The Epistemology of Geometry

Philosophy of Physics Lecture 1, 21 January 2014, Adam Caulton (aepw2@cam.ac.uk)

Geometry is an excellent case study for the general question: *What is the relationship between physics and mathematics?*

1 Euclidean geometry

1. Any two points determine a unique finite straight line (that passes through them).
2. Any finite straight line may be extended to a unique straight line.
3. Any point and a distance (radius) determines a unique circle.
4. All right angles are equal to one another.
5. (*The parallel postulate:*) If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

1-4 are known as *absolute geometry*. Euclid himself may have seen 5 as qualitatively different, since he proved his first 28 propositions without it. An alternative to the parallel postulate (logically equivalent, under the assumption of absolute geometry) is *Playfair's axiom:*

> Any straight line and any point not on that line determine a unique straight line parallel to the first.

2 Non-Euclidean geometries

Bolyai and Lobachevski independently discovered non-Euclidean geometries in 1823. (In fact they had been discovered before by Gauss, but suppressed.) These geometries are obtained by denying the fifth postulate. Specifically, we may either: (i) deny that parallels exist; or (ii) deny that they are unique. Possibility (i) leads to *elliptical* (a.k.a. Riemann) geometry; possibility (ii) leads to *hyperbolic* (a.k.a. Bolyaian or Lobachevskian) geometry.

Geometry	*Curvature*	*# of parallels*	*Sum of traingle's angles*	*Circum./diam.*
Elliptical	positive	0	$> 180°$	$< \pi$
Euclidean	0	1	$180°$	π
Hyperbolic	negative	∞	$< 180°$	$> \pi$

(table adapted from Carnap (1966, 133).)

Non-Euclidean geometries were proved to be consistent by providing a representation of them in Euclidean space. For example in two dimensions:

Elliptical	*Euclidean*	*Hyperbolic 1*	*Hyperbolic 2*	*Hyperbolic 3*
sphere	plane	disc	disc	half-plane
antipodal pair	point	point in disc	point in disc	point in half-plane
great circle	straight line	chord	perp arc	semi-circle on edge

(and there are higher-dimensional analogues. . .)

3 Kant

According to Kant, the postulates of Euclidean geometry are synthetic *a priori* judgements, delivered to us from our pure intution of space.

Kant's distinction between sensibility and understanding allows for two notions of necessity, captured by *a priori* and analytic, respectively. Therefore it is doubtful that Kant would have been concerned with the fact that non-Euclidean geometries are logically consistent—the question is whether they are inuitable. Here Frege was on Kant's side:

> Empirical propositions hold good of what is physically or psychologically actual, the truths of geometry govern all that is spatially intuitable, whether actual or product of our fancy. The wildest visions of delirium, the boldest inventions of legend and poetry, where animals speak and stars stand still, where men are turned to stone and trees turn into men, where the drowning haul themselves up out of swamps by their own topknots – all these remain, so long as they remain intuitable, still subject to the axioms of geometry. Conceptual thought alone can after a fashion shake off this yoke, when it assumes, say, a space of four dimensions or positive curvature. To study such conceptions is not useless by any means; but it is to leave the ground of intuition entirely behind. . . . For purposes of conceptual thought we can always assume the contrary of some one or other of the geometrical axioms, without involving ourselves in any self-contradictions. . . . The fact that this is possible shows that the axioms of geometry are independent of one another and of the primitive laws of logic, and consequently are synthetic. Can the same be said of the fundamental propositions of the science of number? Here, we have only to try denying any one of them, and complete confusion ensues. Even to think at all seems no longer possible. (*The Foundations of Arithmetic*, pp. 20-21.)

Indeed: the interpretations above seek to establish the consistency of non-Euclidean geometries *on the basis of the truth of Euclidean geometry!*

The more pressing questions for Kant are (cf. Coffa (1991, 45)):

1. *How* does pure intuition support the necessity of Euclidean geometry?
2. *Why* must a geometric argument be guided by pure intuition?

4 Helmholtz

Helmholtz made trouble for Kant along the lines of the first question by arguing that *we can intuitively represent non-Euclidean space to ourselves.* He wished to establish that geometry is empirical (i.e. synthetic *a posteriori*); a view that was later supported by Einstein (1921).

> By the much misused expression 'to represent' or 'to be able to think of how something happens', I understand that one could depict the series of sense-impressions one would have if such a thing happened in a particular case, I do not see how one could understand anything else by it without abandoning the whole sense of the expression. ('On the Origin and Significance of the Axioms of Geometry' [1870], in Helmholtz (1921).)

- The mirror universe: how reliable are our intuitions about the geometry of space?
- The auxiliary sphere (the 3-dimensional analogue to *Hyperbolic 1*): judgements of congruence as defined by *habit.*
- Parallel rail tracks (Reichenbach).
- The germ of conventionalism: to test geometrical axioms empirically, we must first know which objects are rigid, which surfaces flat and which edges straight, but "we only decide whether a body is rigid, its side flat and its edges straight, by means of the very propositions whose factual correctness the examination is supposed to show."

