Q1 // derive the following

1. Derive the solution for a homogeneous Pfaffian differential equation in the variable x.
2. Derive the solution for a homogeneous Pfaffian differential equation in the variable y.
3. Derive the solution for a homogeneous Pfaffian differential equation in the variable z.
4. Derive the quasi linear PDE $Ap + Bq = R$ by the elimination of arbitrary function F from the equation $F(u, v) = 0$. where u and v are functions of x, y and z.
5. Derive the second order partial D.E by eliminating the arbitrary functions f and g from the relation $z = f(u) + g(v) + w$, where u, v and w are functions of x and y.
6. Derive the Charpit's auxiliary equation.

Q2 // Theorems

1. Prove that the general solution of linear PDE $Ap + Bq = R$ is $F(u, v) = 0$, where f is an arbitrary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ form a solution of the equation $\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{R}$.
2. State and prove the Integrable Theorem of Pfaffian differential equations.

Exercise:

1- IAN ex $x^2 + y^2 + (z - c)^2 = a^2$ Ans. $yp - xq = 0$

2- IAN ex $x^2 + y^2 = (z - c)^2 \tan^2 a$ Ans. $yp - xq = 0$

3- IAN $(x - a)^2 + (y - b)^2 + z^2 = 1$ Ans. $z^2(p^2 + q^2 + 1) = 1$

4- IAN $z = (x + a)(y + b)$ Ans. $z = pq$

5- IAN $2z = (ax + y)^2 + b$ Ans. $xp + yq = q^2$

6- IAN $ax^2 + by^2 + z^2 = 1$ Ans. $xp + yq = \frac{z^2-1}{z}$

Exercise:

1- IAN $z = xy + f(x^2 + y^2)$ Ans. $xq - yp = x^2 - y^2$

2- IAN $z = x + y + f(xy)$ Ans. $xp - yq = x - y$

3- IAN $z = f(\frac{xy}{z})$ Ans. $xp - yq = 0$

4- IAN $z = f(x - y)$ Ans. $p + q = 0$

5- IAN $z = f(x^2 + y^2 + z^2, z^2 - 2xy)$ Ans. $z(p - q) = y - x$

Exercise:

1- (IAN) Verify that the partial differential equation $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{2z}{x^2}$ is satisfied by $z = \frac{1}{x}\phi(y-x) + \phi'(y-x)$ where ϕ is an arbitrary function.

2- (IAN) If $u = f(x+iy) + g(x-iy)$, where the functions f and g are arbitrary, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

3- (IAN) Show that if f and g are arbitrary functions of a single variable, then

$u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$ is a solution of the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ provided that $\alpha^2 = 1 - v^2/c^2$.

4- (IAN) If $z = f(x^2 - y) + g(x^2 + y)$ where the functions f, g are arbitrary, prove that $\frac{\partial^2 z}{\partial x^2} - \frac{1}{x}\frac{\partial z}{\partial x} = 4x^2\frac{\partial^2 z}{\partial y^2}$

5- (IAN) A variable z is defined in terms of variables x, y as the result of eliminating t from the equations

$$z = \mathrm{t}x + yf(t) + g(t)$$
$$0 = x + yf'(t) + g'(t)$$

Prove that, whatever the functions f and g may be, the equation $rt - s^2 = 0$ is satisfied.

Exercise:

1- (IAN) $z(xp - yq) = y^2 - x^2$ Ans. $dx = dy\,;\ x, y, z\ F(xy, x^2 + y^2 + z^2) = 0$

2- (IAN) $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$

3- (IAN) $px(x + y) = qy(x + y) - (x - y)(2x + 2y + z)$

$dx = dy\colon dx + dy = dx + dy + dz$ Ans. $F(xy, (x+y)(x+y+z)) = 0$

4- (IAN) $y^2 p - xyq = x(z - 2y)$ Ans. $F(x^2 + y^2, yz - y^2) = 0$

5- (IAN) $(y + xz)p - (x + yz)q = x^2 - y^2$ Ans. $y, x, 1\colon x, y, -z$ $F(xy + z, x^2 + y^2 - z^2) = 0$

6- (IAN) $x(x^2 + 3y^2)p - y(3x^2 + y^2)q = 2z(y^2 - x^2)$

Exercise: (IAN)

1- Find the equation of the integral surface of the differential equation
$2y(z-3)p + (2x-z)q = y(2x-3)$ which passes through the circle $z = 0, x^2 + y^2 = 2x$.
Ans.$1, 2y, -2; dx = dz: x + y^2 - 2z = c_1; x^2 - 3x - z^2 + 6z = c_2, x = t \rightarrow c_1 + c_2 = 0$.

