpha_3\partial_3-i\beta m-ie\Phi(t,x_3)]$$
$$\times\phi(t,x_1,x_2,x_3)\,. \tag{5·12}$$

We first make the Fourier transform of $\phi(t,x_1,x_2,x_3)$ in the variable x_2, i.e.,

$$\tilde{\phi}(t,x_1,p,x_3)=(2\pi)^{-1/2}\int_{\boldsymbol{R}}\phi(t,x_1,x_2,x_3)e^{-ipx_2}dp\,,$$

and next expand $\tilde{\phi}(t,x_1,p,x_3)$ in terms of the Hermite functions of the variable

$(eB)^{1/2}(x_1 - p/eB)$. Then we have

$$\phi(t, x_1, x_2, x_3) = (2\pi)^{-1/2} \sum_{n=0}^{\infty} \int_{\boldsymbol{R}} \Omega_n\left((eB)^{1/2}\left(x_1 - \frac{p}{eB}\right)\right) \tilde{\phi}_n(t, x_3; p) e^{ipx_2} dp \qquad (5\cdot13)$$

with

$$\tilde{\phi}_n(t, x_3; p) = \int_{\boldsymbol{R}} \Omega_n\left((eB)^{1/2}\left(x_1 - \frac{p}{eB}\right)\right) \tilde{\phi}(t, x_1, p, x_3)(eB)^{1/2} dx_1 .$$

The $\Omega_n(\xi)$ are the normalized Hermite functions

$$\Omega_n(\xi) = c_n H_n(\xi) e^{-\xi^2/2}, \qquad n = 0, 1, 2, \cdots$$

with normalization constants c_n depending on n, where the functions $H_n(\xi)$ are the Hermite polynomials of order n. They are characterized by the equations

$$(d/d\xi)\Omega_n(\xi) = \left(\frac{n}{2}\right)^{1/2} \Omega_{n-1}(\xi) - \left(\frac{n+1}{2}\right)^{1/2} \Omega_{n+1}(\xi) ,$$

$$\xi \Omega_n(\xi) = \left(\frac{n}{2}\right)^{1/2} \Omega_{n-1}(\xi) + \left(\frac{n+1}{2}\right)^{1/2} \Omega_{n+1}(\xi)$$

with $\Omega_0(\xi) = \pi^{-1/4} e^{-\xi^2/2}$. Further define functions ϕ_n by

$$\phi_n(t, x_3; p) \equiv 2^{-1}(\mathbf{1} + i\alpha_1\alpha_2)\tilde{\phi}_{n-1}(t, x_3; p) + i2^{-1}(\mathbf{1} - i\alpha_1\alpha_2)\tilde{\phi}_n(t, x_3; p) \qquad (5\cdot14)$$

with $\tilde{\phi}_{-1} \equiv 0$. Note that $2^{-1}(\mathbf{1} \pm i\alpha_1\alpha_2)$ are projections of $\boldsymbol{C}^4$ onto two-dimensional subspaces of $\boldsymbol{C}^4$. Substituting (5·13) and (5·14) into the Dirac equation (5·12), we get

$$\partial_t \phi_n(t, x_3; p) = [-\alpha_3\partial_3 - i(m^2 + 2neB)^{1/2}\beta_n - ie\Phi(t, x_3)]\phi_n(t, x_3; p),$$
$$n = 0, 1, \cdots , \qquad (5\cdot15)$$

where β_n is an Hermitian matrix in $M_4(\boldsymbol{C})$ given by

$$\beta_n = (m^2 + 2neB)^{-1/2}(\beta m - (2neB)^{1/2}\alpha_1) .$$

The 4×4 matrices α_3 and β_n satisfy $\alpha_3{}^2 = \beta_n{}^2 = \mathbf{1}$ and $\alpha_3\beta_n + \beta_n\alpha_3 = 0$. Since Eq. (5·15) is a first-order hyperbolic system with two independent variables t and x_3, the theory of the present article can be applied. We may also reduce Eq. (5·15) to an equation with 2×2 matrices as coefficients. In fact, there exists, for each n, a 4×4 matrix N_n such that

$$N_n\alpha_3 N_n{}^{-1} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad N_n\beta_n N_n{}^{-1} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix},$$

where σ_1 and σ_2 are Pauli matrices. Then it is seen that for each n, both $\phi_n{}^{\uparrow} = {}^t((N_n\phi_n)_1, (N_n\phi_n)_2)$ and $\phi_n{}^{\downarrow} = {}^t((N_n\phi_n)_3, (N_n\phi_n)_4)$ satisfy one and the same equation

$$\partial_t \psi(t, x_3) = [-\sigma_3\partial_3 - i(m^2 + 2neB)^{1/2}\sigma_1 - ie\Phi(t, x_3)]\psi(t, x_3),$$
$$t \in \boldsymbol{R},\ x_3 \in \boldsymbol{R} , \qquad (5\cdot16)$$

which is nothing but the two-dimensional Dirac equations (3·1) with mass

$(m^2+2neB)^{1/2}$ and $A\equiv 0$. Therefore the same statements as in Theorem 3.1 hold for Eq. (5·16).

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