> Of course, one could also understand the concept of rigid geometric spatial configurations as a transcendental concept, formed independently of actual experiences and to which these need not necessarily correspond, as in fact our natural bodies do not correspond in an entirely pure and undistorted manner to the concepts that we have abstracted from them inductively. If we adopted this concept of rigidity understood as an ideal, a strict Kantian surely could then regard geometric axioms as *a priori* propositions given through transcendental intuition, and these propositions could not be confirmed or refuted by any experience because one should first have to decide in agreement with them whether given natural bodies should be considered rigid. But we should then add that under this interpretation, geometric axioms would certainly not be synthetic statements in Kant's sense; for they would then only assert an analytic consequence of the concept of rigid geometric configuration necessary for measurement, since one could accept as rigid only those configurations which satisfied the axioms. (*Schriften zur Erkenntnistheorie*, 23-4.)

Here we have a clear statement of the idea that "pure" geometry (analytic *a priori*) can be separated from "physical" geometry (synthetic *a posteriori*).

5 Poincaré

Poincaré's view was that geometry was *neither* synthetic *a priori* nor *a posteriori*, nor analytic *a priori*—rather, it was *conventional* (and therefore without truth-value).

- *Distinguish:* (i) "no fact of the matter"; (ii) under-determination of theory by observation; (iii) conventionalism about truth. Poincaré endorsed (i) and (ii) w.r.t. geometry.
- (ii) in more detail: our geometric observations can be accounted for under *any* geometry—including one of *varying curvature*, provided we are willing also to postulate compensating *universal forces*. Example: Poincaré's disc.
- Universal vs. differential forces.
- "Experiment ... tells us not what is the truest, but what is the most convenient geometry" (1902, 70-71).
- "From among all possible groups, that must be chosen which will be, so to speak, the *standard* to which we shall refer natural phenomena" (1913, 79).

> If we consider physical that deviate sufficiently from the predictions of geometry, then we consider the change, *by an arbitrary convention*, as the resultant of two other component changes. The first component is regarded as a displacement *rigorously* satisfying the laws [of the group of displacements] ... while the second component, which is small, is regarded as a qualitative alteration. ... [Thus] these laws are not imposed by nature upon us, but are imposed by us upon nature. But if we impose them on nature it is because she suffers us to do so. If she offered too much resistance, we should seek in our arsenal for another form which would be more acceptable to her. (1898, 5)

- Poincaré's inference from (ii) to (i) requires the assumption that (a) we have no direct perception of spatial relations; and (b) the (possible) facts of observation *exhaust* the facts (Putnam 1974).

6 Reichenbach

There is controversy as to whether Reichenbach was an empiricist or a conventionalist about geometry. What is certain is that he was inspired a great deal by Poincaré's new category of conventions.

- Mathematics as implicitly defined structures (Hilbert, Schlick, Carnap).
- Coordinative definitions and the "relative *a priori*": constitutive, but not apodictic. Principles that are "not necessary to believe, but if believed they must be believed necessary". (An anticipation of Kuhnian paradigms?)

- Do apparently rigid objects change size as they are transported?

  The problem does not concern a matter of *cognition* but of *definition*. There is no way of knowing whether a measuring rod retains its length when it is transported to another place; a statement of this kind can only be introduced by a definition. For this purpose a coordinative definition is to be used, because two physical objects distant from each other are *defined* as equal in length. It is not the *concept* equality of length which is to be defined, but a *real object* corresponding to it is to be pointed out. A physical structure is coordinated to the concept equality of length, just as the standard meter is coordinated to the concept unit of length. (1958, 16)

- "Conditional definition"—e.g. the definition of *congruence*.

- The elimination of universal forces: conventional or constitutive of physical geometry?

7 Further Reading

- Carnap, R. (1966), *An Introduction to the Philosophy of Science,* edited by Martin Gardner (New York: Dover), Chs. 13-18.

- Coffa, J. A. (1991), *To the Vienna Station: The Semantic Tradition from Kant to Carnap*, edited by L. Wessels (Cambridge: CUP), Ch. 3.

- Einstein, A. (1921), 'Geometry and experience,' lecture before the Prussian Academy of Sciences, January 27, 1921. Available online at: http://www.relativitycalculator.com/pdfs/einstein_geometry_and_experience_1921.pdf

- Helmholtz, H. (1921/1977), *Epistemological Writings: the Paul Hertz/Moritz Schlick Centenary Edition*, translated by M. Lowe, edited by R. S. Cohen & Y. Elkana (Dordrecht: D. Reidel Publishing Company).

- Ben-Menahem, Y. (2006), *Conventionalism: from Poincare to Quine* (Cambridge: CUP), Ch. 3.

- Poincaré, H. (1905/1952), *Science and Hypothesis* (New York: Dover), Chs. 3-5.

- Putnam, H. 'The refutation of conventionalism', *Nous* **8**, pp. 25-40.

- Reichenbach, H. (1958), *The Philosophy of Space & Time*, translated by M. Reichenbach & J. Freund (New York: Dover), §§1-15.