2- Find the general integral of the P.D.E. $(2xy-1)p + (z-2x^2)q = 2(x-yz)$ and also the particular integral which passes through the line $x = 1, y = 0$.

3- Find the integral surface of the equation $(x-y)y^2p + (y-x)x^2q = z(x^2+y^2)$ through the curve $xz = a^3, y = 0$.
Ans. $dx = dy; \frac{1}{x-y}, \frac{-1}{x-y}, \frac{-1}{z}: x^3 + y^3 = c_1; \frac{x-y}{z} = c_2, x = t \rightarrow (x^3+y^3)^2 = a^9 \frac{(x-y)^3}{z^3}$.

4- Find the general solution of the equation $2x(y+z^2)p + y(2y+z^2)q = z^3$ and deduce that $yz(z^2 + yz - 2y) = x^2$ is a solution.
Ans. $\frac{1}{x}dx - \frac{1}{y}dy = \mathrm{dz}; dy = dz: y^2 + z^2 = f(x^2 - y^2)$.

5- Find the general integral of the equation $(x-y)p + (y-x-z)q = z$ and the particular solution through the circle $z = 1, x^2 + y^2 = 1$.

6- Find the general solution of the differential equation
$$x(z+2a)p + (xz + 2yz + 2ay)q = z(z-a)$$
Find also the integral surfaces which pass through the curves:
(a) $y = 0,\ z^2 = 4ax$
(b) $y = 0,\ z^3 + x(z+a)^2 = 0$

7- Find the surface which is orthogonal to the one-parameter system $z = cxy\,(x^2+y^2)$ and which passes through the hyperbola $x^2 - y^2 = a^2, z = 0$.

Ans. $xdx + ydy = -zdz$

8- Find the equation of the system of surfaces which cut orthogonally the cones of the system $x^2 + y^2 + z^2 = cxy$.

Ans. $x, y, z; xdx - ydy = dz$

9- Find the general equation of surfaces orthogonal to the family given by:

a) $x(x^2+y^2+z^2) = c_1y^2$ Ans. $4xdx + 2ydy = dz$
showing that one such orthogonal set consists of the family of spheres given by
b) $x^2 + y^2 + z^2 = c_2z$
If a family exists, orthogonal to both (a) and (b), show that it must satisfy
$$2x(x^2 - z^2)dx + \mathrm{y}(3x^2 + y^2 - z^2)dy + 2z(2x^2 + y^2)dz = 0$$
Show that such a family in fact exists, and find its equation.

10- Show that the integral surface of $(x^2 + y^2 - a^2)(xp + yq) = z(x^2 + y^2)$ are generated by conics, and find the integral surface through the curve $x = 2z, x^2 + y^2 = 4a^2$.
Ans. $\frac{x^2+y^2-a^2}{z^2} = c_1; \frac{y}{x} = c_2; z = t \rightarrow 3z^2(x^2+y^2) = x^2(x^2+y^2-a^2)$

Exercise: (IAN70)

1- $(p^2 + q^2)y = qz$.
2- $p = (z + qy)^2$.
3- $z^2 = pqxy$.
4- $xp + 3yq = 2(z - x^2q^2)$.
5- $px^5 - 4q^3x^2 + 6x^2z - 2 = 0$.
6- $2(y + zq) = q(xp + yq)$.
7- $2(z + xp + yq) = yp^2$.

Exercise: (IAN73)

1- $p + q = pq$.
2- $z = p^2 - q^2$.
3- $zpq = p + q$.
4- $p^2q(x^2 + y^2) = p^2 + q$.
5- $p^2q^2 + x^2y^2 = x^2q^2(x^2 + y^2)$.
6- $pqz = p^2(xq + p^2) + q^2(yp + q^2)$